Total versus quantum correlations in quantum states

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On the premises that total correlations in a bipartite quantum state are measured by the quantum mutual information, and that separation of total correlations into quantum and classical parts satisfies an intuitive dominance relation, we examine to what extent various entropic entanglement measures, such as the distillable entanglement, the relative entropy entanglement, the squashed entanglement, the entanglement cost, and the entanglement of formation, can be regarded as consistent measures of quantum correlations. We illustrate that the entanglement of formation often overestimates quantum correlations and thus is too big to be a genuine measure of quantum correlations. This indicates that the entanglement of formation does not quantify the quantum correlations intrinsic to a quantum state, but rather characterizes the pure entanglement needed to build the quantum state via local operations and classical communication. Furthermore, it has the consequence that, if the additive conjecture for the entanglement of formation is true (as is widely believed), then the entanglement cost, which is an operationally defined measure of entanglement with significant physical meaning, cannot be a consistent measure of quantum correlations in the sense that it may exceed total correlations. Alternatively, if the entanglement cost is dominated by total correlations, as our intuition suggests, then we can immediately disprove the additive conjecture. Both scenarios have their counterintuitive and appealing aspects, and a natural challenge arising in this context is to prove or disprove that the entanglement cost is dominated by the quantum mutual information.

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I. INTRODUCTION

Consider a bipartite quantum state ρ consisting of parts a and b, with marginal states $\rho^a = \mathrm{tr}_b \rho$ (partial trace over part b) and $\rho^b = \mathrm{tr}_a \rho$. A central and fundamental issue in quantum-information theory is to classify and quantify correlations in ρ. For the classification issue, one usually distinguishes between total, quantum, and classical correlations.

For the quantification issue, there are a variety of well-motivated correlation measures, which may be divided into three basic categories. The first tries to quantify *total* correlations which consist of both classical and quantum parts. A prominent example with fundamental informational meaning is the quantum mutual information [1–5]

$$I(\rho) = S(\rho^a) + S(\rho^b) - S(\rho),$$

where $S(\rho) = -\mathrm{tr}\rho \log \rho$ denotes the von Neumann entropy of ρ. We will take the logarithm to the base 2 throughout.

The second tries to capture *quantum* correlations (entanglement) in ρ, and has been widely pursued in the last decade due to the emergence of quantum-information theory [6]. A prototypical entanglement measure, also historically the first and most widely studied, is the entanglement of formation [7–9]

$$E(\rho) = \inf \sum_i \lambda_i S(\mathrm{tr}_a \rho_i).$$

Here the infimum is over all pure-state ensemble realizations $\{\lambda_i, \rho_i\}$ of ρ, that is, $\rho = \Sigma_i \lambda_i \rho_i$ with $\Sigma_i \lambda_i = 1, \quad \lambda_i \geq 0$.

The third tries to characterize classical correlations inherent in ρ, and may be considered as complementary to quantum correlations [10–13].

In spite of considerable progress made by several authors [10–13], it is quite difficult (if not impossible) to separate total correlations into classical and quantum parts in a unique fashion. However, it is still possible to gain some general insights into their relationships. Consider, for example, a Bell state in a two-qubit system

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Clearly, $I(|\Psi^+\rangle\langle\Psi^+|) = 2$; thus total correlations in $|\Psi^+\rangle$, as quantified by the quantum mutual information, are two bits. On the other hand, it is universally accepted that this maximally entangled state contains one unit of quantum correlations, i.e., 1 bit of entanglement [7–9,14]. In this context, we will have a paradox if we think that $|\Psi^+\rangle$ contains only purely quantum correlations, without any classical ones. Due to the remarkable contributions of Refs. [12,15–18], we now have several compelling arguments demonstrating that $|\Psi^+\rangle$ contains $I(|\Psi^+\rangle\langle\Psi^+|) = 2$ bits of information.

(1) Suppose that $|\Psi^+\rangle$ is shared by Alice and Bob; then by superdense coding [15], Alice can communicate 2 bits of information to Bob by manipulating her part of the quantum state and then sending it to Bob.

(2) Apart from the 1 bit of quantum correlation, the additional bit of information has been argued to be related to the negative conditional entropy [16,17].

(3) Groisman *et al.* showed that $|\Psi^+\rangle$ also contains 1 bit of classical information [12]. Indeed, suppose that Alice and Bob want to erase quantum correlations in $|\Psi^+\rangle$. A simple operational procedure is for Alice to flip an unbiased coin. If

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the head comes up, she does nothing and the state $|\Psi^+\rangle$ is left unchanged; if the tail occurs, she applies the Pauli spin operator σ_z to her part of the state, and $|\Psi^+\rangle$ is changed to $|\Psi^-\rangle=(1/\sqrt{2})(|00\rangle-|11\rangle)$. This protocol consumes 1 bit of entropy and renders $|\Psi^+\rangle$ as the mixture

$$\rho=\frac{1}{2}|\Psi^+\rangle\langle\Psi^+|+\frac{1}{2}|\Psi^-\rangle\langle\Psi^-|$$

which is disentangled because it can be rewritten as a mixture of two product states,

$$\rho=\frac{1}{2}|0\rangle\langle 0|\otimes|0\rangle\langle 0|+\frac{1}{2}|1\rangle\langle 1|\otimes|1\rangle\langle 1|.$$

Clearly, ρ is still classically correlated and contains 1 bit of classical correlation [19].

