Operator space entanglement entropy in a transverse Ising chain

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The efficiency of time-dependent density matrix renormalization group methods is intrinsically connected to the rate of entanglement growth. We introduce a measure of entanglement in the space of operators and show, for a transverse Ising spin-1/2 chain, that the simulation of observables, contrary to the simulation of typical pure quantum states, *is* efficient for initial local operators. For initial operators with a finite index in Majorana representation, the operator space entanglement entropy saturates with time to a level which is calculated analytically, while for initial operators with infinite index the growth of operator space entanglement entropy is shown to be logarithmic.

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I. INTRODUCTION

The entanglement is an intrinsic property of composite quantum systems and represents a cornerstone in quantum information theory [1]. It is important to understand the role of quantum entanglement in the classical manipulation of quantum objects and to quantify the degree of entanglement. Although the question of quantification is not clearly resolved, quantum-information theory offers several measures [2,3] of entanglement. Quantum-information theory has also given a new birth or fresh interpretation of a class of methods for numerical simulations of many-body quantum systems which, due to the exponential growth of Hilbert space, cannot be manipulated using exact diagonalization. The methods originally known as density matrix renormalization group (DMRG) [4] deploy the fact that many degrees of freedom are redundant in a quantum-state description; the system is therefore adequately described by taking into account maximally entangled components only. Thus, sufficiently slow growth of the entanglement is of crucial importance. DMRG enjoyed remarkable success in determining the ground-state properties of large one-dimensional quantum models, for which the degree of entanglement scales at most logarithmically with size [5–10]; however, its *time-dependent* version (tDMRG) [11,12] is often plagued by an abundance of entanglement with time evolution [13]. For efficient classical simulations of many-body quantum dynamics using tDMRG it is required that the computational costs grow polynomially in time, meaning that the degree of entanglement of any quantum object which can be represented as an element of a scalable tensor product Hilbert space (either a pure state or a mixed state or operator, etc.) must grow no faster than logarithmically. It was recently shown that this is generically not the case for a *quantum chaotic* Ising spin chain in a tilted magnetic field where the entanglement entropies grow linearly and hence the computation costs increase exponentially in time [13].

In this paper we shall consider the model of a quantum Ising chain in transverse magnetic field which is integrable and an explicit analytical solution exists. Calabrese and Cardy [14] have shown numerically that the growth of entanglement entropy is *linear* for evolution of pure initial states which are ground states of quenched Hamiltonians; see also Ref. [15]. However, from the efficiency of tDMRG for the time evolution of local operators [13] one may conclude that entanglement entropy computed in the space of operators grows only logarithmically. Here we address this problem theoretically using the idea [16] of reformulating the Heisenberg evolution in an algebra of operators in terms of a Schrödinger evolution generated by a different, adjoint Hamiltonian acting on the Hilbert space of operator algebra. We show that operator space entanglement entropy saturates in time for initial local operators of a finite index (precise definitions follow) and explicitly compute the saturation values in the critical case. Further, for initial local operators of infinite index we give accurate numerical evidence and a theoretical hint that the growth is logarithmic (in thermodynamic limit) with prefactor 1/6 in the critical, or 1/3 in the noncritical, case.

We note that, to the best of our knowledge, entanglement in operator space is a concept which has not yet been considered theoretically, and it is clearly not equivalent to the conventional concept of entanglement of density operators as discussed in Sec. IV. Yet it is the one which we expect to be more closely related to the computational complexity of time evolution in infinite interacting quantum systems.

II. FERMION REPRESENTATION OF DYNAMICS IN OPERATOR SPACE

The dynamics of a transverse Ising chain of length $2L$ is described in terms of canonical Pauli matrices σ_j^α for sites $j\in\mathbb{Z}_{2L}\equiv\{-L+1,\dots,0,1,\dots,L\}$ and the Hamiltonian

$$H=\sum_{j=-L+1}^{L-1}\sigma_j^x\sigma_{j+1}^x+h\sum_{j=-L+1}^{L}\sigma_j^z, \qquad (1)$$

