Effects of interactions and noise on tunneling of Bose-Einstein condensates through a potential barrier

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We investigate theoretically the tunneling of a dilute Bose-Einstein condensate through a potential barrier. This scenario is closely related to recent experimental studies of condensates trapped in one-dimensional optical lattices. We derive analytical results for the tunneling rate of the condensate with emphasis on the effects of atom-atom interactions. Furthermore, we consider the effect of fluctuating barrier height to the tunneling rate. We have computed the tunneling rate as a function of the characteristic frequency of the noise. The result is seen to be closely related to the excitation spectrum of the condensate. These observations should be experimentally verifiable.

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I. INTRODUCTION

Tunneling of particles through nanoscale classically forbidden regions is a curious quantum effect which has been under keen study from the early days of modern physics. Not only the tunneling of individual particles, but also tunneling of coherent macroscopic matter waves has drawn wide interest especially since the discovery of the Josephson effect in superconductivity. The experimental realization of Bose-Einstein condensation in dilute atomic gases provides a unique possibility to study the tunneling of coherent matter waves on a macroscopic scale, for example through the barriers of a periodic optical lattice. The tunneling of Bose-Einstein condensates (BECs) through a tilted periodic potential is particularly interesting since it is closely related to the physics of Josephson junctions described effectively by a phase particle moving in a tilted washboard potential.

The behavior of matter waves in periodic potentials has been studied extensively especially in solid-state physics. The atom-atom interactions in gaseous Bose-Einstein condensates give rise to nonlinear effects which make systems consisting of a BEC in an optical lattice quite exceptional [1]: Macroscopic quantum interference effects have been observed in tunneling of a Bose-Einstein condensate through a tilted optical lattice [2]. In addition, tunneling of ultracold atoms through an accelerated optical lattice has been observed [3,4], Josephson junctions and arrays of them have been realized using Bose-Einstein condensates [2,5–7], and superfluidity of condensates has been studied in moving optical lattices [8,9]. The quantized energy levels in the wells of a tilted optical potential have been shown to play a prominent role in the tunneling, giving rise to tunneling resonances [10–13]. The frequencies of collective modes, which affect the tunneling rate, have been studied both theoretically [14] and experimentally [15,16] for magnetically trapped BECs in optical lattices. The coherent flow of a BEC through a double potential barrier has been studied theoretically, and transport resonances have been found [17]. Also, motion of BECs through disordered regions has been investigated [18,19].

In this work, we study the dynamics of a single-component dilute atomic Bose-Einstein condensate in a tilted optical lattice in the zero temperature limit. In practice, the origin of the tilt in the potential can be due to an external magnetic field, gravity, or acceleration of the optical lattice. We derive an analytic expression for the tunneling rate through a potential barrier using a semiclassical approximation. The result for the tunneling rate shows explicitly the dependence on the chemical potential of the condensate, the effective atom-atom interaction strength, and the form of the external potential in the tunneling region. Furthermore, we study the effect of noise on the interwell tunneling rate through a tilted sinusoidal potential. Finally, we investigate the effect of a harmonic drive on the tunneling rate and verify that the resulting resonances originate from the excitation of the lowest-lying eigenmodes of the system.

II. MEAN-FIELD THEORY

In the zero temperature limit, the time evolution of the condensate order parameter is determined by the Gross-Pitaevskii (GP) equation

$$\mathcal{H}(\mathbf{r})\Psi(\mathbf{r},t) = i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t), \tag{1}$$

where $\mathcal{H}(\mathbf{r})=-\frac{\hbar^2}{2m}\nabla^2+V_{\mathrm{ext}}(\mathbf{r})+g|\Psi(\mathbf{r},t)|^2$, m is the particle mass, $V_{\mathrm{ext}}(\mathbf{r})$ is the external potential, and $g=4\pi\hbar^2 a_s/m$ the atom-atom interaction coupling constant given in terms of the s-wave scattering length a_s. Stationary states of the system are solutions of the form $\Psi(\mathbf{r},t)=\Psi(\mathbf{r})e^{-i\mu t/\hbar}$, where the value of the chemical potential μ is fixed by the normalization condition $\int|\Psi(\mathbf{r})|^2 d\mathbf{r}=N$, where N is the number of particles in the condensate. With large enough particle numbers or interaction strengths such that the kinetic energy term may be neglected in the GP equation, the ground state of the system is approximately of the Thomas-Fermi form

