Entanglement transformations using separable operations

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(Received 11 May 2007; published 11 September 2007)

We study conditions for the deterministic transformation $|\psi\rangle \rightarrow |\phi\rangle$ of a bipartite entangled state by a separable operation. If the separable operation is a local operation with classical communication (LOCC), Nielsen's majorization theorem provides necessary and sufficient conditions. For the general case, we derive a necessary condition in terms of products of Schmidt coefficients, which is equivalent to the Nielsen condition when either of the two factor spaces is of dimension 2, but is otherwise weaker. One implication is that no separable operation can reverse a deterministic map produced by another separable operation, if one excludes the case where the Schmidt coefficients of $|\psi\rangle$ are the same as those of $|\phi\rangle$. The question of sufficient conditions in the general separable case remains open. When the Schmidt coefficients of $|\psi\rangle$ are the same as those of $|\phi\rangle$, we show that the Kraus operators of the separable transformation restricted to the supports of $|\psi\rangle$ on the factor spaces are proportional to unitaries. When that proportionality holds and the factor spaces have equal dimension, we find conditions for the deterministic transformation of a collection of several full Schmidt rank pure states $|\psi_j\rangle$ to pure states $|\phi_j\rangle$.

DOI: 10.1103/PhysRevA.76.032310 PACS number(s): 03.67.Mn

I. INTRODUCTION

A separable operation Λ on a bipartite quantum system is a transformation of the form

$$\rho' = \Lambda(\rho) = \sum_m (A_m \otimes B_m)\rho(A_m^\dagger \otimes B_m^\dagger), \tag{1}$$

where ρ is an initial density operator on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. The Kraus operators $A_m \otimes B_m$ are arbitrary product operators satisfying the closure condition

$$\sum_m A_m^\dagger A_m \otimes B_m^\dagger B_m = I \otimes I. \tag{2}$$

The extension of (1) and (2) to multipartite systems is obvious, but here we will consider only the bipartite case. To avoid technical issues the sums in (1) and (2) and the dimensions of $\mathcal{H}_A$ and $\mathcal{H}_B$ are assumed to be finite.

Various kinds of separable operations play important roles in quantum information theory. When m takes on only one value, the operators A_1 and B_1 are (or can be chosen to be) unitary operators, and the operation is a *local unitary* transformation. When every A_m and every B_m is proportional to a unitary operator, we call the operation a *separable random unitary channel*. Both of these are members of the well-studied class of *local operations with classical communication* (LOCC), which can be thought of as an operation carried out by Alice on $\mathcal{H}_A$ with the outcome communicated to Bob. He then uses this information to choose an operation that is carried out on $\mathcal{H}_B$, with outcome communicated to Alice, who uses it to determine the next operation on $\mathcal{H}_A$, and so forth. For a precise definition and a discussion, see [[1], Sec. XI]. While any LOCC is a separable operation, i.e., can be written in the form (1), the reverse is not true: there are separable operations that fall outside the LOCC class [2].

Studying properties of general separable operations seems worthwhile because any results obtained in this way then apply to the LOCC subcategory, which is harder to characterize from a mathematical point of view. However, relatively little is known about separable operations, whereas LOCC has been the subject of intensive studies, with many important results. For example, a LOCC applied to a pure entangled state $|\psi\rangle$ [i.e., $\rho = |\psi\rangle\langle\psi|$ in (1)] results in an ensemble of pure states (labeled by m) whose average entanglement cannot exceed that of $|\psi\rangle$ [[1], Sec. XV D]. One suspects that the same is true of a general separable operation Λ, but this has not been proved. All that seems to be known is that Λ cannot "generate" entanglement when applied to a product pure state or a separable mixed state: the outcome (as is easily checked) will be a separable state.

