Ramsey interferometry with a two-level generalized Tonks-Girardeau gas

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We propose a solvable generalization of the Tonks-Girardeau model that describes a coherent one-dimensional (1D) gas of cold two-level bosons which interact with two external fields in a Ramsey interferometer. They also interact among themselves by idealized, infinitely strong contact potentials, with interchange of momentum and internal state. We study the corresponding Ramsey fringes and the quantum projection noise which, essentially unaffected by the interactions, remains that for ideal bosons. The dual system of this gas, an ideal gas of two-level fermions coupled by the interaction with the separated fields, produces the same fringes and noise fluctuations. The cases of time-separated and spatially separated fields are studied. For spatially separated fields the fringes may be broadened slightly by increasing the number of particles, but only for large particle numbers far from present experiments with Tonks-Girardeau gases. The uncertainty in the determination of the atomic transition frequency diminishes, essentially with the inverse root of the particle number. The difficulties to implement the model experimentally and possible shortcomings of strongly interacting 1D gases for frequency standards and atomic clocks are discussed.

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I. INTRODUCTION

A basic feature of the observed interference fringes in a standard Ramsey experiment is that their width is determined by the inverse of the time taken by the atoms to cross the intermediate drift region. For precision measurement purposes, as in atomic clocks, this motivates the use of very slow (ultracold) atoms, and therefore the development of laser cooling techniques has changed the entire prospects of frequency standards [1]. Experimentally, atomic velocities of the order of 1 cm/s and smaller can be achieved, and space-based clocks are in development to eliminate gravitational effects in the motion of such slow particles [2]. Laser cooled atoms are also interesting in metrology and interferometry because of the possibility to achieve narrow velocity distributions and avoid averaging effects. In addition, fundamentally new effects may arise by using coherent few-body or many-body states as input in the form of condensates or otherwise: for example, there exist proposals to beat the limitations imposed by quantum projection noise using entanglement [3,4].

In spite of the above, the motto "the slower the better" in the context of atomic clocks has actually a limited domain beyond which quantum motional phenomena may affect strongly and eventually deform totally the usual Ramsey pattern. If the slow atom moves initially along the x axis and the fields are oriented perpendicularly along the y axis, there are two origins of modification of the standard Ramsey result [5]. First, the absorption of a photon leads to a transverse momentum transfer on the atom, such that the excited state separates in space from the ground state. This is negligible for microwaves but not for optically induced (one- or two-photon) transitions. The effect can be understood classically by means of energy conservation and momentum conservation in y direction. It has been studied in detail by Bordé and co-workers [6] and multibeam setups have been implemented to correct for this separation in order to observe quantum interference [7–9]. Second, the field acts as a barrier for the longitudinal motion of the atom, and quantum reflection and tunneling may occur. Thus, momentum in x direction is not conserved as a consequence of the x dependence of the fields. For microwave fields and the corresponding Rabi frequencies these quantized motion effects are tiny for present atomic velocities but may become important for deeply ultracold particles. Moreover, in view of a proclaimed near-future accuracy of frequency standards of 10^{-18} [10], even those tiny effects have to be studied beyond the limits of the standard theoretical description of the Ramsey pattern. Recently, we have given an exact quantum result of the Ramsey fringes for guided atoms as a function of the detuning including quantum tunneling and reflection by means of two-channel recurrence relations [11].

Apart from quantum motion effects affecting ensembles of independent particles, other effects are due to the importance of quantum statistics and interactions. The use of a Bose-Einstein condensate for an atomic clock immediately comes to mind, but the improvements associated with low velocities and narrow velocity distribution may be compensated by negative effects, such as collisional shifts and instabilities leading to the separation of the gas cloud [12,13].

A natural candidate for further exploration is the Tonks-Girardeau (TG) regime of impenetrable, tightly confined bosons subjected to "contact" interactions [14,15], since

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some of its properties are complementary to those of the condensates. In particular, the TG requirement of strong contact interactions implies similarities between the bosonic system and a "dual" system of freely moving fermions, with all local correlation functions and the elementary excitations of the TG bosons and the dual system of fermions being actually equal. Other important feature of the TG gas is its one dimensional (1D) character. Olshanii showed [16–18] that when a bosonic vapor is confined in a wave guide with transverse trapping so tight and temperature so low that the transverse vibrational excitation quantum $\hbar\omega_\perp$ is larger than available longitudinal zero point and thermal energies, the effective dynamics becomes one dimensional, and accurately described by a 1D Hamiltonian with δ-function interactions $g_{1D}\delta(x_j-x_\ell)$, where x_j and x_ℓ are 1D longitudinal position variables. This is the Lieb-Liniger (LL) model, exactly solved in 1963 by a Bethe ansatz method [19]. The coupling constant g_{1D} can be tuned along a broad range of values by varying the magnetic field (and thus the three dimensional s-wave scattering length) or the confinement (confinement induced resonances [16,17]) near a Feschbach resonance; this allows in particular to reach the Tonks-Girardeau regime of impenetrable bosons, which corresponds to the $g_{1D}\to\infty$ limit of the LL model. The limiting regime has been realized experimentally [20,21], and was solved exactly in 1960 [14,15] by the so-called Fermi-Bose mapping to the ideal Fermi gas.

In metrology and atomic interferometry, the tight 1D confinement along a waveguide is a simplifying feature since no transversal motional branches have to be considered with the possible bonus of an increased signal. Nevertheless, the confinement is by itself problematic for frequency standard applications, since it is carried out by means of magnetic or optical interactions which will in principle perturb the internal state levels of the atom. Several schemes have been proposed to mitigate this problem and compensate or avoid the shifts due to magnetic [12] or optical interactions [22,23], and we shall assume hereafter that such a compensation is implemented.