$$\Psi_{\mathrm{TF}}(\mathbf{r}) = \sqrt{\frac{\mu - V_{\mathrm{ext}}(\mathbf{r})}{g}}. \tag{2}$$

This approximation is typically accurate in the interior of the condensate, but fails close to the surface of the cloud.

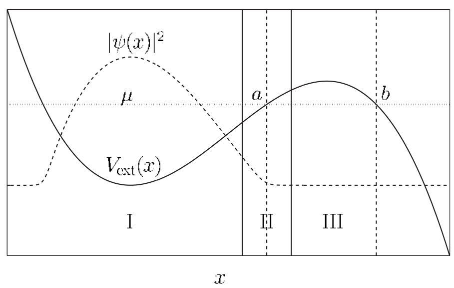

FIG. 1. A schematic diagram of the tunneling problem. The external potential $V_{\text{ext}}(x)$ is represented by the solid curve, and the square of the wave function $|\psi(x)|^2$ by the dashed curve. The chemical potential μ is denoted by the horizontal dotted line and the corresponding classical turning points a and b by the vertical dashed lines. The region II in the vicinity of the left-hand classical turning point where the order parameter is approximately of the Airy form is bounded by the vertical solid lines. The size of this region has been exaggerated for clarity.

Small-amplitude oscillations of the system can be studied by linearizing the GP equation for a given stationary state. This leads to the Bogoliubov eigenvalue equation

$$\begin{pmatrix} \mathcal{L}(\mathbf{r}) & \mathcal{M}(\mathbf{r}) \\ -\mathcal{M}^*(\mathbf{r}) & -\mathcal{L}(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u_q(\mathbf{r}) \\ v_q(\mathbf{r}) \end{pmatrix} = \hbar\omega_q \begin{pmatrix} u_q(\mathbf{r}) \\ v_q(\mathbf{r}) \end{pmatrix} \tag{3}$$

characterizing the eigenfrequencies ω_q and quasiparticle wave functions $u_q(\mathbf{r})$ and $v_q(\mathbf{r})$ of the eigenmode q. Above $\mathcal{L}(\mathbf{r})=\mathcal{H}(\mathbf{r})-\mu+2g|\Psi(\mathbf{r})|^2$ and $\mathcal{M}(\mathbf{r})=g\Psi^2(\mathbf{r})$.

We assume a cylindrical trap geometry and a tight harmonic confinement in the radial direction such that the radial harmonic oscillator length is much smaller than the characteristic length scale of density variations in the axial direction. Hence the radial density profile of low energy states is well approximated by a Gaussian. In this limit the GP equation simplifies to the one-dimensional form

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_{\text{ext}}(x) + \tilde{g}|\psi(x)|^2\right]\psi(x) = i\hbar\frac{\partial}{\partial t}\psi(x), \tag{4}$$

where $\tilde{g}$ is an effective interaction strength and the one-dimensional wave function is normalized to unity: $\int|\psi(x)|^2 dx=1$. In fact, the forthcoming analysis is essentially independent of the exact radial profile of the condensate, as long as the radial confinement is tight enough. A different radial distribution results merely in a different relation between the coupling constants $\tilde{g}$ and g, and hence the qualitative features of the results hold even if the condition of harmonic radial confinement is not satisfied.