with open-boundary conditions, which can be diagonalized by means of Jordan-Wigner transformation and introduction of Majorana fermion operators [16,17], $X_n=(\prod_{j<n}\sigma_j^z)\sigma_n^x$ and $Y_n=(\prod_{j<n}\sigma_j^z)\sigma_n^y$, fulfilling the anticommutation relations $\{X_i,X_j\}=\{Y_i,Y_j\}=2\delta_{ij}$ and $\{X_i,Y_j\}=0$. Heisenberg equations of motion $dA/dt=i[H,A]$ for Majorana operators can be written

$$\dot{X}_n = 2(Y_{n-1} - hY_n), \quad \dot{Y}_n = -2(X_{n+1} - hX_n). \tag{2}$$

An operator corresponding to an arbitrary physical observable can be written as a superposition of products of Majorana operators X_j, Y_j: namely, $P_{\mathbf{n},\mathbf{n}'} = X_{-L+1}^{n_{-L+1}} Y_{-L+1}^{n'_{-L+1}} \cdots X_L^{n_L} Y_L^{n'_L}$ with powers $n_j, n'_j \in \{0,1\}$. A set of 4^{2L} operators $\{|P_{\mathbf{n},\mathbf{n}'}\rangle_\sharp\}$ spans an orthonormal basis of a Hilbert space: namely, the matrix alebra $\mathfrak{A} = \mathbb{C}^{2^{2L}\times 2^{2L}}$, with an inner product ${}_\sharp\langle A|B\rangle_\sharp = 2^{-2L}\mathrm{tr}A^\dagger B$. $\mathfrak{A}$ can be conveniently interpreted as a Fock space of $2L$ *adjoint fermions* (we shall call them a-fermions) with pseudospin (distinguishing between Majorana X_j and Y_j operators). An arbitrary operator A is then an a-fermion state $|A\rangle_\sharp = \Sigma_{\mathbf{n},\mathbf{n}'} a_{\mathbf{n},\mathbf{n}'} |P_{\mathbf{n},\mathbf{n}'}\rangle_\sharp$. a-fermion operators over $\mathfrak{A}$, $\hat{c}_{j\uparrow}$, and $\hat{c}_{j\downarrow}$ can be introduced by

$$\hat{c}_{j\uparrow}|P_{\mathbf{n},\mathbf{n}'}\rangle_\sharp = n_j |X_j P_{\mathbf{n},\mathbf{n}'}\rangle_\sharp, \tag{3}$$

$$\hat{c}_{j\downarrow}|P_{\mathbf{n},\mathbf{n}'}\rangle_\sharp = n'_j |Y_j P_{\mathbf{n},\mathbf{n}'}\rangle_\sharp, \tag{4}$$

satisfying canonical anticommutation relations.

The index of an operator $P_{\mathbf{n},\mathbf{n}'}$ is defined as $I_{\mathbf{n},\mathbf{n}'} = \Sigma_j (n_j + n'_j)$ and for index-1 operators Eq. (2) is rewritten:

$$\frac{d}{dt}|X_n\rangle_\sharp = 2(\hat{c}^\dagger_{n-1\downarrow} - h\hat{c}^\dagger_{n\downarrow})\hat{c}_{n\uparrow}|X_n\rangle_\sharp, \tag{5}$$

$$\frac{d}{dt}|Y_n\rangle_\sharp = -2(\hat{c}^\dagger_{n+1\uparrow} - h\hat{c}^\dagger_{n\uparrow})\hat{c}_{n\downarrow}|Y_n\rangle_\sharp, \tag{6}$$

which can be interpreted as a Schrödinger equation $(d/dt)|A\rangle_\sharp = -i\hat{\mathcal{H}}|A\rangle_\sharp$ for the adjoint Hamiltonian

$$\hat{\mathcal{H}} = 2i \sum_{n\in\mathbb{Z}_{2L}} [(\hat{c}^\dagger_{n-1\downarrow}\hat{c}_{n\uparrow} - \hat{c}^\dagger_{n+1\uparrow}\hat{c}_{n\downarrow}) + h(\hat{c}^\dagger_{n\uparrow}\hat{c}_{n\downarrow} - \hat{c}^\dagger_{n\downarrow}c_{n\uparrow})]. \tag{7}$$

Since the adjoint time evolution is a *homomorphism*, the Schrödinger equation extends to an arbitrary element of the operator algebra $|A\rangle_\sharp \in \mathfrak{A}$. Note that the number of a-fermions, $\hat{\mathcal{N}} = \Sigma_{n,s}\hat{c}^\dagger_{n,s}\hat{c}_{n,s}$, is conserved, unlike the number of ordinary Majorana fermions.