(4) Horodecki *et al.* provided an alternative argument showing that $|\Psi^+\rangle$ contains 1 bit of classical correlation and 1 bit of quantum correlation, which are complementary to each other [18].

The above arguments can be straightforwardly generalized to any pure bipartite state $|\Psi\rangle$; the conclusion is that $|\Psi\rangle$ contains $I(|\Psi\rangle\langle\Psi|)=2S(\mathrm{tr}_a|\Psi\rangle\langle\Psi|)$ bits of total correlations, which can be divided as $E(|\Psi\rangle\langle\Psi|)=S(\mathrm{tr}_a|\Psi\rangle\langle\Psi|)$ bits of quantum correlations and $I(|\Psi\rangle\langle\Psi|)-E(|\Psi\rangle\langle\Psi|)=S(\mathrm{tr}_a|\Psi\rangle\times\langle\Psi|)$ bits of classical correlations. Consequently, any pure state contains equal amounts of quantum and classical correlations, which is just the reduced von Neumann entropy.

But how about mixed states? Since any mixed state has pure-state ensemble realizations, and mixing usually decreases quantum correlations, it seems plausible to assume that, for mixed states, the magnitude of quantum correlations should be dominated by that of classical correlations. This is further supported by the observation that a general state may contain classical correlations without any quantum ones (e.g., a separable state), but not vice versa [10,12,20].

Now suppose that total correlations of a mixed state ρ, as measured by the quantum mutual information $I(\rho)$, have the following decomposition:

$$I(\rho)=C(\rho)+Q(\rho) \tag{1}$$

where $C(\rho)$ represents classical correlations while $Q(\rho)$ represents quantum ones. Based on the above reasoning and inspired by the suggestions of several authors [10,12,13], it seems natural and desirable to postulate that $C(\rho)\geq Q(\rho)$, which will be further elaborated in Sec. III. Consequently, taking into account Eq. (1), we are led to

$$Q(\rho)\leq\frac{1}{2}I(\rho). \tag{2}$$

In this paper, we will just accept the above intuitive postulate as an assumption of our reasoning, and will investigate the implications and consequences of it when we take $Q(\rho)$ as various entanglement measures introduced in quantum-information theory [7–9,14,21–23]. In this respect, we distinguish entanglement measures quantifying correlations of a quantum state *from within*, such as the distillable entanglement, from those that quantify correlations *from outside*, such as the entanglement cost and the entanglement of formation. While it is reasonable to expect that the distillable entanglement should be a good candidate for $Q(\rho)$, it is not *a priori* clear whether the entanglement cost and the entanglement of formation are suitable. More specifically, we will examine to what extent the entanglement of formation and the entanglement cost can be interpreted as measures of quantum correlations in a mixed state. We will show that neither can be used as an inherent quantum correlation measure (they may be used as upper bounds or envelopes of quantum correlations) in the sense that, if we take $Q(\rho)=E(\rho)$ or the entanglement cost $E_c(\rho)$ in Eq. (1), then it will often violate inequality (2), thus contradicting our intuitive requirement.

Furthermore, it may even happen that

$$E(\rho)>I(\rho) \tag{3}$$

for certain ρ, that is, quantum correlations, as measured by the entanglement of formation, may exceed total correlations, as measured by the quantum mutual information. Thus the entanglement of formation is too big to be a genuine measure of quantum correlations. A dramatic consequence of inequality (3) is that, if the additive conjecture for entanglement of formation is true (as is widely believed) [24], then the entanglement cost coincides with the entanglement of formation [25], and thus cannot be regarded as a consistent measure of quantum correlations either, in spite of its operational meaning and informational significance. Alternatively, if the entanglement cost is a consistent measure of quantum correlations in the sense that it is dominated by total correlations (as our intuition requires), then the additive conjecture is immediately disproved.

It is known that the distillable entanglement and the squashed entanglement satisfy inequality (2) when substituted into it for $Q(\rho)$ [23].

The rest of this paper is arranged as follows. In Sec. II, we review a variety of arguments justifying the quantum mutual information as a measure of total correlations since this is our basic point of departure. We then illustrate the counterintuitive relationships between the quantum mutual information and the entanglement of formation via several paradigmatic examples such as the isotropic states and the Werner states in Sec. III. We also indicate a reason for this "paradox" and point out a weak point of the entanglement of formation in quantifying quantum correlations. In Sec. IV, we compare the quantum mutual information with other entropic entanglement measures such as the distillable entanglement, the relative entropy entanglement, the squashed entanglement, and the entanglement cost. Finally, we conclude with some discussions in Sec. V.

II. QUANTUM MUTUAL INFORMATION AS TOTAL CORRELATIONS

The name "quantum mutual information" has its origin in Cerf and Adami [4], but this quantity dates back at least to Stratanovich's correlation entropy [1]. It has been implicitly used to study information transfer in quantum measurements by Zurek [2], and is called the "index of correlation" by

Barnett and Phoenix in their study of correlations in quantum states [3].

Our point of departure in this paper is to maintain the quantum mutual information as a measure (or, more generally, as an upper bound) of total correlations. Owing to the considerable efforts of many authors, now we have numerous heuristic as well as mathematical and physical justifications for this point of view [4,5,10,12,13,17,18,26–28]. To put our investigation in perspective, let us summarize some of these arguments.