III. SEMICLASSICAL APPROXIMATION TO TUNNELING RATES

In this section, we derive an analytic expression for the tunneling rate through a potential barrier. We assume that the stationary tunneling state $\psi(x)$ is approximately of the Thomas-Fermi form in region I, see Fig. 1. The left- and right-hand classical turning points, denoted by a and b, respectively, in the region of the barrier are solutions to the equation $V(x)=\mu$. If the barrier is strong enough, the nonlinear term in the GP equation can be neglected to a fair approximation in region II and beyond. We linearize the potential in the vicinity of the left-hand turning point a, which amounts to approximating the order parameter $\psi(x)$ by an Airy function in region II. Furthermore, the order parameter $\psi(x)$ is approximately given by the semiclassical form in the tunneling region, labeled as III. Finally, the tunneling rate is determined by the amplitude of the order parameter $\psi(x)$ at the right-hand turning point b.

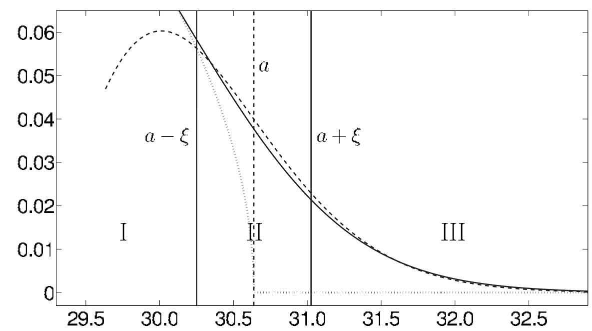

FIG. 2. Approximations used in the derivation of the analytic expression for the decay rate Γ. The Thomas-Fermi (TF) form for the ground state is given by the dotted line, the Airy function is given by the dashed line, and the actual stationary tunneling state wave function is given by the solid line. The points $a-\xi$ and $a+\xi$ are denoted by the solid vertical lines, and the left-hand classical turning point a is denoted by the vertical dashed line. The TF approximation is used in region I, $x\leq a-\xi$, the Airy solution in region II, $a-\xi\leq x\leq a+\xi$, and the semiclassical approximation in region III, $a+\xi\leq x\leq b$.

For sufficiently large values of the interaction strength $\tilde{g}$, the order parameter of the ground state of the system is well approximated by the Thomas-Fermi profile in the interior of the condensate. This approximation fails near the surface of the cloud where the kinetic energy term in the Gross-Pitaevskii equation is comparable to the interaction term. We denote the point at which these two terms are equal for $\psi_{\text{TF}}(x)$ by $a-\xi$, see Fig. 2. By linearizing the potential around a such that $V(x)-\mu\approx F(x-a)$, where F is a constant, it follows that $\xi=\frac{1}{2}\sqrt[3]{\hbar^2/(mF)}$. When the nonlinear term is neglected, the stationary GP equation reads

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + F(x-a)\right]\psi(x) = 0, \quad a-\xi\leq x\leq a+\xi. \tag{5}$$

The solution is given by the Airy function $\psi(x)=C_1\text{Ai}[(2mF/\hbar^2)^{1/3}(x-a)]$, where the second linearly independent solution is discarded due to its divergence for large values of the argument. This form yields a fair approximation to the actual wave function in the interval $[a-\xi,a+\xi]$, and it is fitted to the Thomas-Fermi form at $x=a-\xi$, yielding $C_1\approx\sqrt{\frac{F\xi}{\tilde{g}}}/\text{Ai}[-2^{-2/3}]$.

For high barriers the nonlinear term of the GP equation can be neglected to a good approximation in the region of the barrier and hence the GP equation reduces there to the ordinary Schrödinger equation. An approximate semiclassical solution in the region of the barrier is thus given by

$$\psi(x) = \frac{C_2}{\sqrt{\kappa(x)}} e^{-\int_a^x \kappa(y) \mathrm{d}y}, \quad a + \xi \leq x \lesssim b, \tag{6}$$

where $\kappa(x) = \sqrt{2m[V(x)-\mu]/\hbar^2}$. By fitting this form to the Airy function at $a+\xi$, we obtain

$$C_2 \approx \sqrt{\frac{F}{2\tilde{g}}} e^{1/3} \frac{\mathrm{Ai}[2^{-2/3}]}{\mathrm{Ai}[-2^{-2/3}]}. \tag{7}$$