III. OPERATOR SPACE ENTANGLEMENT ENTROPY

It is clear that classical simulability of quantum states is quantified by the entanglement entropy of half-half (or worst case) bipartition of the lattice. However, for simulability of quantum observables (or density operators of mixed states), the decisive quantity is an analog of entanglement entropy defined for an arbitrary element of operator algebra $\mathfrak{A} \ni |A\rangle_\sharp = \Sigma_{\mathbf{n},\mathbf{n}'} a_{\mathbf{n},\mathbf{n}'} |P_{\mathbf{n},\mathbf{n}'}\rangle_\sharp$, with the adjoint reduced density matrix

$$R_{(n_{-L+1},n'_{-L+1},\ldots,n_0,n'_0),(m_{-L+1},m'_{-L+1},\ldots,m_0,m'_0)} = \sum_{n_1,n'_1,\ldots,n_L,n'_L} a_{(n_{-L+1},\ldots,n_0,n_1,\ldots,n_L),(n'_{-L+1},\ldots,n'_0,n'_1,\ldots,n'_L)} \times a^*_{(m_{-L+1},\ldots,m_0,n_1,\ldots,n_L),(m'_{-L+1},\ldots,m'_0,n'_1,\ldots,n'_L)}, \tag{8}$$

namely,

$$S = -\mathrm{tr}R\ln R. \tag{9}$$

For a spin-1/2 chain it is perhaps more natural to use a set of 4^{2L} Pauli operators $Q_{s_{-L+1},\ldots,s_L} = \sigma^{s_{-L+1}}_{-L+1}\cdots\sigma^{s_L}_L$, where $s_j \in \{0,x,y,z\}$, $\sigma^0 \equiv \mathbb{1}$, as a physical basis of operator algebra $\mathfrak{A}$, and define bipartition and entanglement entropy with respect to $Q_{\mathbf{s}}$. However, it is easy to show that the result is identical to Eq. (9) since the transformation between the bases $\{P_{\mathbf{n},\mathbf{n}'}\}$ and $\{Q_{\mathbf{s}}\}$ is a simple *permutation* of multiindices $(\mathbf{n},\mathbf{n}') \leftrightarrow \mathbf{s}$ (with multiplications by ± 1), and even though it is *nonlocal* it maps first L a-fermions to only first L spins and vice versa.

Let us now try to compute *time dependence* of operator space entanglement entropy $S(t)$ for some simple initial operators A. For convenience, we introduce staggered a-fermion operators $\hat{w}_j$, $j \in \mathbb{Z}_{4L} = \{-2L+1,\ldots,0,1,\ldots,2L\}$, such that $\hat{w}_{2n-1} \equiv \hat{c}_{n\uparrow}$ and $\hat{w}_{2n} \equiv \hat{c}_{n\downarrow}$. Any operator acting solely in a space of first L a fermions (or first L spins) can be expressed in terms of $2L$ anticommuting operators $\hat{w}_j$ with $j \in \mathbb{Z}^-_{2L} \equiv \{-2L+1,\ldots,-1,0\}$. We follow Ref. [10] and express 2^{2L} eigenvalues of adjoint reduced density matrix R, as $\rho_{\mathbf{n}} = \Pi_j[n_j\gamma_j + (1-n_j)(1-\gamma_j)]$, $n_j \in \{0,1\}$, where γ_j are eigenvalues of time-dependent $2L\times 2L$ correlation matrix

$$\Gamma_{mn}(t) = {}_\sharp\langle A|\hat{w}^\dagger_m(t)\hat{w}_n(t)|A\rangle_\sharp, \quad m,n \in \mathbb{Z}^-_{2L}. \tag{10}$$

Then, the entanglement entropy (9) simply reads

$$S(t) = \sum_j e(\gamma_j), \quad e(x) \equiv -x\ln x - (1-x)\ln(1-x). \tag{11}$$

This procedure (see [14] for details) results in an efficient numerical method which essentially only requires diagonalization of the $2L$-dimensional matrix Γ for the solution of a quantum problem on 2^{4L}-dimensional Hilbert space.