(1) Quantum mutual information is a straightforward generalization of the classical mutual information, and reduces to the latter in commutative cases. The classical mutual information quantifies all correlations in a bivariate classical state (probability distribution), and is most beautifully exhibited in the Shannon noisy channel coding theorem [29,30]. Put in other words, the classical analog of quantum mutual information is a well-established measure of all correlations, and thus it is plausible to also regard the quantum mutual information as a good measure of all correlations.

(2) Based on the idea in Landauer's erasure principle that the amount of information equals the amount of entropy required for its erasure [31], Groisman *et al.* presented an operational justification [12]: They showed that the amount of randomness (noise) needed to erase completely the correlations in a bipartite quantum state is precisely quantified by the quantum mutual information.

(3) Schumacher and Westmoreland gave a justification based on a communication consideration: If a bipartite quantum state is used as the key for a one-time-pad cryptographic communication system, then the maximum amount of information that can be sent securely is the quantum mutual information [28].

On the premise that the quantum mutual information is regarded as a measure of total correlations, we can investigate to what extent the various entanglement measures introduced in the last decade, such as the distillable entanglement, the relative entropy entanglement, the squashed entanglement, the entanglement cost, and the entanglement of formation, etc., can be used to quantify quantum correlations. Clearly, a simple requirement is that if an entanglement measure well quantifies quantum correlations (which constitute a part of the total correlations), then it should not exceed the total correlations. We will see that the distillable entanglement, the relative entropy entanglement, and the squashed entanglement meet this requirement. However, we will show that the entanglement of formation does not meet this simple requirement, and thus demonstrate that the entanglement of formation cannot be regarded as a good measure of quantum correlations. If, moreover, the additive conjecture for the entanglement of formation turns out to be true, then the entanglement cost cannot be regarded as a consistent measure of quantum correlations either.

III. TOTAL CORRELATIONS VS ENTANGLEMENT OF FORMATION

Though we still do not have universally accepted measures of quantum correlations and classical correlations, it seems reasonable to make the following heuristic and intuitive assumption concerning their relationship.

(A) For any quantum state, its classical correlations should not be less than its quantum correlations.

Some plausible arguments supporting (A) are as follows.

(1) For any pure state, as discussed in Sec. I, both its quantum correlations and classical correlations are equal to the reduced von Neumann entropy. Thus assumption A is trivially satisfied in this case.

(2) For any mixed state, since it can be represented as classical mixing (i.e., convex combination, incoherent superposition) of pure states, and classical mixing often increases classical correlations and decreases quantum ones, it is intuitive, based on (1), to assume that classical correlations in a mixed state are not less than quantum correlations. More specifically, because for pure states total correlations are equally divided between quantum and classical parts (both are equal to the reduced von Neumann entropy), while when a pure state is becoming more and more mixed, its quantumness degrades and its classicality increases, assumption A seems a natural guess.

(3) One often observes that a bipartite state can have classical correlations without any quantum ones, but not vice versa. For instance, any separable state trivially satisfies assumption A, since its quantum correlations are zero, while in general it possesses classical correlations.

(4) It is known that, if we measure quantum correlations by the distillable entanglement or the squashed entanglement, then assumption A holds [11,23], and, for the relative entropy entanglement, it is an open problem whether assumption A holds. The entanglement of formation violates assumption A, but, as we will show, the entanglement of formation cannot be a consistent measure for quantum correlations since it can even exceed the total correlations itself. Moreover, the entanglement of formation is not monogamous [20], which is another weak point in using it as an entanglement measure.

(5) Assumption A is conjectured by Henderson and Vedral [10].

(6) Groisman *et al.* have suggested that assumption A is intuitive, and actually have also put it forward as a conjecture [12].

Now we specify to concrete measures of correlations. Since both the quantum mutual information and the entanglement of formation are designed for the task of quantifying correlations in a quantum state, and both are based on the same notion of the von Neumann entropy, we are naturally led to investigate the relationships between them in the context of assumption A, and to examine to what extent various existing measures of entanglement may serve as good quantities for quantum correlations.

If we wish to use the entanglement of formation as a measure of quantum correlations while in the meantime taking the quantum mutual information as a measure of total correlations, then it is natural to define

$$C(\rho) = I(\rho) - E(\rho)$$

as a measure of classical correlations. Consequently, assumption A has the form $C(\rho) \geq E(\rho)$, and therefore we are led to

$$E(\rho) \leq \frac{1}{2}I(\rho). \quad (4)$$

Even if it is maintained that, for a general mixed state, the total correlations cannot be neatly divided into quantum and classical parts due to their intertwining, as long as the quantum mutual information is accepted as an upper bound for total correlations (i.e., classical correlations+quantum correlations≤quantum mutual information, which is actually satisfied by all the examples in Ref. [10]), the result is still the inequality (4).

Unfortunately, the above inequality is often violated as the following examples illustrate. This is a peculiar feature of the entanglement of formation if we interpret it as a measure of quantum correlations and accept the assumption A.