To the right of the right-hand classical turning point the semiclassical solution is of the form

$$\psi(x) = \frac{C_3}{\sqrt{k(x)}} e^{i\int_b^x k(y)\mathrm{d}y}, \quad x \gtrsim b, \tag{8}$$

where $k(x) = \sqrt{2m[\mu - V(x)]/\hbar^2}$. The particle current corresponding to this solution at the right-hand turning point is given by $\Gamma \equiv \lim_{x\to b} j(x) = \frac{\hbar}{m}|C_3|^2$. By applying the connection formulas given, e.g., in Ref. [20], we obtain $|C_3| = |C_2 e^{-W(\mu)}|$, where

$$W(\mu) = \int_a^b \kappa(y) dy \tag{9}$$

measures the strength of the barrier at the chemical potential μ. Hence the tunneling rate through the barrier takes the form [21]

$$\Gamma = \frac{\hbar}{m} \frac{F}{\tilde{g}} \frac{e^{2/3}}{2} \left(\frac{\mathrm{Ai}[2^{-2/3}]}{\mathrm{Ai}[-2^{-2/3}]} \right)^2 e^{-2W(\mu)} \tag{10}$$

$$\approx 0.161 \frac{\hbar}{m} \frac{F}{\tilde{g}} e^{-2W(\mu)}. \tag{11}$$

The rate thus carries an exponential identical to that of the standard WKB result with the energy eigenvalue of the Schrödinger equation replaced by the chemical potential μ. The prefactor, usually called the *attempt frequency*, explicitly includes the coupling constant $\tilde{g}$.

To demonstrate the validity of the approximations employed above, we have calculated the tunneling rate Γ also numerically by integrating the time-dependent Gross-Pitaevskii equation. The instantaneous decay rate $\Gamma(t)$ can be deduced from the fraction of particles left in the condensate $N(t+dt)/N(t) = e^{-\Gamma(t)dt}$ after a time step dt. Hence by normalizing the wave function $\psi(x)$ after each time step, the decay rate for a constant interaction strength $\tilde{g}$ is given by

$$\Gamma = -\left\langle \frac{\ln[N(t+dt)/N(t)]}{dt} \right\rangle_t, \tag{12}$$

where $\langle \cdot \rangle_t$ denotes the time average. The potential employed was a single barrier of a tilted optical lattice

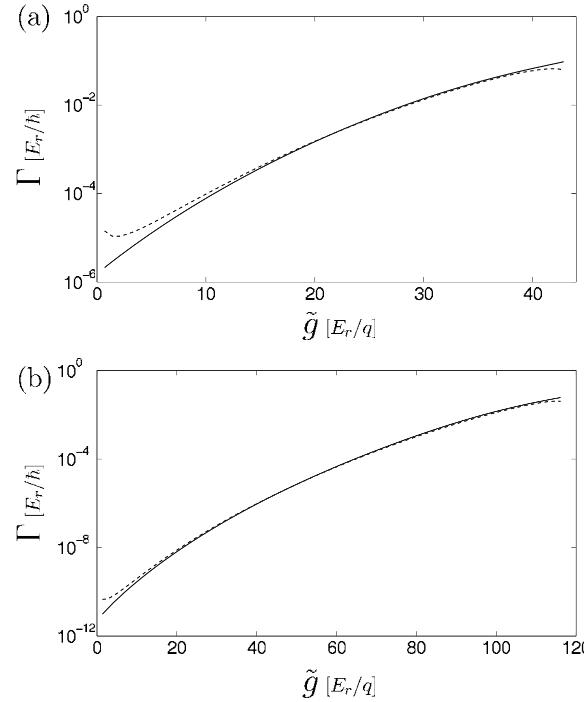

FIG. 3. Comparison of decay rates Γ obtained from simulation (solid line) and by using the analytic expression of Eq. (10) (dashed line) for two values of the barrier height parameter, $s=50$ (a) and $s=100$ (b). The expression fails to produce the correct decay rate for small values of $\tilde{g}$ and for values of the chemical potential close to the barrier height.