The possible applications in interferometry are a strong motivation for current research in TG gases. Interference effects have been examined so far in a few publications in which internal states have not played any role [24,25]. Indeed, a TG model including internal states and an external interaction coupling them has not been discussed, although optically guided systems with free spin subjected to potentials for singlet and triplet interactions have been studied by means of effective LL models [26,27]. Note also that a model applicable to a two-level LL gas coupled by an on-resonance laser has been solved by nested Bethe ansatz [28].

In this paper, we investigate the implications in Ramsey interferometry of a model in the spirit of the original (structureless) TG gas, but with internal structure. The idealized, strong contact interactions of the model allow to achieve essential solvability of the dynamical problem in the Ramsey two-field excitation setup by simple quadrature: the collisions are characterized by internal state and momentum exchange, which reduce to the usual impenetrable constraint for collisions in the same internal channel. The difficulties and somewhat extreme requirements to implement the generalized TG gas may have negative implications in metrology applications which will also be discussed.

We shall study different configurations for the two fields, both in space and time domains. They are conceptually different and the mathematical treatment is different too. For reasonable parameters, however, the results turn out to be very similar.

II. TWO-LEVEL TONKS-GIRARDEAU GAS WITH EXCHANGE, CONTACT INTERACTIONS

A. Notation and contact interactions

We shall propose here a generalization of the Tonks-Girardeau gas for two-level impenetrable atoms. First we shall need to review or introduce some notation and basic concepts. In one dimension the state of a single two-level atom may be written as a two-component "spinor"

$$\Phi_n(x_1)=\sum_{b=g,e}\phi_n^{(b)}(x_1)|b\rangle, \tag{1}$$

where $n=1,2,3,\ldots$. is a label to distinguish different spinors and b is a generic index for the internal bound state, which may be g (ground) or e (excited). (Remark 1: Note that in general g and e do not necessarily correspond to states with definite values of the component of the electronic spin in one direction, i.e., the word "spinor" is here synonym of "two-component wave vector;" Remark 2: the subindex n will later on correspond to states prepared as harmonic oscillator eigenstates of a longitudinal trap.) One-particle states may be combined to form two-particle ones with the form

$$\Phi_{nn'}(x_1,x_2)=\sum_{b,b'}\phi_n^{(b)}(x_1)\phi_{n'}^{(b')}(x_2)|bb'\rangle, \tag{2}$$

and similarly for more particles. The convention in $|bb'\rangle$ is that b is for particle 1 and b' for particle 2. This will in some equations be indicated even more explicitly adding a particle subscript to the internal state label, b_1, b_2, etc.

To discuss the contact interactions of the model consider now the usual Pauli operators acting on one-particle internal state vectors,

$$\sigma_X=|g\rangle\langle e|+|e\rangle\langle g|,$$

$$\sigma_Y=i(|g\rangle\langle e|-|e\rangle\langle g|),$$

$$\sigma_Z=|e\rangle\langle e|-|g\rangle\langle g|, \tag{3}$$

and the corresponding three-component operator $\hat{\mathbf{S}}_j=\boldsymbol{\sigma}_j/2$ for particle j analogous to the spin-1/2 angular momentum operator. If $\mathbf{S}=\mathbf{S}_1+\mathbf{S}_2$, $\mathbf{S}^2$ has eigenvalues $S(S+1)$ with $S=0$ and $S=1$ corresponding to singlet and triplet subspaces spanned by $|-\rangle\equiv(|eg\rangle-|ge\rangle)/\sqrt{2}$ and $\{|gg\rangle,|ee\rangle,|+\rangle\equiv(|eg\rangle+|ge\rangle)/\sqrt{2}\}$, respectively.

Assume now the following Hamiltonian

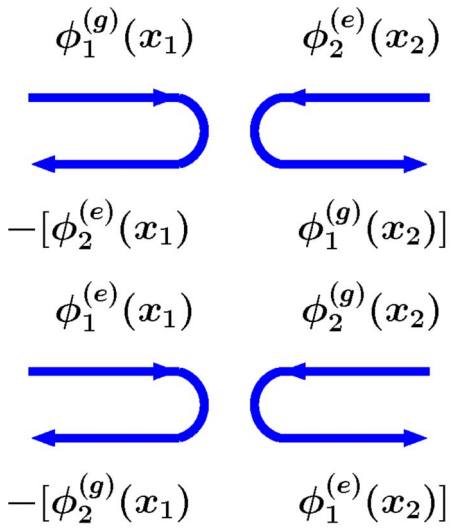

FIG. 1. (Color online) Diagrammatic representation of the collisions for particles with different internal states in the sector $x_1 < x_2$: the particles interchange their momentum and internal state picking up an additional phase (minus sign).

$$\hat{H}_{\text{coll}} = -\frac{\hbar^2}{2m}\sum_{j=1}^{2}\partial_{x_j}^2 + v_s(x_{12})\hat{P}_{12}^s + v_t(x_{12})\hat{P}_{12}^{t,el}. \tag{4}$$

Here $x_{12}=x_1-x_2$, $\hat{P}_{12}^s=|-\rangle\langle-|=\frac{1}{4}-\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2$ is the projector onto the subspaces of singlet functions, and $\hat{P}_{12}^{t,el}=|gg\rangle\langle gg|+|ee\rangle\times\langle ee|+|+\rangle\langle+|$ is a projector onto the triplet subspace restricted to elastic processes.