Example 1 (isotropic states). Let a system be described by $C^d \otimes C^d$ and let ρ_x be an isotropic state [32,33]

$$\rho_x = \frac{1-x}{d^2-1}1 + \frac{d^2x-1}{d^2-1}|\Psi\rangle\langle\Psi|, \quad x \in [0,1].$$

Here $|\Psi\rangle=(1/\sqrt{d})\Sigma_{i=1}^d|ii\rangle$ with $\{|i\rangle\}$ constituting an orthonormal basis for C^d. It is known that if $d\geq 3$ and $x\in[4(d-1)/d^2,1]$, then

$$E(\rho_x) = (x-1)\frac{d}{d-2}\log(d-1) + \log\ d.$$

The above result was first established in Ref. [32] and later confirmed in Ref. [34].

To evaluate the quantum mutual information $I(\rho_x)$, we note that the reduced states are $\rho_x^a=\rho_x^b=(1/d)1$, and the spectra of ρ_x consist of $(1-x)/(d^2-1)$ with multiplicity d^2-1 and x with multiplicity 1. Consequently, the quantum mutual information of ρ_x is

$$\begin{aligned} I(\rho_x) &= S(\rho_x^a) + S(\rho_x^b) - S(\rho_x) \\ &= 2\log d + (d^2-1)\frac{1-x}{d^2-1}\log\frac{1-x}{d^2-1} + x\log x \\ &= 2\log d - (1-x)\log(d^2-1) - H(x). \end{aligned}$$

Here $H(x)=-x\log x-(1-x)\log(1-x)$ is the binary entropy function. Now it is straightforward to evaluate that, for any $x\in(0,1)$ [noting that $4(d-1)/d^2\to 0$ when $d\to\infty$], it holds that

$$\lim_{d\to\infty}\left(E(\rho_x) - \frac{1}{2}I(\rho_x)\right) = \frac{1}{2}H(x) > 0.$$

Consequently, for any fixed $x\in(0,1)$, when d is *sufficiently large*, the entanglement of formation $E(\rho_x)$ will be larger than half of the quantum mutual information $I(\rho_x)$, and thus violates inequality (4).

Example 2 (Werner states). Let w_x be the Werner state [19,33]

$$w_x = \frac{d-x}{d^3-d}1 + \frac{dx-1}{d^3-d}F, \quad x \in [-1,1],$$

acting on $C^d\otimes C^d$. Here $F=\Sigma_{i,j=1}^d|ij\rangle\langle ji|$ is the flip operator with $\{|ij\rangle\}$ an orthonormal basis of product states for the composite system. For $x\in[-1,0]$, the entanglement of formation of w_x is [9,33]

$$E(w_x) = H\left(\frac{1}{2}(1-\sqrt{1-x^2})\right).$$

To evaluate the quantum mutual information $I(w_x)$, note that $w_x^a=w_x^b=(1/d)1$, and the spectra of w_x consist of $(1-x)/(d^2-d)$ with multiplicity $(d^2-d)/2$, and $(1+x)/(d^2+d)$ with multiplicity $(d^2+d)/2$. Therefore, the quantum mutual information is

$$\begin{aligned} I(w_x) &= S(w_x^a) + S(w_x^b) - S(w_x) \\ &= 2\log d + \frac{1-x}{2}\log\frac{1-x}{d^2-d} + \frac{1+x}{2}\log\frac{1+x}{d^2+d} \\ &= \log d - \frac{1-x}{2}\log(d-1) - \frac{1+x}{2}\log(d+1) + 1 \\ &\quad - H\left(\frac{1-x}{2}\right). \end{aligned}$$

Put $f_d(x)=\frac{1}{2}I(w_x)-E(w_x)$; then

$$f_\infty(x) := \lim_{d\to\infty} f_d(x) = \frac{1}{2} - \frac{1}{2}H\left(\frac{1-x}{2}\right) - H\left(\frac{1}{2}(1-\sqrt{1-x^2})\right).$$

We depict the graphs of $f_d(x)$ when $x\in(-1,0)$ for $d=2,3,4,16$ in Fig. 1. We see that, for $d=2$, $E(w_x)$ already exceeds $\frac{1}{2}I(w_x)$ for certain x, and with increasing dimension d, $E(w_x)$ exceeds $\frac{1}{2}I(w_x)$ for more and more x. In particular, for any x, $E(w_x)$ will ultimately exceed $\frac{1}{2}I(w_x)$ when d is sufficiently large because $f_\infty(x)<0$ for $x\in(-1,0)$.

We have shown by two simple examples that the entanglement of formation may exceed half of the quantum mutual information. This is quite counterintuitive if we regard the entanglement of formation as a measure of quantum correlations. Furthermore, Hayden discovered a more striking phenomenon regarding the relation between the quantum mutual information and the entanglement of formation [35]: The latter may even exceed the former itself. However, he exhibited this only with an abstract construction in an asymptotic regime, that is, he indicated the existence of a subspace of a large composite quantum system such that a mixed state supported on this subspace has a smaller quantum mutual information than the entanglement of formation. Here we consolidate his remarkable observation through the following explicit example.