$$V_{\mathrm{ext}}(x) = \begin{cases} E_r[s\cos^2(qx) - cx], & 0 \leq x \leq x_{\mathrm{m}} \\ \mathrm{const}, & x \geq x_{\mathrm{m}}, \end{cases} \tag{13}$$

where q is the wave number, $E_r = \hbar^2 q^2/2m$ is the recoil energy, and $x_{\mathrm{m}} = [3\pi + \arcsin(\frac{c}{sq})]/2q$ is the location of the first minimum of the potential to the right of the barrier. The Dirichlet boundary conditions were applied at $x=0$ and $x=3\pi/q$. In order to absorb the tunneled particles, we used an additional imaginary potential $c_1 \tanh[c_2(x-c_3)]$ in the region to the right of the barrier, where the constants c_1 and c_2 were chosen such that the absorption is sufficiently fast and no backward reflection of tunneled particles occurs [24]. Figure 3 shows comparison of the decay rates Γ obtained from the computation and by using the analytic expression, Eq. (10). The rates are shown as functions of the effective interaction strength $\tilde{g}$ for two values of the barrier height parameter, $s=50$ (a) and $s=100$ (b). The agreement is good for a wide range of decay rates, but the derived formula fails as the chemical potential approaches the barrier height. This is due to failure of the WKB approximation in the tunneling region and the linearized potential approximation close to the top of the barrier. The analytic expression fails to produce the correct decay rate also for small values of $\tilde{g}$ due to failure of the Thomas-Fermi approximation in region I. We have tested the validity of Eq. (10) also for the cubic potential with very similar results as above.

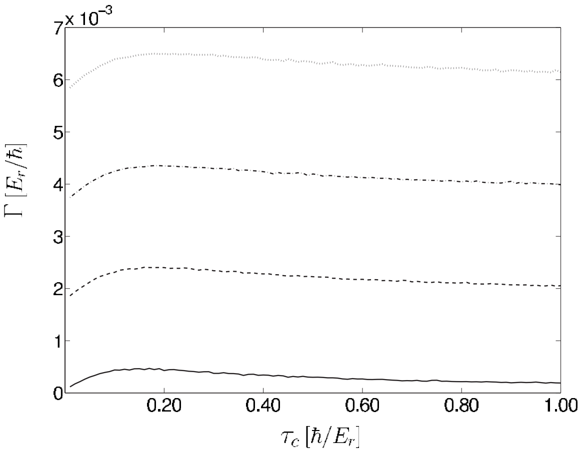

FIG. 4. Decay rate Γ as a function of the RTN correlation time τ_c for four values of the effective interaction strength $\tilde{g}/(E_r/q)=0$, 6.91, 8.98, and 10.4.

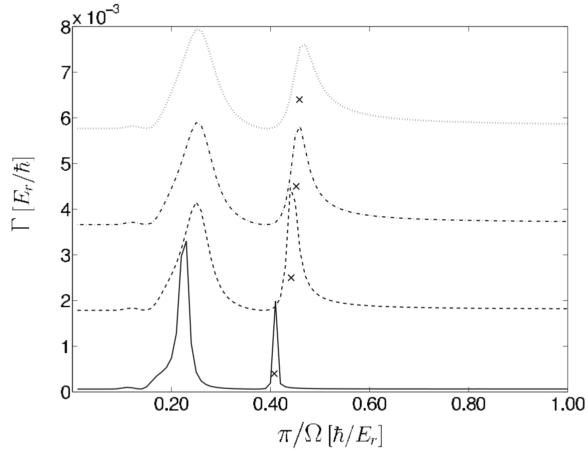

FIG. 5. Decay rate Γ as a function of π/Ω, where Ω is the frequency of the harmonic drive, for four values of the effective interaction strength $\tilde{g}/(E_r/q)=0$, 6.91, 8.98, and 10.4. The crosses denote the frequency of the dipole mode for each interaction strength.