If a LOCC is applied to a pure (entangled) state $|\psi\rangle$, Lo and Popescu [3] have shown that the same result, typically an ensemble, can be achieved using a different LOCC (depending both on the original operation and on $|\psi\rangle$) in which Alice carries out an appropriate operation on $\mathcal{H}_A$ and Bob a unitary, depending on that outcome, on $\mathcal{H}_B$. This in turn is the basis of a condition due to Nielsen [4] which states that there is a LOCC operation deterministically (probability 1) mapping a given bipartite state $|\psi\rangle$ to another pure state $|\phi\rangle$ if and only if $|\phi\rangle$ majorizes $|\psi\rangle$ [5].

In this paper we derive a necessary condition for a separable operation to deterministically map $|\psi\rangle$ to $|\phi\rangle$ in terms of their Schmidt coefficients, the inequality (5). While it is weaker than Nielsen's condition (unless either $\mathcal{H}_A$ or $\mathcal{H}_B$ is two dimensional, in which case it is equivalent), it is not trivial. In the particular case that the Schmidt coefficients are the same, i.e., $|\psi\rangle$ and $|\phi\rangle$ are equivalent under local unitaries, we show that all the A_m and B_m operators in (1) are proportional to unitaries, so that in this case the separable operation is also a random unitary channel. For this situation we also study the conditions under which a whole *collection* $\{|\psi_j\rangle\}$ of pure states are deterministically mapped to pure states, a problem which seems not to have been previously

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studied either for LOCC or for more general separable operations.

The remainder of the paper is organized as follows. Section II has the proof, based on an inequality by Minkowski, p. 482 of [6], of the relationship between the Schmidt coefficients of $|\psi\rangle$ and $|\phi\rangle$ when a separable operation deterministically maps $|\psi\rangle$ to $|\phi\rangle$, and some consequences of this result. In Sec. III we derive and discuss the conditions under which a separable random unitary channel will map a collection of pure states to pure states. A summary and some discussion of open questions will be found in Sec. IV.

II. LOCAL TRANSFORMATIONS OF BIPARTITE ENTANGLED STATES

We use the term *Schmidt coefficients* for the non-negative coefficients $\{\lambda_j\}$ in the Schmidt expansion

$$|\psi\rangle = \sum_{j=0}^{d-1} \lambda_j |\bar{a}_j\rangle \otimes |\bar{b}_j\rangle, \tag{3}$$

of a state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, using appropriately chosen orthonormal bases $\{|\bar{a}_j\rangle\}$ and $\{|\bar{b}_j\rangle\}$, with the order chosen so that

$$\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{d-1} \geq 0. \tag{4}$$

The number r of positive (nonzero) Schmidt coefficients is called the *Schmidt rank*. We call the subspace of $\mathcal{H}_A$ spanned by $|\bar{a}_0\rangle, |\bar{a}_1\rangle, \ldots, |\bar{a}_{r-1}\rangle$, i.e., the basis kets for which the Schmidt coefficients are positive, the $\mathcal{H}_A$ support of $|\psi\rangle$, and that spanned by $|\bar{b}_0\rangle, |\bar{b}_1\rangle, \ldots, |\bar{b}_{r-1}\rangle$ its $\mathcal{H}_B$ support.

Our main result is the following.

Theorem 1. Let $|\psi\rangle$ and $|\phi\rangle$ be two bipartite entangled states on $\mathcal{H}_A \otimes \mathcal{H}_B$ with positive Schmidt coefficients $\{\lambda_j\}$ and $\{\mu_j\}$, respectively, in decreasing order, and let r be the Schmidt rank of $|\psi\rangle$. If $|\psi\rangle$ can be transformed to $|\phi\rangle$ by a deterministic separable operation, then:

(i) The Schmidt rank of $|\phi\rangle$ is less than or equal to r.