Example 3. Consider the Werner state in example 2, and put $g_d(x)=I(w_x)-E(w_x)$. Simple manipulation shows that

$$g_\infty(x) := \lim_{d\to\infty} g_d(x) = 1 - H\left(\frac{1-x}{2}\right) - H\left(\frac{1}{2}(1-\sqrt{1-x^2})\right).$$

We depict the graphs of $g_d(x)=I(w_x)-E(w_x)$ for $d=6,16,\infty$ in Fig. 2. We see that the entanglement of formation may exceed the quantum mutual information when $d\geq 6$ (numerical calculation shows this does not occur for $d\leq 5$), and, for any fixed $x\in(-1,0)$, the entanglement of formation $E(w_x)$

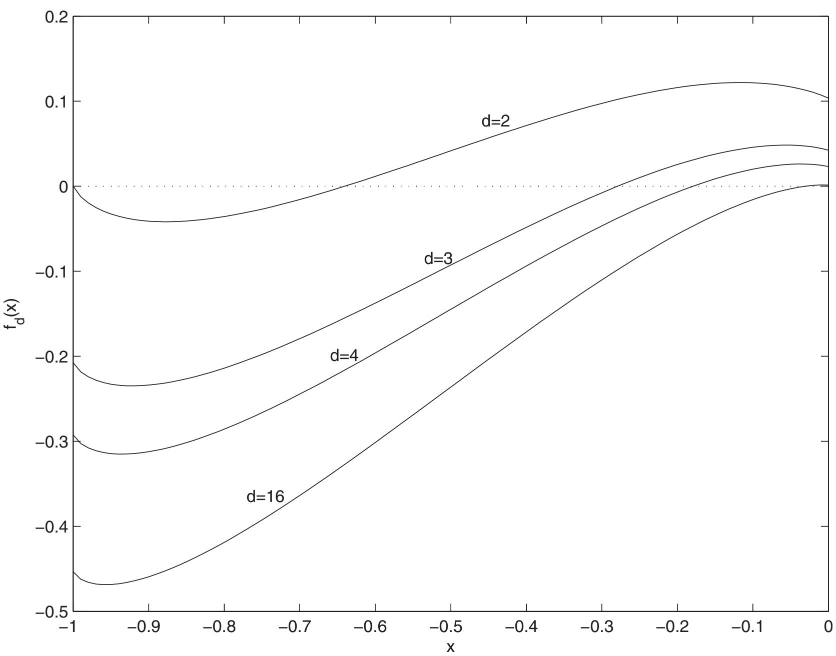

FIG. 1. Graphs of $f_d(x) = (1/2)I(w_x) - E(w_x)$ versus $x \in [-1, 0]$ for $d = 2, 3, 4, 16$. We see that the entanglement of formation $E(w_x)$ exceeds half of the quantum mutual information $(1/2)I(w_x)$ for some x, and with increase of the dimension d, this happens for an increasing interval of x.

will exceed the quantum mutual information $I(w_x)$ ultimately as long as d is sufficiently large, because $g_\infty(x) < 0$ for any $x \in (-1, 0)$.

The lesson from the above counterintuitive behavior is that we cannot interpret the entanglement of formation as a measure of quantum correlations if we accept that the quantum mutual information is an appropriate measure (or an upper bound) for total correlations. Are there any intuitive reasons for the phenomena illuminated in the above example concerning the "wrong" dominance relation between the quantum mutual information and the entanglement of formation? We maintain that total correlations are well quantified by the quantum mutual information. Then the peculiarity resides in the very definition of the entanglement of formation, which allows only pure-state decompositions. After all, the counterintuitive behavior disappears if, in defining the entanglement of formation, we replace the reduced von Neumann entropy by either half of the quantum mutual information or the maximal classical mutual information induced by local measurements, and take the infimum over all ensemble realizations (including mixed ones rather than restricting to pure ones) of the state.

In summary, the entanglement of formation is a good measure of quantum correlations only for pure states, and in this case it is precisely the reduced von Neumann entropy. For mixed states, the entanglement of formation often overestimates quantum correlations and thus can only be interpreted as an *upper bound* for quantum correlations.

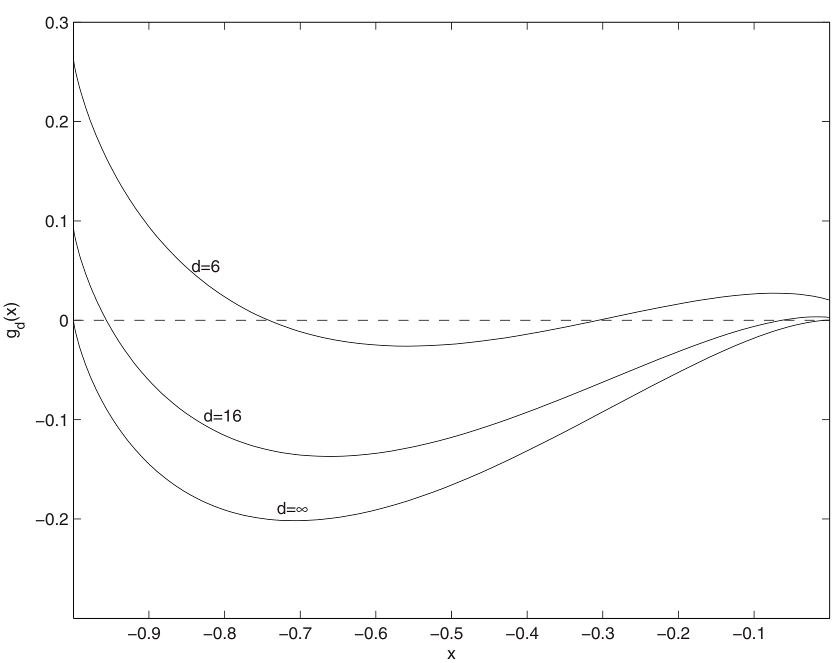

FIG. 2. Graphs of $g_d(x) = I(w_x) - E(w_x)$ for $d = 6, 16, \infty$. We see that the entanglement of formation $E(w_x)$ exceeds the quantum mutual information $I(w_x)$ for some x, and with increase of the dimension d, this happens for an increasing interval of x.