The internal Hilbert space can be written as the sum of singlet and triplet subspaces as $\mathcal{H}_s\oplus\mathcal{H}_t$. Suppose now that the reflection amplitude for relative motion in such representation takes the values $+1,-1$ in singlet and triplet subspaces respectively. The particles are impenetrable and these values correspond to a hard wall potential v_t at $x_{12}=0$ for all collisions in triplet channels $|gg\rangle\rightarrow|gg\rangle$, $|ee\rangle\rightarrow|ee\rangle$, or $|+\rangle\rightarrow|+\rangle$, whereas v_s has, in addition to the hard wall at $x_{12}=0$, a well of width l and depth V, so that the reflection amplitude becomes $R=+1$ in the limit in which the well is made infinitely narrow and the well infinitely deep, keeping $(2mV/\hbar^2)^{1/2}l=\pi/2$ [26,29–31]. (Notice that in these references the well applies to the triplet subspace and not to the singlet subspace as here. This infinite well potential appears in the so-called fermionic Tonks-Girardeau gas [26,29,30].)

Translated into the g,e basis and for the sector $x_1<x_2$ this implies that in all collisions between atoms in g or e and well-defined momenta, they interchange their momenta (the relative momentum changes sign), as well as their internal state, with the outgoing wave function picking up a minus sign because of the hard-core reflection, as shown in Fig. 1. For $x_1<x_2$ and equal internal states such collision is represented by

$$e^{ikx_1}e^{ik'x_2}|bb\rangle - [e^{ik'x_1}e^{ikx_2}]|bb\rangle, \tag{5}$$

whereas for $b\neq b'$,

$$e^{ikx_1}e^{ik'x_2}|bb'\rangle - [e^{ik'x_1}e^{ikx_2}]|b'b\rangle. \tag{6}$$

In the diagonal case of equal internal states the spatial part vanishes at contact, $x_1=x_2$, whereas in the nondiagonal case it does not, but note that in Eq. (6) only the "external region" is considered, disregarding the infinitely narrow well region.

We shall proceed now to discuss a possible implementation of these contact interactions. First of all, undesired inelastic collisions are expected to be substantially reduced by confinement at low collision energies, so as to limit the internal space to the two internal levels of the model and to the simple processes considered [32]. The internal levels of ^{133}Cs clocks, $|F=3,m_F=0\rangle$ and $|F=4,m_F=0\rangle$, combine spin-down and -up components, and the different triplet elastic channels may have different scattering amplitudes. We may again rely on strong confinement and low energies to expect that the inner electronic cores will produce effective reflection coefficients close to -1, independently of the internal state.

A second and important condition is the need to achieve strongly attractive singlet collisions. For two bosons in the singlet subspace, the space wave function is antisymmetric, so that s-wave scattering is forbidden; therefore the interactions are governed to leading order by a 3D p-wave scattering amplitude and can be enhanced by a 1D odd-wave confinement-induced Feschbach resonance (CIR), which allows in principle to engineer v_s and achieve a strong attraction as required above. For two spin-polarized interacting fermions [26,29,30,33], the spatial symmetry of the subspaces is interchanged with respect to the bosonic case, and one requires a p-wave scattering in the triplet subspace. However, in the rest of the paper we shall focus on the former, bosonic case, whose dual, auxiliary system of free fermions we introduce next.

B. Two noninteracting fermions

Let us consider now a fermionic state made of two noninteracting one-particle states with the form

$$\Psi_F(x_1,x_2) = \frac{1}{\sqrt{2}}\begin{vmatrix}\Phi_1(x_1) & \Phi_1(x_2)\\ \Phi_2(x_1) & \Phi_2(x_2)\end{vmatrix} = \frac{1}{\sqrt{2}}\det_{n,j=1}^2\Phi_n(x_j)$$
$$= \frac{1}{\sqrt{2}}\sum_{b_1,b_2=g,e}\begin{vmatrix}\phi_1^{(b_1)}(x_1) & \phi_1^{(b_2)}(x_2)\\ \phi_2^{(b_1)}(x_1) & \phi_2^{(b_2)}(x_2)\end{vmatrix}|b_1b_2\rangle,$$

with the state sign changing by switching particles 1 and 2 *and* the internal states. We insist that these fermions do not interact among themselves, but they could interact with an external potential as we shall see later. To construct the state we are simply combining two Hartree products of one-particle two-component wave vectors Φ_n, each vector corresponding to the solution of the Schrödinger equation for a single fermion with internal structure. Ψ_F is thus a solution of the Schrödinger equation for two fermions without mutual interaction.

More explicitly,

$$
\begin{aligned}
2^{1/2}\Psi_F(x_1,x_2) = &[\phi_1^{(g)}(x_1)\phi_2^{(g)}(x_2) - \phi_2^{(g)}(x_1)\phi_1^{(g)}(x_2)]|gg\rangle \\
&+ [\phi_1^{(e)}(x_1)\phi_2^{(e)}(x_2) - \phi_2^{(e)}(x_1)\phi_1^{(e)}(x_2)]|ee\rangle \\
&+ [\phi_1^{(g)}(x_1)\phi_2^{(e)}(x_2) - \phi_2^{(g)}(x_1)\phi_1^{(e)}(x_2)]|ge\rangle \\
&+ [\phi_1^{(e)}(x_1)\phi_2^{(g)}(x_2) - \phi_2^{(e)}(x_1)\phi_1^{(g)}(x_2)]|eg\rangle. \qquad (7)
\end{aligned}
$$

It might at first be surprising that the spatial part for the nondiagonal components $|ge\rangle$ or $|eg\rangle$ does not vanish at $x_1 = x_2$, but one can check that the state is indeed fermionic noticing that, with our notation, the interchange of particle label and internal state leaves in each term the internal state vectors $|bb'\rangle$ unchanged. Thus the interchange symmetry operation simply changes the sign of the total state.