IV. TUNNELING UNDER A FLUCTUATING BARRIER

We have also investigated the effect of fluctuations in the amplitude of the optical potential to the tunneling rate Γ through a single barrier of the washboard potential, Eq. (13), by integrating the time dependent GP equation. The random telegraph noise (RTN) is used to model thermal fluctuations in the amplitude of the optical potential. In this model the strength parameter s is assigned a value $s_0+\Delta$ or $s_0-\Delta$. The jumping time τ between these values is chosen from the distribution $\frac{1}{\tau_c}e^{-\tau/\tau_c}$, where the correlation time τ_c gives the expectation value for the jump to occur. The distribution of the number of jumps occurring in a given time interval $|t-t'|$ is Poissonian, yielding $\langle \eta(t)\eta(t')\rangle=\Delta^2 e^{-2|t-t'|/\tau_c}$ for the correlation function of the RTN signal. A similar model for the standard Schrödinger dynamics has been discussed in detail by one of the authors [27].

Figure 4 shows the decay rate Γ as a function of the correlation time τ_c for four values of the effective interaction $\tilde{g}$. The parameters in the simulation were chosen as $s_0=20$, $c=2q$, and $\Delta=0.5$. The effective interaction strengths were set to the values $\tilde{g}/(E_r/q)=0$, 6.91, 8.98, and 10.4. Clearly, the decay rate increases rapidly as the effective interaction strength $\tilde{g}$ grows due to increase in the chemical potential and the consequent reduction of the opacity of the barrier. For small values of the correlation time the effect of the noise is averaged out, and the decay rate approaches the value obtained without the noise. The decay becomes faster as the correlation time increases, reaching its maximum around $\tau_c\approx 0.20\ \hbar/E_r$, and starts to decrease gradually for larger correlation times. Qualitatively, this feature has also been observed in Ref. [27], but here the nonmonotonic behavior can be understood on the grounds of the noise signal exciting the collective eigenmodes of the system, as seen in the next section. However, RTN does not produce sharp resonance peaks because of its wide frequency spectrum.

V. TUNNELING UNDER A HARMONIC DRIVE

In order to investigate the effect of the eigenmode excitations to the tunneling rate Γ under external perturbations, we have calculated the decay rate as a function of the frequency of a harmonic drive, for which the perturbing potential is of the form $V_{\mathrm{pert}}(t)=\Delta\sin(\Omega t)$. Here $\Omega=\pi/\tau_c$ is the frequency of the drive chosen such that on the average a RTN signal corresponding to τ_c jumps twice in one cycle.

Figure 5 shows the tunneling rate Γ as a function of the inverse driving frequency of the harmonic perturbation for four interaction strengths. The parameters were chosen as in Fig. 4. Two resonance peaks are clearly visible [28]. According to simulations, the amplitude of the harmonic perturbation affects only the magnitude, but not the location of the resonance peaks. The right-hand side peak turns out to be due to the excitation of the dipole mode of the condensate: For each interaction strength, the location of the maximum of this peak corresponds accurately to the eigenfrequency of the dipole mode. The left-hand side peak is located close to the frequency of the quadru- and octupole modes.

In order to further investigate the origin of the resonance peaks shown in Fig. 4, we have measured the quasiparticle populations in the lowest-lying Bogoliubov eigenmodes during the time evolution. We define the population amplitude of the qth mode as

$$p_q(t)=\int [u_q^*(x)-v_q(x)][\psi(x,t)-c\psi_{\mathrm{gs}}(x)]dx, \qquad (14)$$

where $c=\int\psi_{\mathrm{gs}}(x)^*\psi(x,t)dx$, and ψ_{gs} is the ground state of the system without the perturbation. The population of the qth mode is then given by $\langle|p_q(t)|^2\rangle_t$, where $\langle\cdot\rangle_t$ denotes the time average. Figures 6 and 7 show the quasiparticle populations of the lowest-lying modes in the RTN and harmonic perturbation schemes, respectively. In the RTN scheme, the dipole mode is most strongly excited except in the high frequency limit $\tau_c\rightarrow 0$. The nonmonotonic dependence of the dipole mode population on the correlation time explains the similar behavior in the decay rate Γ in Fig. 4. Under harmonic perturbation the dipole mode is strongly excited around τ_c