(ii)

$$\prod_{j=0}^{r-1} \lambda_j \geq \prod_{j=0}^{r-1} \mu_j. \tag{5}$$

(iii) If (5) is an equality with both sides positive, the Schmidt coefficients of $|\psi\rangle$ and $|\phi\rangle$ are identical, $\lambda_j = \mu_j$, and the operators A_m and B_m restricted to the $\mathcal{H}_A$ and $\mathcal{H}_B$ supports of $|\psi\rangle$, respectively, are proportional to unitary operators.

(iv) The reverse deterministic transformation of $|\phi\rangle$ to $|\psi\rangle$ by a separable operation is only possible when the Schmidt coefficients are identical, $\lambda_j = \mu_j$.

Proof. For the proof it is convenient to use map-state duality (see [7,8] and [[9], Chap. 11]) defined in the following way. Let $\{|b_j\rangle\}$ be an orthonormal basis of $\mathcal{H}_B$ that will remain fixed throughout the following discussion. Any ket $|\chi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be expanded in this basis in the form

$$|\chi\rangle = \sum_j |\alpha_j\rangle \otimes |b_j\rangle, \tag{6}$$

where the $\{|\alpha_j\rangle\}$ are the (unnormalized) expansion coefficients. We define the corresponding dual map $\chi : \mathcal{H}_B \to \mathcal{H}_A$ to be

$$\chi = \sum_j |\alpha_j\rangle\langle b_j|. \tag{7}$$

Obviously, any map from $\mathcal{H}_B$ to $\mathcal{H}_A$ can be written in the form (7), and can thus be transformed into a ket on $\mathcal{H}_A \otimes \mathcal{H}_B$ by the inverse process: replacing $\langle b_j|$ with $|b_j\rangle$. The transformation depends on the choice of basis $\{|b_j\rangle\}$, but this will not matter, because our results will in the end be independent of this choice. Note in particular that the *rank* of the operator χ is exactly the same as the *Schmidt rank* of $|\chi\rangle$.

For a separable operation that deterministically maps $|\psi\rangle$ to $|\phi\rangle$ (or, to be more specific, $|\psi\rangle\langle\psi|$ to $|\phi\rangle\langle\phi|$) it must be the case that

$$(A_m \otimes B_m)|\psi\rangle = \sqrt{p_m}|\phi\rangle \tag{8}$$

for every m, as otherwise the result of the separable operation acting on $|\psi\rangle$ would be a mixed state. (One could also include a complex phase factor depending on m, but this can be removed by incorporating it in A_m—an operation is not changed if the Kraus operators are multiplied by phases.) By using map-state duality we may rewrite (8) in the form

$$A_m \psi \bar{B}_m = \sqrt{p_m}\phi, \tag{9}$$

where by $\bar{B}_m$ we mean the *transpose* of this operator in the basis $\{|b_j\rangle\}$—or, to be more precise, the operator whose matrix in this basis is the transpose of the matrix of B_m. From (9) one sees at once that, since the rank of a product of operators cannot be larger than the rank of any of the factors, the rank of ϕ cannot be greater than that of ψ. When translated back into Schmidt ranks this proves (i).

For the next part of the proof let us first assume that $\mathcal{H}_A$ and $\mathcal{H}_B$ have the same dimension d, and that the Schmidt ranks of both $|\psi\rangle$ and $|\phi\rangle$ are equal to d; we leave until later the modifications necessary when these conditions are not satisfied. In light of the previous discussion of (9), we see that $\bar{B}_m$ has rank d, so is invertible. Therefore one can solve (9) for A_m, and if the solution is inserted in (2) the result is

$$I \otimes I = \sum_m p_m[\psi^{-1\dagger}\bar{B}_m^{-1\dagger}(\phi^\dagger\phi)\bar{B}_m^{-1}\psi^{-1}] \otimes [B_m^\dagger B_m]. \tag{10}$$