C. Mapping to a bosonic wave function

An associated bosonic system of interacting atoms, totally symmetric under $(x_i,b_i) \leftrightarrow (x_j,b_j)$ permutations may be now obtained by means of the Bose-Fermi mapping, $\Psi_B(x_1,x_2) = \mathcal{A}\Psi_F(x_1,x_2)$, where the antisymmetric unit function is $\mathcal{A} = \mathrm{sgn}(x_2 - x_1)$.

For the sector $x_1 < x_2$ we could use Eq. (7) directly, or rearrange it slightly to construct the bosonic wave function as

$$
\begin{aligned}
2^{1/2}\Psi_B(x_1,x_2) = &[\phi_1^{(g)}(x_1)\phi_2^{(g)}(x_2) - \phi_2^{(g)}(x_1)\phi_1^{(g)}(x_1)]|gg\rangle \\
&+ [\phi_1^{(e)}(x_1)\phi_2^{(e)}(x_2) - \phi_2^{(e)}(x_1)\phi_1^{(e)}(x_2)]|ee\rangle \\
&+ [\phi_1^{(g)}(x_1)\phi_2^{(e)}(x_2)]|ge\rangle - [\phi_2^{(e)}(x_1)\phi_1^{(g)}(x_2)] \\
&\times|eg\rangle + [\phi_1^{(e)}(x_1)\phi_2^{(g)}(x_2)]|eg\rangle \\
&- [\phi_2^{(g)}(x_1)\phi_1^{(e)}(x_2)]|ge\rangle. \qquad (8)
\end{aligned}
$$

In the complementary sector $x_1 > x_2$ we would obtain the same form except for a global minus sign consistent now with the bosonic character. The resulting bosonic state is discontinuous at contact and cannot represent noninteracting bosons. Comparing Eq. (8) with Eqs. (5) and (6), we may interpret the bosonic function as the result of contact collisions between impenetrable atoms that interchange internal state and momenta, exactly as described in Sec. II A. In other words, Ψ_B is a bosonic solution of the two-body Schrödinger equation with Hamiltonian (4), with an infinite wall for the triplet potential, whereas the singlet potential, in addition to the infinite wall has an infinitely narrow and deep well. Ψ_B represents the wave function only outside the narrow well. Figure 1 provides a pictorial representation of the last two terms in Eq. (8) within the sector $x_1 < x_2$.

D. Generalization for *N* atoms

We have in summary constructed a bosonic wave function for a system of two particles subjected to contact interactions with internal state and momentum interchange, using a dual system of two noninteracting fermions and the antisymmetric unit function. The generalization to *N*-atoms is straightforward. The state for *N* noninteracting fermions with internal structure takes the form

$$
\begin{aligned}
\Psi_F(x_1,\ldots,x_N) &= \frac{1}{\sqrt{N!}}\det_{n,j=1}^{N}\Phi_n(x_j) \\
&= \frac{1}{\sqrt{N!}}\sum_{b_1,\cdots,b_N=g,e}\begin{vmatrix} \phi_1^{(b_1)}(x_1) & \cdots & \phi_1^{(b_N)}(x_N) \\ \vdots & \ddots & \vdots \\ \phi_N^{(b_1)}(x_1) & \cdots & \phi_N^{(b_N)}(x_N) \end{vmatrix} \\
&\quad \times|b_1\cdots b_N\rangle,
\end{aligned}
$$

and

$$
\Psi_B(x_1,\ldots,x_N) = \mathcal{A}\Psi_F(x_1,\ldots,x_N), \qquad (9)
$$

where

$$
\mathcal{A} = \prod_{1\le j<k\le N}\mathrm{sgn}(x_k - x_j), \qquad (10)
$$

is the bosonic solution of the time-dependent or stationary Schrödinger equation for the Hamiltonian

$$
\hat{H}_{\mathrm{coll}} = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\partial_{x_j}^2 + \sum_{1\le j<\ell\le N}[v_s(x_{j\ell})\hat{P}_{j\ell}^{s} + v_t(x_{j\ell})\hat{P}_{j\ell}^{t,el}], \qquad (11)
$$

with the same contact interactions as before.

The density profile, normalized to *N* particles, which gives the appearance of the cloud, is defined by

$$
\rho_N(x) = N\int \|\Psi_B(x_1,\ldots,x_N)\|^2 dx_2\ldots dx_N. \qquad (12)
$$

Provided that the one-particle spinor states Φ_n are orthonormal, as they will always be hereafter, the density profile reads

$$
\rho_N(x) = \sum_{b=g,e}\sum_{n=1}^{N}|\phi_n^{(b)}(x)|^2 = \sum_{b=g,e}\rho_N^{(b)}(x), \qquad (13)
$$

where the density profile for each of the channels defined by the two internal levels is given by

$$
\rho_N^{(b)}(x) = \sum_{n=1}^{N}|\phi_n^{(b)}(x)|^2. \qquad (14)
$$

The simplicity achieved by our model parallels that of the usual (structureless) TG gas in the sense that an *N*-body wave function with interactions is obtained from freely moving one-body states. Even more, this property is preserved by adding an interaction affecting the individual atoms only and coupling the internal levels. This is precisely the type of interaction that we find in the Ramsey interferometer.