The time-dependent a-fermion operators $\hat{w}_m(t)$ are obtained from *linear* Heisenberg-type equations $\dot{\hat{w}}_m = -i[\hat{w}_m, \hat{\mathcal{H}}]$: namely, $\dot{\hat{w}}_{2j} = 2(\hat{w}_{2j+1} - h\hat{w}_{2j-1})$ and $\dot{\hat{w}}_{2j-1} = 2(-\hat{w}_{2j-2} + h\hat{w}_{2j})$. The solution of such Heisenberg equations, written as $\dot{\hat{w}}_m = -i\Sigma_n G_{mn}\hat{w}_n$, is obtained by diagonalizing a $2L\times 2L$ matrix $G = V\Lambda V^\dagger$ which yields

$$\hat{w}_m(t) = \sum_n \left(\sum_k V_{mk}e^{-it\Lambda_k}V^*_{nk}\right)\hat{w}_n. \tag{12}$$

However, in the "critical case" $h=1$, the time evolution of $\hat{w}_m(t)$ can be solved exactly and some analytical solutions can be given. Namely, in such a case the sets of Heisenberg equations are identical and are solved via a discrete sine transform with $V_{mk} = i^m\sqrt{\frac{2}{4L+1}}\sin\left[\frac{(m+2L)k\pi}{4L+1}\right]$,

$$\hat{w}_m(t) = \sum_n \left[\sum_{k=0}^{4L} V_{mk}e^{i4\cos[k\pi/(4L+1)]t}V^*_{nk}\right]\hat{w}_n. \tag{13}$$

In the following we shall be interested in the results in the *thermodynamic limit* (TL) $L\to\infty$. The infinite sum over k in Eq. (13) is transformed onto an integral which yields $\hat{w}_m(t) = \Sigma_{n\in\mathbb{Z}}\Phi_{nm}(4t)\hat{w}_n$ in terms of Bessel functions

$\Phi_{ab}(x) \equiv J_{a-b}(x)$. The correlation matrix elements are therefore (using $\hat{n}_b = \hat{w}_b^\dagger \hat{w}_b$)

$$\Gamma_{mn}(t) = \sum_{b \in \mathbb{Z}} \Phi_{bm}(4t)\Phi_{bn}(4t)_\sharp\langle A|\hat{n}_b|A\rangle_\sharp \quad m,n = 0,-1,-2\ldots . \tag{14}$$

We also assume that the initial operator A is *local*—i.e., a product of *finite* number of Pauli matrices σ_j^α. This implies that (i) either A has a *finite* index—i.e., $\langle \hat{n}_b\rangle_\sharp \equiv {}_\sharp\langle A|\hat{n}_b|A\rangle_\sharp = 0$, for $|b| > b_0$ for some $b_0 \in \mathbb{Z}^+$—or (ii) A has an infinite index and $\langle \hat{n}_b\rangle_\sharp = 1$ for $b < -b_0$ and $\langle \hat{n}_b\rangle_\sharp = 0$ for $b > b_0$ (such as, e.g., $A = \sigma_1^x$). Then, as $L \to \infty$, an arbitrary large fixed finite piece of correlation matrix can be asymptotically written as

$$\Gamma_{mn}(t) = \sum_{b \in \mathbb{Z}} J_{b-m}(4t)J_{b-n}(4t)\langle \hat{n}_b\rangle_\sharp . \tag{15}$$

Note that Γ_{mn} has effectively finite rank $\sim x = 4t$: namely, $\Gamma_{m,n} \sim \delta_{mn}$, for $-m,-n > x$. For brevity we shall be omitting the argument of Bessel functions always equal to $x = 4t$.

A. Initial operators of finite index

First, we focus on the case (i) of finite index initial operators A. It was conjectured in [13] that in such cases the entanglement entropy in thermodynamic limit saturates in time. Using the a-fermion algebra we are now able to calculate the exact saturation value since the right-hand side in (15) is a finite sum. Consider Γ_{mn} as a real matrix over $\mathbb{R}^\infty$ with canonical basis $\{|m\rangle, m \in \mathbb{Z}^-\}$ and write a set of *nonorthogonal* vectors $\{|\psi_\alpha\rangle\}$: namely, $\langle m|\psi_\alpha\rangle = (-1)^\alpha J_{\alpha-m}(4t) = \langle \psi_\alpha|m\rangle$. Let us write initial operator of finite index (K) as $A = O_{j_1}\cdots O_{j_K}$ where $O_{2j-1} \equiv X_j$ and $O_{2j} \equiv Y_j$. Then we have $\Gamma_{mn} = J_{j_1-m}J_{j_1-n} + \cdots + J_{j_K-m}J_{j_K-n}$, or

$$\Gamma_{mn}(t) = \langle m|\psi_{j_1}\rangle\langle\psi_{j_1}|n\rangle + \cdots + \langle m|\psi_{j_K}\rangle\langle\psi_{j_K}|n\rangle. \tag{16}$$