The Minkowski inequality ([6], p. 482) for a sum of positive semidefinite operators on a D-dimensional space is

$$\left[\det\left(\sum_m Q_m\right)\right]^{1/D} \geq \sum_m (\det Q_m)^{1/D}, \tag{11}$$

with equality if and only if all Q_m's are proportional, i.e., $Q_i = f_{ij}Q_j$, where the f_{ij} are positive constants. Since $A_m^\dagger A_m \otimes B_m^\dagger B_m$ is a positive operator on a $(D=d^2)$-dimensional space, (10) and (11) yield

$$1 \geq \left[\det\left(\sum_m p_m[\psi^{-1\dagger}\bar{B}_m^{-1\dagger}(\phi^\dagger\phi)\bar{B}_m^{-1}\psi^{-1}]\otimes[B_m^\dagger B_m]\right)\right]^{1/d^2}$$

$$\geq \sum_m \left(\det\left[p_m[\psi_{-1}^\dagger\bar{B}_m^{-1\dagger}(\phi^\dagger\phi)\bar{B}_m^{-1}\psi^{-1}]\otimes[B_m^\dagger B_m)\right]\right)^{1/d^2}$$

$$= \sum_m p_m \frac{\det(\phi^\dagger\phi)^{1/d}}{\det(\psi^\dagger\psi)^{1/d}} = \frac{\det(\phi^\dagger\phi)^{1/d}}{\det(\psi^\dagger\psi)^{1/d}}, \tag{12}$$

which is equivalent to

$$\det(\psi^\dagger\psi) \geq \det(\phi^\dagger\phi). \tag{13}$$

The relation $\det(A\otimes B)=(\det A)^b(\det B)^a$, where a,b are the dimensions of A and B, was used in deriving (12). Since (13) is the square of (5), this proves part (ii).

If (5) is an equality with both sides positive, $\det(\phi^\dagger\phi)/\det(\psi^\dagger\psi)=1$ and the inequality (12) becomes an equality, which implies that all positive operators in (11) are proportional, i.e.,

$$A_m^\dagger A_m \otimes B_m^\dagger B_m = f_{mn} A_n^\dagger A_n \otimes B_n^\dagger B_n, \tag{14}$$

where the f_{mn} are positive constants. Setting $n=1$ in (14) and inserting it in (2), one gets

$$\left(\sum_m f_{m1}\right) A_1^\dagger A_1 \otimes B_1^\dagger B_1 = I \otimes I. \tag{15}$$

This implies that both $A_1^\dagger A_1$ and $B_1^\dagger B_1$ are proportional to the identity, so A_1 and B_1 are proportional to unitary operators, and of course the same argument works for every m. Since local unitaries cannot change the Schmidt coefficients, it is obvious that $|\psi\rangle$ and $|\phi\rangle$ must share the same set of Schmidt coefficients, that is, $\lambda_j=\mu_j$, for every j, and this proves (iii).

To prove (iv), note that, if there is a separable operation carrying $|\psi\rangle$ to $|\phi\rangle$ and another carrying $|\phi\rangle$ to $|\psi\rangle$, the Schmidt ranks of $|\psi\rangle$ and $|\phi\rangle$ must be equal by (i), and (5) is an equality, so (iii) implies equal Schmidt coefficients.

Next let us consider the modifications needed when the Schmidt ranks of $|\psi\rangle$ and $|\phi\rangle$ might be unequal, and are possibly less than the dimensions of $\mathcal{H}_A$ or $\mathcal{H}_B$, which need not be the same. As noted previously, (9) shows that the Schmidt rank of $|\phi\rangle$ cannot be greater than that of $|\psi\rangle$. If it is less, then the right side of (5) is zero, because at least one of the μ_j in the product will be zero, so part (ii) of the theorem is automatically satisfied, part (iii) does not apply, and (iv) is trivial. Thus we only need to discuss the case in which the Schmidt ranks of $|\psi\rangle$ and $|\phi\rangle$ have the same value r. Let P_A and P_B be the projectors on the $\mathcal{H}_A$ and $\mathcal{H}_B$ supports $\mathcal{S}_A$ and $\mathcal{S}_B$ of $|\psi\rangle$ (as defined at the beginning of this section), and let $\mathcal{T}_A$ and $\mathcal{T}_B$ be the corresponding supports of $|\phi\rangle$. Note that each of these subspaces is of dimension r. Since $(P_A \otimes P_B)|\psi\rangle=|\psi\rangle$, (8) can be rewritten as