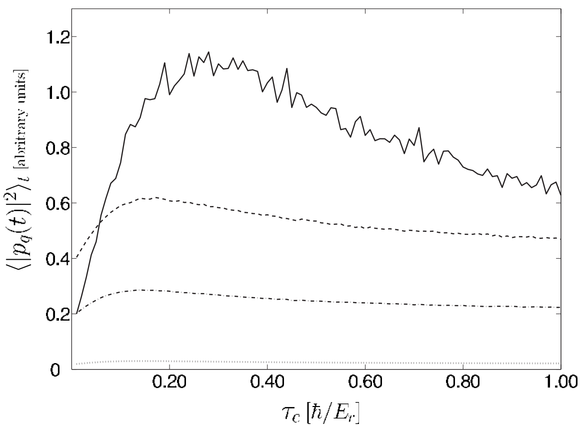

FIG. 6. Time-averaged population $\langle |p_q(t)|^2 \rangle_t$ of the dipole (solid), quadrupole (dashed), octupole (dash-dot), and hexadecapole (dotted) modes as a function of the correlation time τ_c of the RTN signal. The effective interaction was set to the value $\tilde{g}=3.46E_r/q$.

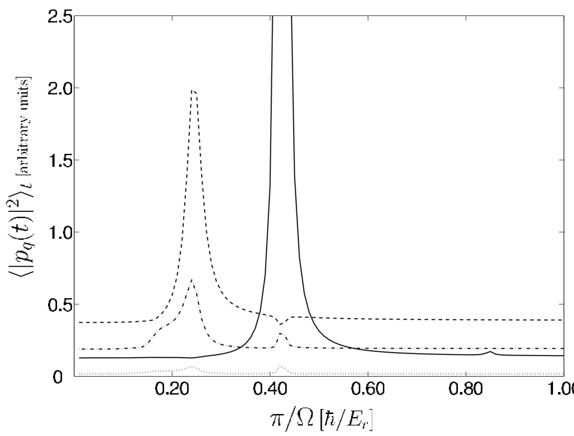

FIG. 7. Time-averaged populations $\langle |p_q(t)|^2 \rangle_t$ of the dipole (solid), quadrupole (dashed), octupole (dash-dot), and hexadecapole (dotted) modes as functions of π/Ω, where Ω is the frequency of the harmonic drive. The effective interaction was set to the value $\tilde{g}=3.46E_r/q$.

$\approx 0.40\ \hbar/E_r$, and both the quadru- and octupole modes around $\tau_c \approx 0.20\hbar/E_r$, which give rise to the resonance peaks in the decay rate Γ shown in Fig. 5. The tiny humps in the population of the quadru-, octu-, and hexadecapole modes around $\tau_c \approx 0.40\hbar/E_r$ are probably due to the strongly excited dipole mode at this frequency and the nonlinearity of the system.

VI. CONCLUSIONS

In this work we have derived an analytic expression for the tunneling rate of a Bose-Einstein condensate through a potential barrier. The result depends only on the particle mass, chemical potential of the system and the external potential. In deriving the expression it was assumed that the potential barrier and the effective atom-atom interactions are sufficiently strong, and that the potential is approximately linear in the vicinity of the left-hand classical turning point. We also studied the effect of fluctuations in the height of a potential barrier to the tunneling rate through it using the random telegraph noise. The tunneling rate was computed as a function of the characteristic frequency of the noise for several values of the effective interaction strength. In order to interpret the results, we calculated the tunneling rate also in the presence of harmonic perturbation in the height of the barrier. We observed tunneling resonances at frequencies close to the Bogoliubov eigenfrequencies of the condensate, and concluded that the resonances are due to excitation of the eigenmodes of the system.

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