This means that the rank of Γ_{mn} is bounded by K; in fact, it is K and its nontrivial eigenspaces are spanned by $\{|\psi_{j_k}\rangle, 1 \le k \le K\}$. Let $\{|\phi_k\rangle, 1 \le k \le K\}$ be an orthonormalized set obtained from $\{|\psi_{j_k}\rangle, 1 \le k \le K\}$ by a standard Gramm-Schmidt procedure, for which the only input is the set of scalar products $\langle\psi_\alpha|\psi_\beta\rangle = \Sigma_{k\in\mathbb{Z}^-} J_{k-\alpha}J_{k-\beta}$ which can be in TL evaluated analytically for *any* t in terms of finite sums and approach the long-time asymptotics $\langle\psi_\alpha|\psi_\beta\rangle|_{t=\infty} = (1/2)\delta_{\alpha\beta} - \sin[\pi(\alpha-\beta)/2]/[\pi(\alpha-\beta)]$. The nonvanishing part of the spectrum $\{\gamma_j\}$ of the correlation matrix (16) entering Eq. (11) is thus given by the eigenvalues of the $K \times K$ matrix

$$\tilde{\Gamma}_{kl} = \langle\phi_k|\psi_{j_1}\rangle\langle\psi_{j_1}|\phi_l\rangle + \cdots + \langle\phi_k|\psi_{j_K}\rangle\langle\psi_{j_K}|\phi_l\rangle. \tag{17}$$

Thus, Eq. (17) is our main result for the case of finite-index initial operators. For illustration, let us calculate the asymptotic value $S(t\to\infty)$ for the simplest two cases: (a) $A = O_j$, e.g., $A = \cdots\sigma_{-2}^z\sigma_{-1}^z\sigma_0^x$, and (b) $A = O_jO_{j+1}$, e.g., $A = \sigma_1^z$. In case (a), $K=1$, the result is $\gamma_1 = \langle\psi_j|\psi_j\rangle$ with $\gamma_1|_{t=\infty} = \frac{1}{2}$ which gives the entanglement entropy of $S(\infty) = \ln 2$. In case (b), $K=2$, we have $\gamma_{1,2} = \frac{1}{2}[\langle\psi_j|\psi_j\rangle + \langle\psi_{j+1}|\psi_{j+1}\rangle \pm \sqrt{(\langle\psi_j|\psi_j\rangle + \langle\psi_{j+1}|\psi_{j+1}\rangle)2 + 4\langle\psi_j|\psi_{j+1}\rangle^2}]$ with $\gamma_{1,2}|_{t=\infty} = \frac{1}{2} \pm \frac{1}{\pi}$ and $S(\infty) = 2\ln(\gamma_1^{-\gamma_1}\gamma_2^{-\gamma_2})$. Both results agree excellently with numerical solutions of Heisenberg equations for $\hat{w}_j(t)$ shown in Fig. 1, for the case $h=1$, whereas saturation is observed for any h.