$$(A'_m \otimes B'_m)|\psi\rangle = \sqrt{p_m}|\phi\rangle, \tag{16}$$

where

$$A'_m = A_m P_A, \quad B'_m = B_m P_B \tag{17}$$

are the operators A_m and B_m restricted to the supports of $|\psi\rangle$. In fact, A'_m maps $\mathcal{S}_A$ onto $\mathcal{T}_A$, and B'_m maps $\mathcal{S}_B$ onto $\mathcal{T}_B$, as this is the only way in which (16) can be satisfied when $|\phi\rangle$ and $|\psi\rangle$ have the same Schmidt rank. Finally, by multiplying (2) by $P_A\otimes P_B$ on both left and right one arrives at the closure condition

$$\sum_m A'^\dagger_m A'_m \otimes B'^\dagger_m B'_m = P_A \otimes P_B. \tag{18}$$

Thus if we use the restricted operators A'_m and B'_m we are back to the situation considered previously, with $\mathcal{S}_A$ and $\mathcal{T}_A$ (which are isomorphic) playing the role of $\mathcal{H}_A$, and $\mathcal{S}_B$ and $\mathcal{T}_B$ the role of $\mathcal{H}_B$, and hence the previous proof applies. ■

Some connections between LOCC and the more general category of separable operations are indicated in the following corollaries.

Corollary 1. When $|\psi\rangle$ is majorized by $|\phi\rangle$, so there is a deterministic LOCC mapping $|\psi\rangle$ to $|\phi\rangle$, there does not exist a separable operation that deterministically maps $|\phi\rangle$ to $|\psi\rangle$, unless these have equal Schmidt coefficients (are equivalent under local unitaries).

This is nothing but (iv) of Theorem 1 applied when the $|\psi\rangle$ to $|\phi\rangle$ map is LOCC, and thus separable. It is nonetheless worth pointing out because majorization provides a very precise characterization of what deterministic LOCC operations can accomplish, and the corollary provides a connection with more general separable operations.

Corollary 2. If either $\mathcal{H}_A$ or $\mathcal{H}_B$ is two dimensional, then $|\psi\rangle$ can be deterministically transformed to $|\phi\rangle$ if and only if this is possible using LOCC, i.e., $|\psi\rangle$ is majorized by $|\phi\rangle$.

The proof comes from noting that, when there are only two nonzero Schmidt coefficients, the majorization condition is $\mu_0 \geq \lambda_0$, and this is equivalent to (5).

III. SEPARABLE RANDOM UNITARY CHANNEL

A. Condition for deterministic mapping

Any quantum operation (trace-preserving completely positive map) can be thought of as a quantum channel, and if the Kraus operators are proportional to unitaries, the channel is bistochastic (maps I to I) and is called a random unitary channel or a random external field in Sec. 10.6 of [9]. Thus a separable operation in which the A_m and B_m are proportional to unitaries U_m and V_m, so that (1) takes the form

$$\rho' = \Lambda(\rho) = \sum_m p_m (U_m \otimes V_m)\rho(U_m \otimes V_m)^\dagger, \tag{19}$$

with the $p_m>0$ summing to 1, can be called a separable random unitary channel. We shall be interested in the case in which $\mathcal{H}_A$ and $\mathcal{H}_B$ have the same dimension d, and in which the separable unitary channel deterministically maps not just one but a collection $\{|\psi_j\rangle\}$, $1\leq j\leq N$, of pure states of full Schmidt rank d to pure states. This means that (8) written in the form

$$(U_m \otimes V_m)|\psi_j\rangle \doteq |\phi_j\rangle \tag{20}$$

must hold for all j as well as for all m. The symbol $\doteq$ means that the two sides can differ by at most a complex phase. Here such phases cannot simply be incorporated in U_m or V_m, because (20) must hold for all values of j, even though they are not relevant for the map carrying $|\psi_j\rangle\langle\psi_j|$ to $|\phi_j\rangle\langle\phi_j|$.