IV. TOTAL CORRELATIONS VS OTHER ENTROPIC ENTANGLEMENT MEASURES

Apart from the entanglement of formation, there are several other entanglement measures, such as the distillable entanglement E_d, the entanglement cost E_c, the relative entropy entanglement E_r [8,21,22], and the squashed entanglement E_s [23], which are all information-theoretically motivated and coincide with the reduced von Neumann entropy on pure states. We call these entropic entanglement measures, to emphasize their difference from other geometric measures of entanglement such as the negativity and the entanglement of robustness [36,37].

Since the quantum mutual information is an entropic measure of total correlations, and if the above entanglement measures can be regarded as quantifying quantum correlations, they should not exceed half of the quantum mutual information, or at least should not exceed the quantum mutual information itself.

Let us first recall the precise meaning of the various entanglement measures, to highlight whether the involved quantity is quantifying entanglement in a quantum state from within or from outside. For the latter, we have to be careful in interpreting it as a measure of quantum correlations *inherent* in a mixed state.

(1) Distillable entanglement $E_d(\rho)$ [7,8]. The distillable entanglement quantifies how many maximally entangled Bell states $|\Psi^+\rangle$ can be extracted from a bipartite state ρ by local operations and classical communication (LOCC); thus it is an entanglement measure characterizing entanglement *from within*. To state the precise mathematical definition, for any fixed $\epsilon > 0$, let

$$D_\epsilon = \left\{ \frac{m}{n}: \text{ there exists a LOCC } \Lambda \text{ such that } \|\Lambda(\rho^{\otimes n}) - (|\Psi^+\rangle\langle\Psi^+|)^{\otimes m}\| < \epsilon \right\},$$

where $\|\cdot\|$ may either be the trace distance or the Bures distance or any other equivalent distance. Now the entanglement distillation is defined as

$$E_d(\rho) = \lim_{\epsilon\to 0,\ n\to\infty} \sup \left\{ \frac{m}{n}: \frac{m}{n} \in D_\epsilon \right\}.$$

(2) Entanglement cost $E_c(\rho)$ [7,8]. The entanglement cost quantifies how many maximally entangled Bell states are used to create ρ by means of LOCC; thus it is an entanglement measure characterizing quantum correlations in a state *from outside*, a kind of envelope. This is dual to the entanglement of distillation, and can be mathematically expressed as

$$E_c(\rho) = \lim_{\epsilon\to 0,\ n\to\infty} \sup \left\{ \frac{m}{n}: \frac{m}{n} \in C_\epsilon \right\},$$

where

$$C_\epsilon = \left\{ \frac{m}{n}: \text{ there exists an LOCC } \Lambda \text{ such that } \|\rho^{\otimes n} - \Lambda(|\Psi^+\rangle\langle\Psi^+|^{\otimes m})\| < \epsilon \right\}.$$

(3) Entanglement of formation $E(\rho)$ [7–9]. This is already defined in the Introduction, and may be considered as a somewhat restricted variant of the entanglement cost. Its regularized version coincides with the entanglement cost [25].

(4) Relative entropy entanglement $E_r(\rho)$ [21]. This is actually a kind of (pseudo)distance measure, but it is so intimately related to the von Neumann entropy that we would rather classify it as an entropic entanglement measure. It is defined as

$$E_r(\rho) = \inf D(\rho\|\sigma)$$

where the infimum is over the set consisting of all separable bipartite states σ, and $D(\rho\|\sigma) = \mathrm{tr}\rho(\log\rho - \log\sigma)$ is the quantum relative entropy.

(5) Squashed entanglement [23]. The squashed entanglement is motivated by classical cryptography, and is defined as

$$E_s(\rho) = \inf\left\{ \frac{1}{2} I(\rho_{abe}|\rho_e) : \mathrm{tr}_e \rho_{abe} = \rho_{ab} \right\},$$

where

$$I(\rho_{abe}|\rho_e) = S(\rho_{ae}|\rho_e) + S(\rho_{be}|\rho_e) - S(\rho_{abe}|\rho_e)$$

is the conditional quantum mutual information, while $S(\rho_{ae}|\rho_e) = S(\rho_{ae}) - S(\rho_e)$, etc., are conditional quantum entropies.

From Refs. [11,21,23], we have the following observations.

(1) $E_d(\rho) \le \frac{1}{2} I(\rho)$.
(2) $E_s(\rho) \le \frac{1}{2} I(\rho)$.
(3) $E_r(\rho) \le I(\rho)$.

For the relative entropy entanglement, though it is trivial that $E_r(\rho) \le I(\rho)$ because $I(\rho) = D(\rho\|\rho^a \otimes \rho^b)$ (quantum relative entropy), we do not know whether $E_r(\rho) \le \frac{1}{2} I(\rho)$.