III. QUANTUM PROJECTION NOISE IN THE TWO-LEVEL GENERALIZED TONKS-GIRARDEAU GAS

Itano *et al.* [34] studied the quantum projection noise for a Hartree product state of the form $|b_1,\ldots,b_N\rangle = \otimes_{i=1}^{N}|b_i\rangle$. This noise is, in other words, the fluctuation of the number of excited atoms for measurements made in the *N*-body system. In Ramsey interferometry, the error in the determination of

the atomic frequency depends on the ratio between the (root of the) fluctuation in the number of excited atoms and the derivative of the signal (proportional to the number of excited atoms) with respect to detuning.

Here we shall obtain the noise associated with the state $\Psi_B(x_1,\ldots,x_N)$. We shall follow [34] and introduce the operator

$$\hat{S}_Z = \sum_{i=1}^{N} \hat{S}_{i_Z} = \frac{1}{2}\sum_{i=1}^{N} (|e_i\rangle\langle e_i| - |g_i\rangle\langle g_i|), \tag{15}$$

where it is assumed, as usual, that each term in the summation is multiplied by the identity operator for all the other atoms.

The quantum projection noise of a signal is proportional to the variance

$$(\Delta S_Z)^2 = \langle \hat{S}_Z^2\rangle - \langle \hat{S}_Z\rangle^2, \tag{16}$$

and expressions for both terms will now be worked out. First, notice that S_Z commutes with $\mathcal{A}$ so that, using $\mathcal{A}^2=1$, we may compute the expectation values substituting Ψ_B by Ψ_F, i.e., for the more easily tractable, dual fermionic system. Since Ψ_F is antisymmetric it follows that

$$\langle \hat{S}_Z\rangle = N\langle \Psi_B|\hat{S}_{1_Z}|\Psi_B\rangle \tag{17}$$

and

$$\langle \hat{S}_Z^2\rangle = \sum_{i=1}^{N}\langle \hat{S}_{i_Z}^2\rangle + \sum_{i,ji\neq j}\langle \hat{S}_{i_Z}\hat{S}_{j_Z}\rangle = N\langle \hat{S}_{1_Z}^2\rangle + N(N-1)\langle \hat{S}_{1_Z}\hat{S}_{2_Z}\rangle. \tag{18}$$

Equation (17) takes the form

$$N\langle \Psi_B|\hat{S}_{1_Z}|\Psi_B\rangle = N\int \prod_i dx_i \Psi_B^*(x_1,\ldots,x_N)\hat{S}_{1_Z}\Psi_B(x_1,\ldots,x_N) = \sum_{n=1}^{N}\langle \Phi_n|\hat{S}_{1_Z}|\Phi_n\rangle = \frac{1}{2}\sum_{n=1}^{N}\alpha_n, \tag{19}$$

where $\alpha_n = p_n^{(e)} - p_n^{(g)}$ is the probability difference for the excited and ground state in state n.

In Eq. (18), note that $S_{1_Z}^2 = \mathbf{1}_N/4$ and therefore $\langle \hat{S}_{1_Z}^2\rangle = 1/4$ for the normalized state $\Psi_B(x_1,\ldots,x_N)$. The cross term can be evaluated as

$$N(N-1)\langle \hat{S}_{1_Z}\hat{S}_{2_Z}\rangle = \sum_{n,m}(\langle \Phi_n|\hat{S}_{1_Z}|\Phi_n\rangle\langle \Phi_m|\hat{S}_{2_Z}|\Phi_m\rangle - \langle \Phi_n\Phi_m|\hat{S}_{1_Z}\hat{S}_{2_Z}|\Phi_m\Phi_n\rangle) = \frac{1}{4}\sum_{n,m}(\alpha_n\alpha_m - \Delta_{nm}), \tag{20}$$

where the Δ_{nm} terms are positive and defined as

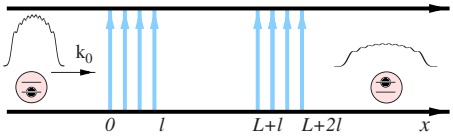

FIG. 2. (Color online) Schematic setup for Ramsey interferometry of guided atoms in the spatial domain. The atoms are prepared in the ground state and the probability of excitation is measured after passing the two fields.

$$\Delta_{nm} = \left|\int dx[\phi_n^{(e)}(x)]^*\phi_m^{(e)}(x) - [\phi_n^{(g)}(x)]^*\phi_m^{(g)}(x)\right|^2 = |\langle \phi_n^{(e)}|\phi_m^{(e)}\rangle - \langle \phi_n^{(g)}|\phi_m^{(g)}\rangle|^2. \tag{21}$$

Combining these results, the variance simply reads

$$(\Delta S_Z)^2 = \frac{N}{4} - \frac{1}{4}\sum_{n,m}\Delta_{nm}. \tag{22}$$

If the dependence of single particle expectation value $\langle \Phi_n|\hat{S}_{1_Z}|\Phi_n\rangle$ on n can be neglected, so that $\alpha_n \simeq \alpha$ for all n,

$$(\Delta S_Z)^2 = \frac{N}{4}\left(1 - \frac{1}{N}\sum_n \alpha_n^2\right) - \frac{1}{4}\sum_{n,mn\neq m}\Delta_{nm} \simeq \frac{N}{4}(1-\alpha^2) - \frac{1}{4}\sum_{n,mn\neq m}\Delta_{nm} < (\Delta S_Z)_0^2, \tag{23}$$

where we have identified a term $(\Delta S_Z)_0^2 \equiv \frac{N}{4}(1-\alpha^2)$ corresponding to the quantum noise for the Hartree product state in Ref. [34], and a negative correction for the strongly interacting bosonic TG gas.