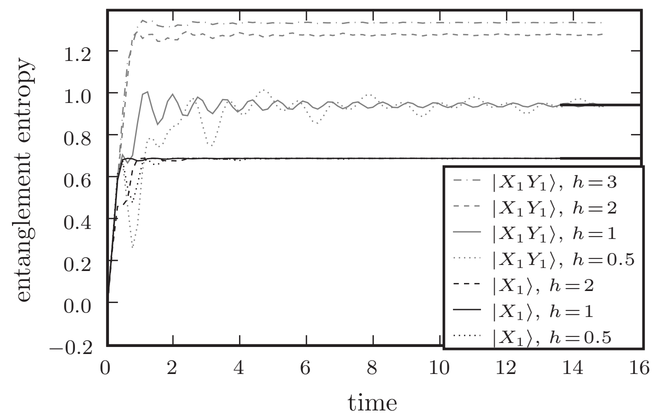

FIG. 1. Entanglement entropy for finite-index operators $|X_1\rangle_\sharp$ (black) and $|X_1Y_1\rangle_\sharp = i|\sigma_1^z\rangle_\sharp$ (gray) compared to theoretic saturation value for $t\to\infty$ and $h=1$ (thick lines). Three different values of magnetic field are considered: $h=1$ (solid curve), $h=0.5$ (dotted curve), $h=2$ (dashed curve), and $h=3$ (dash-dotted curve). We set $2L=200$, such that no finite-size effects were noticable.

B. Initial operators of infinite index

Second, we consider the case (ii) of infinite index initial operator A. In the thermodynamic limit, local spin operators such as σ_1^x are products of infinite number of Majorana operators X_n, Y_n, in particular $|\sigma_1^x\rangle_\sharp = |\cdots X_{-1}Y_{-1}X_0Y_0X_1\rangle_\sharp$, and the previous discussion does not apply. As conjectured in [13] time complexity for such initial operators only grows polynomially in time which corresponds to logarithmic growth of the entanglement entropy. Let us define an infinite index operator $F = \cdots X_{-1}Y_{-1}X_0Y_0$ which corresponds to a half-filled Fermi sea $|F\rangle_\sharp$ of a fermions. Any operator of interest here can be written as $A = FB$ where B is a finite-index operator; again, $|A\rangle_\sharp$ can be intepreted as an a-Fermi sea with a finite number of particle and hole excitations. Figure 2 shows results for $S(t)$, Eq. (11), based on numerical solution of Heisenberg equations for $\hat{w}_j(t)$ for $2L$ up to 800 such that no finite-size effects are noticeable in the figure. For any initial operator of the form $A = FB$, we observe a clean asymptotic logarithmic growth

$$S(t) \asymp c\ln t + c',$$

$$\text{where } c = \begin{cases} 1/6 & \text{if } |h| = 1, \\ 1/3 & \text{if } |h| \neq 1, \end{cases} \tag{18}$$

and c' is a constant which, for given h, only depends on the choice of finite-index operator B. Note an intriguing similarity with the size scaling of entanglement entropy of the ground state of Eq. (1) [5,18]. An analytical explanation of this interesting phenomenon may be nontrivial, so in the following we limit ourselves to the case of critical field $h=1$ and consider only the simplest initial operator of infinite in-

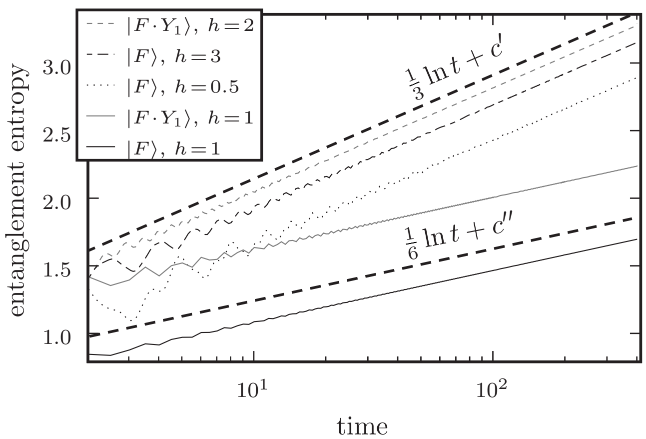

FIG. 2. Entanglement entropy for infinite-index operators $|F\rangle_\sharp$ (black) and $|FY_1\rangle_\sharp = |\sigma_1^y\rangle_\sharp$ (gray) and different magnetic fields (same styling as in Fig. 1) as they appear in the legend from top to bottom. Thick dashed lines correspond to $(1/3)\ln t$ and $(1/6)\ln t$.

dex: namely, $A=F$ where the problem can be connected to the theory of block Toeplitz determinants. In order to avoid negative indices, we redefine the correlation matrix as $\Gamma'_{mn} \equiv \Gamma_{-m,-n}$, so from Eq. (15) follows

$$\Gamma'_{mn} = (-1)^{m+n}\sum_{b=0}^{\infty} J_{b-m}J_{b-n}, \quad m,n = 0,1,\ldots \quad . \tag{19}$$