For the entanglement cost, we will show that the relation $E_c(\rho) \le \frac{1}{2} I(\rho)$ cannot be true in general in examples 4 and 5. However, we do not know whether it is dominated by the quantum mutual information or not, that is, whether $E_c(\rho) \le I(\rho)$. But from the very operational definition of the entanglement cost, it seems quite natural and intuitive to conjecture that the entanglement cost should be dominated by the quantum mutual information. If this conjecture turns out to be true, then we can immediately disprove the additive conjecture for the entanglement of formation [24], since this latter conjecture implies that the entanglement of formation and the entanglement cost are identical [25], but we know from example 3 that the entanglement of formation may exceed the quantum mutual information for certain states.

Example 4. Consider a mixture of two Bell states, $|\Psi^\pm\rangle = (1/\sqrt{2})(|00\rangle \pm |11\rangle)$, in a two-qubit system,

$$\rho_x = x|\Psi^-\rangle\langle\Psi^-| + (1-x)|\Psi^+\rangle\langle\Psi^+|, \quad x \in [0, 1/2],$$

for which we have [38]

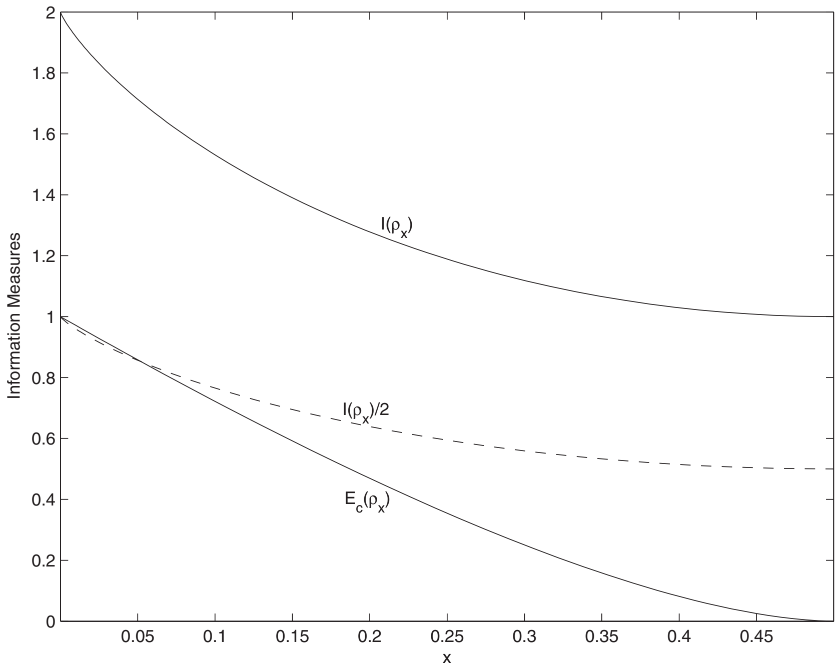

FIG. 3. Graphs of $I(\rho_x)$, $E_c(\rho_x)$, and $(1/2)I(\rho_x)$ versus $x\in[0,1/2]$ for the state ρ_x in example 4. We observe that $E_c(\rho_x) > (1/2)I(\rho_x)$ for $x\in(0,0.0521)$.

$$E_c(\rho_x) = H\left(\frac{1}{2} + \sqrt{x(1-x)}\right).$$

The quantum mutual information can be easily evaluated as

$$I(\rho_x) = 2 - H(x).$$

We depict the graphs of $I(\rho_x)$, $E_c(\rho_x)$, and $(1/2)I(\rho_x)$ in Fig. 3, and see that $E_c(\rho_x) > \frac{1}{2}I(\rho_x)$ for $x\in(0,0.0521)$.

Example 5. Consider a qubit-qutrit system described by $C^2\otimes C^3$. Let V be the subspace spanned by

$$|v_1\rangle = \frac{1}{\sqrt{3}}(|0_a\rangle|2_b\rangle - \sqrt{2}|1_a\rangle|0_b\rangle),$$

$$|v_2\rangle = \frac{-1}{\sqrt{3}}(|1_a\rangle|2_b\rangle - \sqrt{2}|0_a\rangle|1_b\rangle),$$

where $\{|0_a\rangle,|1_a\rangle\}$ constitute an orthonormal basis for C^2 and $\{|0_b\rangle,|1_b\rangle,|2_b\rangle\}$ for C^3. Let

$$\rho_x = x|v_1\rangle\langle v_1| + (1-x)|v_2\rangle\langle v_2|, \quad x\in[0,1].$$

Then by the result of Ref. [38] (example 3 there), the entanglement cost is given by

$$E_c(\rho_x) = H(1/3).$$

On the other hand, the reduced states are

$$\rho_x^a = \frac{2-x}{3}|0_a\rangle\langle 0_a| + \frac{1+x}{3}|1_a\rangle\langle 1_a|,$$

$$\rho_x^b = \frac{1}{3}|2_b\rangle\langle 2_b| + \frac{2x}{3}|0_b\rangle\langle 0_b| + \frac{2(1-x)}{3}|1_b\rangle\langle 1_b|,$$

and thus the quantum mutual information can be evaluated as

$$\begin{aligned} I(\rho_x) &= -\frac{2-x}{3}\log\frac{2-x}{3} - \frac{1+x}{3}\log\frac{1+x}{3} - \frac{1}{3}\log\frac{1}{3} - \frac{2x}{3}\log\frac{2x}{3} \\ &\quad - \frac{2(1-x)}{3}\log\frac{2(1-x)}{3} - H(x) \\ &= 2\log 3 - \frac{2}{3} - \frac{2-x}{3}\log(2-x) - \frac{1+x}{3}\log(1+x) \\ &\quad + \frac{x}{3}\log x + \frac{1-x}{3}\log(1-x). \end{aligned}$$

The graphs of $I(\rho_x)$, $E_c(\rho_x)$, and $\frac{1}{2}I(\rho_x)$ are depicted in Fig. 4. We see that $E_c(\rho_x) > \frac{1}{2}I(\rho_x)$ for $x\in(0,1)$.