IV. THE RAMSEY INTERFEROMETER

Ramsey interferometry with guided ultracold atoms has recently been discussed in Ref. [11]. Here we consider a system of N two-level atoms in the Tonks-Girardeau regime, initially confined in their ground internal states in a harmonic trap of frequency ω. All energy scales are supposed to be much smaller than the transverse excitation energy $\hbar\omega_\perp$, so that the radial degrees of freedom are frozen out and the system is effectively one dimensional. The cloud is prepared in the ground state, and released by switching off the trap along the x axis at time $t=0$ ($\omega=0$ for $t>0$); a momentum kick $\hbar k_0$ is also applied, so that the cloud moves along the x axis towards the two separated oscillating fields localized between 0 and l and between $l+L$ and $2l+L$ (Fig. 2). The initial state is prepared far from the first field. We thus have to take into account the spatial width (root of the variance) of the highest state, $\delta_N=[(N+1/2)\hbar/(m\omega)]^{1/2}$, and choose the central initial position of the harmonic trap $x_0<0$ so that $|x_0| \gg \delta_N$.

In an oscillating-field-adapted interaction picture (which does not affect the collisional Hamiltonian) and using the Lamb-Dicke (see the next section), dipole and rotating-wave approximations the Hamiltonian is, for each of the particles,

$$H = \frac{\hat{p}^2}{2m} - \hbar\Delta|e\rangle\langle e| + \frac{\hbar}{2}\Omega(\hat{x})(|g\rangle\langle e| + |e\rangle\langle g|), \qquad (24)$$

where the first term counts for the kinetic energy of the atom, $\Delta=\omega_L-\omega_{12}$ is the detuning between the oscillating field frequency and atomic transition frequency, and $\Omega(x)$ is the position-dependent Rabi frequency. For the explicit x dependence we assume mesa functions, $\Omega(x)=\Omega$ for $x\in[0,l]$ and $x\in[l+L,2l+L]$ and zero elsewhere. In addition, we have to include the interparticle interactions but this is done implicitly by means of the wave function (9) and its boundary conditions at contact.

The Ramsey pattern is defined by the dependence on the detuning of the probability of excited atoms after the interaction with the two field regions. From Eq. (14) it follows that $P_e=\frac{1}{N}\Sigma_{n=1}^{N}p_n^{(e)}$, which is a remarkably simple result for an N-body system with external and interparticle interactions. Once a particle incident from the left and prepared in the state $e^{ik_0(x-x_0)}\phi_n(x-x_0)|g\rangle$ at $t=0$ has passed completely through both fields, the probability amplitude for it to be in the excited state is

$$\phi_n^{(e)}(x,t) = \frac{1}{\sqrt{2\pi}}\int dk e^{iqx-ik^2\hbar t/(2m)}T_{ge}(k)\tilde{\phi}_n(k), \qquad (25)$$

where $\tilde{\phi}_n(k)$ is the wave number representation of the kicked nth harmonic eigenstate,

$$\tilde{\phi}_n(k) = \frac{(-i)^n}{\sqrt{2^n n!}}\left(\frac{2\delta_0^2}{\pi}\right)^{1/4} e^{-\delta_0^2(k-k_0)^2}e^{-ikx_0}H_n[\sqrt{2}\delta_0(k-k_0)], \qquad (26)$$

the momentum in the excited state is $q=\sqrt{k^2+2m\Delta/\hbar}$, the spatial width of the $n=0$ state is $\delta_0=[\hbar/(2m\omega)]^{1/2}$, H_n the Hermite polynomials, and T_{ge} is the "double-barrier" transmission amplitude for the excited state corresponding to atoms incident in the ground state (the excited state probability for monochromatic incidence in the ground state is $\frac{q}{k}|T_{ge}|^2$). The full quantum treatment of T_{ge} can be done by means of the two-channel recurrence relations connecting it with one-field transmission and reflection amplitudes [11].

Our numerical simulations are for $l=1$ cm, $L=10$ cm, $N=10$, and $v_0=1$ cm/s. Figure 3 shows that the variation of the excitation probability for different harmonic eigenstates is negligible in the scale shown, and in fact the curves for the central fringe are indistinguishable from the semiclassical result of Ramsey (which assumes classical motion for the center of mass, uncoupled from the internal levels),

$$P_{e,scl}(\Delta) = \frac{4\Omega^2}{\Omega'^2}\sin^2\left(\frac{\Omega'\tau}{2}\right)\left[\cos\left(\frac{\Omega'\tau}{2}\right)\cos\left(\frac{\Delta T}{2}\right) - \frac{\Delta}{\Omega'}\sin\left(\frac{\Omega'\tau}{2}\right)\sin\left(\frac{\Delta T}{2}\right)\right]^2, \qquad (27)$$

where $\tau=l/v_0$, $T=L/v_0$, and $\Omega'=(\Omega^2+\Delta^2)^{1/2}$.