It should be noted that the correlation matrix can be factorized $\Gamma'_{mn} = \Sigma_{l=0}^{\infty}\Psi_{ml}\Psi_{ln}$ as a square of a matrix $\Psi_{mn} = (-1)^n J_{m-n}$. Note that Ψ_{mn} is a real symmetric infinite block Toeplitz matrix

$$\Psi = \begin{pmatrix} \Pi_0 & \Pi_1 & \ddots \\ \Pi_{-1} & \Pi_0 & \ddots \\ \ddots & \ddots & \ddots \end{pmatrix}, \quad \Pi_l \equiv \begin{pmatrix} J_{2l} & J_{2l+1} \\ -J_{2l-1} & -J_{2l} \end{pmatrix}. \tag{20}$$

Following Ref. [10] we express the time-dependent entanglement entropy (11) in terms of a formula involving Block Toeplitz determinant:

$$S = \frac{1}{2\pi i}\int_{\Xi} e(\lambda^2)\left[\frac{d}{d\lambda}\ln\det(\lambda\mathbb{1} - \Psi)\right]d\lambda, \tag{21}$$

where Ξ is a closed curve in complex plane enclosing unit disk, avoiding point 1 and interval $[-1,0]$. Note that eigenvalues of infinite dimensional matrix Ψ come in pairs λ, $-\lambda$ with accumulation points ± 1. For any $\epsilon > 0$ there is only a finite number $N_\epsilon(t) \sim t$ of eigenvalues of Ψ which are not in ϵ vicinity of ± 1. However, we find numerically that most of these eigenvalues cluster around 0 and only $\sim \ln t$ of them lie outside ϵ vicinity of 0 which are the only eigenvalues contributing to entanglement entropy result (18).

At the present state of the theory of block Toeplitz determinants—in connection to the theory of integrable Fredholm operators and the Riemann-Hilbert problem [19]—formula (21) can be explicitly evaluated (see, e.g., Ref. [20]) provided the *matrix symbol* $\Phi(z) = \lambda\mathbb{1} - \Sigma_{k\in\mathbb{Z}}\Pi_k z^k$ admits explicit Wiener-Hopf factorizatons $\Phi(z) = U^+(z)U^-(z) = V^-(z)V^+(z)$ where the matrix functions $U^\pm(z)$ and $V^\pm(z)$ are analytic inside $(+)$ or outside $(-)$ the unit circle. Even though the matrix symbol has an appealing explicit form

$$\Phi(z) = \begin{pmatrix} \lambda - f\bar{f} + g\bar{g} & f\bar{g}/z - g\bar{f} \\ zg\bar{f} - f\bar{g} & \lambda + f\bar{f} - g\bar{g} \end{pmatrix}, \tag{22}$$

where $f = f(z)$, $\bar{f} = f(z^{-1})$, $g = g(z)$, $\bar{g} = g(z^{-1})$, and $f(z) \equiv \cosh(2t\sqrt{z})$ and $g(z) \equiv \sinh(2t\sqrt{z})/\sqrt{z}$ are *entire* analytic functions, its Wiener-Hopf factorizaton is at present unknown and poses a challenging problem.

IV. CONCLUSIONS

We have studied complexity of the time evolution of initial local operators under dynamics given by the transverse Ising chain. Such complexity can be characterized in terms of the entanglement entropy of operators treated as elements of a product Hilbert space corresponding to a bipartition of a chain and is directly related to the time efficiency of simulation methods such as tDMRG. Note that operator space entanglement entropy, of, say, a density operator, is *not* simply related to a traditional notion of entanglement of the corresponding mixed state. For example, consider a macroscopic convex combination of 2^L product states. This corresponds to a *nonentangled* mixed state but has a macroscopic operator space entanglement entropy $\sim L$ and hence it is difficult to simulate classically. Thus it seems that the traditional concept of *state entanglement* is not sufficient to characterize the classical complexity of quantum operators. In this paper we have shown, in parts analytically and numerically, that the operator space entanglement entropy of the transverse Ising model grows at most logarithmically for initial operators which are local products of Pauli matrices. This result has to be contrasted with a linear growth of entanglement entropy for time evolution of pure states [14]. An explanation of the deeper physical reasons for this dramatic effect is needed.

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