V. DISCUSSION

We have investigated the relationships between the quantum mutual information and various entanglement measures by maintaining that total correlations are well quantified by the quantum mutual information. While we see that the distillable entanglement and the squashed entanglement are good measures of quantum correlations, we also observe that, for mixed states, the entanglement of formation often exceeds half of the quantum mutual information, and for the Werner states, the entanglement of formation may even exceed the quantum mutual information itself. This has the consequence that we cannot interpret the entanglement of formation as a good measure of *quantum correlations* and in the meantime regard the quantum mutual information as measuring *total correlations*. The origin of this counterintuitive behavior seems to be related to the pure-state decompositions in the very definition of the entanglement of formation. Moreover, if the additive conjecture is true, then it will imply that the entanglement cost may also exceed total correlations and thus cannot be regarded as a genuine quantum

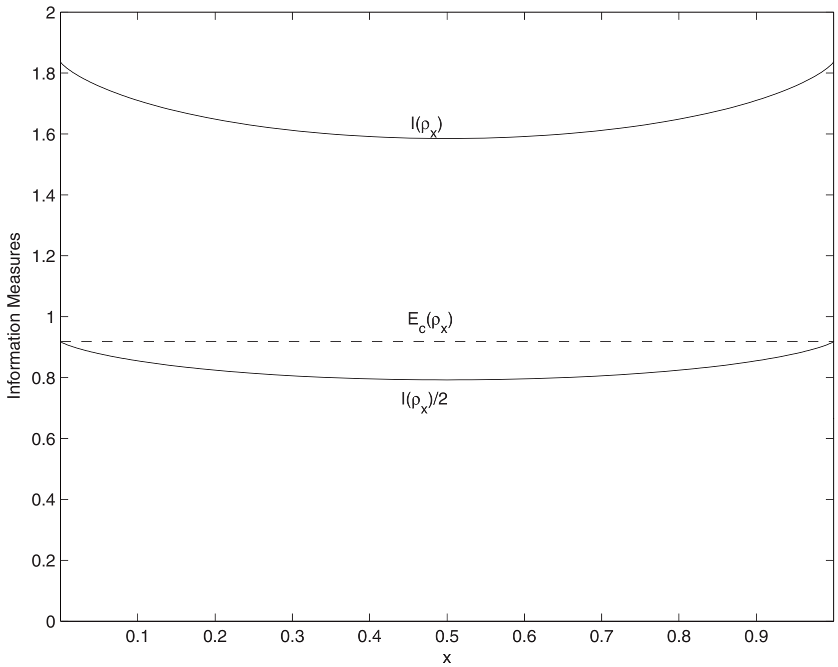

FIG. 4. Graphs of $I(\rho_x)$, $E_c(\rho_x)$, and $(1/2)I(\rho_x)$ versus $x\in[0,1]$ for the state ρ_x in example 5. We observe that $E_c(\rho_x) > (1/2)I(\rho_x)$ for $x\in(0,1)$.

correlation measure either. On the other hand, if the entanglement cost is dominated by total correlations as measured by the quantum mutual information, then the additive conjecture is immediately disproved. Consequently, it will be interesting to put the following two *contradictory* conjectures together.

(1) The entanglement of formation is additive [24], and thus coincides with the entanglement cost [25].

(2) The entanglement cost is a consistent measure of quantum correlations in the sense that it cannot exceed total correlations, that is, $E_c(\rho)\leq I(\rho)$ for any state ρ.

The first conjecture is appealing in that it is widely believed to be true (and indeed verified for some particular cases), and is proved to be equivalent to several other celebrated conjectures concerning channel capacity [24]. It is counterintuitive in that it will imply that the entanglement cost, a physically motivated operational entanglement measure, cannot be a consistent measure of quantum correlations if we accept the well-established quantum mutual information as a measure of total correlations. The second conjecture, being contradictory to the first one, is appealing or counterintuitive just for the opposite reasons. Thus an important issue in this context is to explore the relationships between the quantum mutual information and the entanglement cost.

We emphasize that, in order to highlight some counterintuitive features of the entanglement of formation, we have used the plausible assumption A, which states that classical correlations are larger than quantum correlations. This is certainly less convincing than the assumption that total correlations are larger than quantum correlations. We have provided only some heuristic arguments supporting it. To make sense of this assumption, we clearly have to specify the measures for quantum correlations and classical correlations. Thus, to what extent the assumption is plausible depends on the context. In any case, it may be interesting to use this assumption as a criterion to classify various entanglement measures. Furthermore, we may turn the argument around, refute or consolidate assumption A by checking whether a certain good entanglement measure violates it. For instance, if one insists that the entanglement of formation is a consistent measure of quantum correlations, then one will certainly regard assumption A as paradoxical. The relationships between quantum correlations and classical ones are extremely subtle, and we can hardly expect a single inequality such as assumption A to capture all their features.

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