There is in principle a broadening of the central fringe by increasing n due to the momentum broadening of vibrationally excited states. This effect may be expected however to be quite small for the few-body states of our calculations,

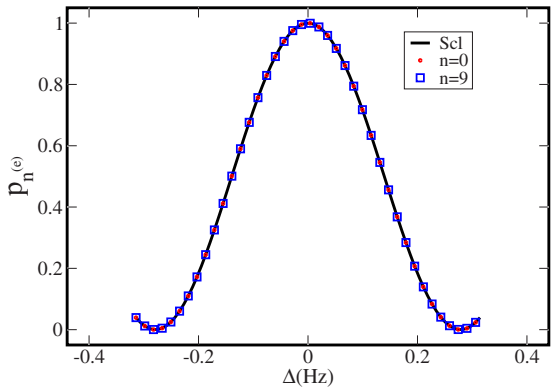

FIG. 3. (Color online) Central fringe for Ramsey interferometry in the spatial domain. The agreement is shown for the $n=0,9$ single-particle wave packets and the semiclassical result, for ^{133}Cs atoms, with $\hbar k_0/m=1$ cm/s, $l=1$ cm, $L=10$ cm, $\delta_0=20$ μm ($\omega\simeq 0.6$ Hz), and $t=15$ s.

$N=10$, which is in fact of the order of current experiments with TG gases ($N\approx 15,50$ in Refs. [20,21]). The width (root of the variance) of the velocity distribution around the central velocity $v_0=\hbar k_0/m$ for the nth state is

$$\Delta_v = \left[\left(n+\frac{1}{2}\right)\frac{\omega\hbar}{m}\right]^{1/2} = \sqrt{2n+1}\frac{\hbar}{2m\delta_0}, \qquad (28)$$

where we have used the spatial width of the $n=0$ state, $\delta_0=[\hbar/(2m\omega)]^{1/2}$. This will not affect significantly the width of the central fringe (proportional to the inverse of the crossing time T) as long as $\Delta_v/v_0\ll 1$. For $v_0=1$ cm/s, $\delta_0=20$ μm, the mass of ^{133}Cs, and $N=10$, this ratio is $\sim 5\times 10^{-3}$. N should be $\sim 4\times 10^5$ to obtain a ratio of order one, but this means four orders of magnitude more particles than in the existing experiments.

The error to estimate the atomic frequency from the Ramsey pattern depends on the ratio [34]

$$r = \frac{\Delta S_Z}{|\partial\langle S_Z\rangle/\partial\Delta|}, \qquad (29)$$

which we calculate at half height of the central interference peak. We compute ΔS_Z with Eq. (22). Since, according to the previous discussion, the excitation probabilities are essentially independent of n, Eq. (23) is an excellent approximation. Moreover, the correction to $(\Delta S_Z)_0$ due the particle correlations is negligible, with a relative error $[(\Delta S_Z)_0 - \Delta S_Z]/\Delta S_Z \sim 10^{-10}$ in our calculations. Since, in addition, the derivative in Eq. (29) is very well approximated by the semiclassical result, the ratio r essentially coincides with that for freely moving, uncorrelated particles [34] and, for $L\gg l$ it gives $1/(T\sqrt{N})$ for all Δ, see Fig. 4.

V. RAMSEY INTERFEROMETRY IN THE TIME DOMAIN FOR GUIDED ATOMS

An alternative to the previous setup is the separation of the fields in time rather than space but, at variance with the usual procedure, keeping the gas confined transversally at all

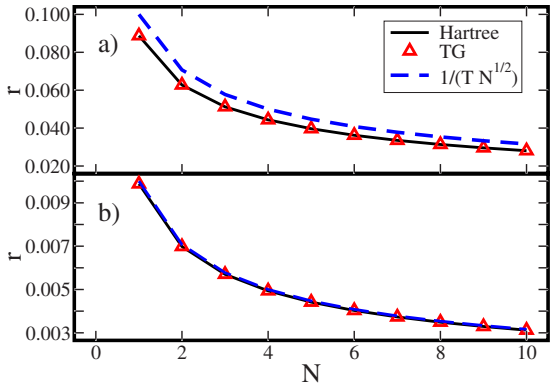

FIG. 4. (Color online) Quantum projection noise ratio, Eq. (29), for the Ramsey interferometry in the spatial domain. The two-level Tonks-Girardeau gas ratio is essentially that of a Hartree product (uncorrelated atoms). The same parameters as in Fig. 3 are used, with (a) $L=10$ cm, (b) $L=100$ cm.

times as required for the $1D$ regime of the TG gas, Fig. 5. Because of the tight confinement the transverse vibrational excitation is negligible so that the Ramsey pattern is given by the standard expression irrespective of the value of n. The whole TG gas therefore produces the usual Ramsey pattern (27) as we shall see in more detail.

A two level atom in a cigar shape trap with characteristic frequencies ω_x, ω_y, and ω_z ($\omega_x \ll \omega_y \sim \omega_z$) interacting with a (classical) laser field directed in the perpendicular y direction is described (in a laser adapted interaction picture) by the Hamiltonian

$$H = \sum_{i=x,y,z} \hbar\omega_i \left(a_i^\dagger a_i + \frac{1}{2} \right) - \hbar\Delta |e\rangle\langle e| + \frac{\hbar\Omega}{2} [e^{i\eta_y (a_y + a_y^\dagger)} \sigma_+ + \text{H.c.}], \tag{30}$$

where $\sigma_+ = |e\rangle\langle g|$. The Rabi frequency Ω is here a constant, independent of x, and $a_i^\dagger$ (a_i) are the creation (annihilation) operators of the vibrational modes in the direction of the subscript. The parameter $\eta_y = k_L y_0$ is known as the Lamb-Dicke (LD) parameter, with $y_0 = \sqrt{\hbar/2m\omega_y}$ being the extension of the atomic ground state in y direction. The vibrational levels in the longitudinal x direction are not coupled by the field if the x dependence of the field is negligible in the scale δ_N of the cloud. Also, motion in the z direction remains uncoupled.

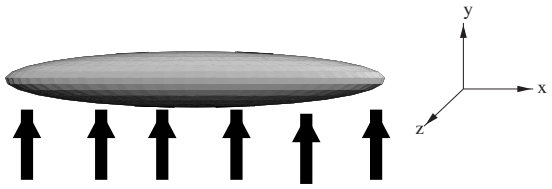

FIG. 5. Schematic setup for Ramsey interferometry in time domain. The TG gas is confined in a cigar-shaped trap and illuminated by a laser in y direction.

A. Lamb-Dicke regime

A particular interesting limit when dealing with trapped atoms interacting with laser fields is the so-called Lamb-Dicke regime. In this regime, the extension of the atom's wave function in the direction of the field is much smaller than the laser wavelength, i.e., $\eta_y \ll 1$. If the LD regime is assumed, it is natural to approximate the exponentials in the coupling term of the Hamiltonian (30) as $e^{\pm i\eta_y (a_y + a_y^\dagger)} \approx 1$, giving an approximate Hamiltonian

$$H_{LD} = \sum_{i=x,y,z} \hbar\omega_i \left(a_i^\dagger a_i + \frac{1}{2} \right) - \hbar\Delta |e\rangle\langle e| + \frac{\hbar\Omega}{2} (\sigma_+ + \sigma_-), \tag{31}$$

which does not couple the well-separated vibrational levels in the transversal y direction (the recoil frequency is much smaller than the trapping frequency). Within this approximation the number operators $n_i = a_i^\dagger a_i$ are some constant of motion for $i = x, y, z$ and thus the dynamics of the system is independent of the vibrational modes, reproducing the usual Ramsey fringe pattern (27) when time separated pulses are applied.

B. TG regime

A tight transversal confinement is needed in order to reach the TG regime, which is achieved when the dimensionless parameter $\gamma = mg/\hbar^2 n \gg 1$. Here $n \approx N/\delta_N$ is the linear density of the gas and g the 1D interaction strength, which is given by $g = 2\hbar^2 a/m y_0^2$, with a being the s-wave scattering length [35]. We may then write the criterion for being in the TG regime as $2a/n y_0^2 \gg 1$ or

$$\gamma = \frac{2a\delta_N}{N y_0^2} \gg 1. \tag{32}$$

If ^{133}Cs atoms in a trap with frequency $\omega_y \approx 2\pi \times 1$ MHz are considered, the transversal confinement turns out to be $y_0 \approx 6$ nm. For $\delta_0 \approx 20$ μm, a scattering length of $a \approx 100 a_0$, a_0 being the Bohr radius, and $N=10$, $\gamma \approx 2 \times 10^3$, well in the TG regime.

If the hyperfine transition of the ^{133}Cs atom at 9.192 GHz is driven, a LD parameter of $\eta_y = k_L y_0 \approx 10^{-6}$ is obtained for the transversal confinement of $y_0 \approx 6$ nm previously estimated, which is well inside the LD regime. The TG condition then imposes the LD condition for microwave transitions. For optical transitions, LD parameters of 0.05–0.1 are obtained, that can also be considered to be into the LD regime.

VI. SUMMARY AND DISCUSSION

In this paper we have explored the behavior of atomic clocks or frequency standards based on ultracold atoms in the strong interaction, Tonks-Girardeau, 1D regime. This is motivated by the benefits of increased flight times and narrow velocity widths of ultracold atoms, and the similarities between TG interacting bosons and freely moving fermions,

which suggest that collisional shifts and instabilities found for condensates may be avoided.

A solvable model of N bosons in 1D with contact interactions that interchange the momentum and internal state of the two-level atoms subjected to two oscillating fields has been worked out. For parameters in the ultracold regime, the system behaves similarly for spatial or temporal separation of the fields, and according to the semiclassical Ramsey pattern for independent, freely moving particles. Moreover, the quantum projection noise remains close to that of an ensemble of independent atoms.

For the two-level Tonks-Girardeau gas, the interactions do not worsen the quality of the Ramsey pattern but have the additional advantage of dramatically reducing the three-body correlation function [36,37] and therefore enhancing the stability of the gas with respect to the ideal case. We thus expect strongly interacting gases to play a remarkable role in interferometry with ultracold atoms in waveguides.

Our results are based on an idealized model and much remains to be done before a definite assessment can be made of the usefulness of the strongly interacting regime of ultracold guided atoms in time-frequency metrology. We have argued that strong confinement and a p-wave Feschbach resonance may provide the necessary conditions to implement the contact interactions of the model, but strong confinement may induce level perturbations difficult to compensate. In any case we hope that the present model will motive further analysis of atom interferometry with internal states in the strongly interacting regime. At the modeling level, one of the possibilities is to study other phase relations between singlet and triplet reflection amplitudes (which do not give the simple structure for the wave function found here); other possibilities are more complex mappings or mappings from free bosons to interacting fermions [38]; and there is of course the need to jump from the modeling level to an actual realistic atomic level scenario and investigate the parameters and species leading to a closer realization of contact interactions.

ACKNOWLEDGMENTS

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