Theorem 2. Let $\{|\psi_j\rangle\}$, $1 \leq j \leq N$, be a collection of states of full Schmidt rank on a tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ of two spaces of equal dimension, and let Λ be the separable random unitary channel defined by (19). Let ψ_j and ϕ_j be the operators dual to $|\psi_j\rangle$ and $|\phi_j\rangle$—see (6) and (7).

(i) If every $|\psi_j\rangle$ from the collection is deterministically mapped to a pure state, then

$$U_m^\dagger U_n \psi_j \psi_k^\dagger \doteq \psi_j \psi_k^\dagger U_m^\dagger U_n \tag{21}$$

for every m, n, j, and k.

(ii) If (21) holds for a fixed m and every n, j, and k, it holds for every m, n, j, and k. If in addition at least one of the states from the collection $\{|\psi_j\rangle\}$ is deterministically mapped to a pure state by Λ, then every state in the collection is mapped to a pure state.

(iii) Statements (i) and (ii) also hold when (21) is replaced with

$$V_m^\dagger V_n \psi_j^\dagger \psi_k \doteq \psi_j^\dagger \psi_k V_m^\dagger V_n. \tag{22}$$

Proof. (i) By map-state duality, (20) can be rewritten as

$$U_m \psi_j \bar{V}_m \doteq \phi_j, \tag{23}$$

where $\bar{V}_m$ is the transpose of V_m—see the remarks following (9). By combining (23) with its adjoint with j replaced by k, and using the fact that $\bar{V}_m$ is unitary, we arrive at

$$U_m \psi_j \psi_k^\dagger U_m^\dagger \doteq \phi_j \phi_k^\dagger. \tag{24}$$

Since the right side is independent of m, so is the left, which means that

$$U_n \psi_j \psi_k^\dagger U_n^\dagger \doteq U_m \psi_j \psi_k^\dagger U_m^\dagger. \tag{25}$$

Multiply on the left by $U_m^\dagger$ and on the right by U_n to obtain (21).

(ii) If (25), which is equivalent to (21), holds for $m=1$ it obviously holds for all values of m. Now assume that $|\psi_1\rangle$ is mapped by Λ to a pure state $|\phi_1\rangle$, so (23) holds for all m when $j=1$. Take the adjoint of this equation and multiply by $\bar{V}_m$ to obtain

$$\psi_1^\dagger U_m^\dagger \doteq \bar{V}_m \phi_1^\dagger. \tag{26}$$

Set $k=1$ in (25), and use (26) to rewrite it as

$$U_n \psi_j \bar{V}_n \phi_1^\dagger \doteq U_m \psi_j \bar{V}_m \phi_1^\dagger. \tag{27}$$

Since by hypothesis $|\psi_1\rangle$ has Schmidt rank d, the same is true of ψ_1, and since U_m and $\bar{V}_m$ in (23) are unitaries, ϕ_1 and thus also $\phi_1^\dagger$ has rank d and is invertible. Consequently, (27) implies that

$$U_n \psi_j \bar{V}_n \doteq U_m \psi_j \bar{V}_m, \tag{28}$$

and we can define ϕ_j to be one of these common values, for example, $U_1 \psi_j \bar{V}_1$. Map-state duality transforms this ϕ_j into $|\phi_j\rangle$ which, because of (28), satisfies (20).

(iii) The roles of U_m and V_m are obviously symmetrical, but our convention for map-state duality makes ψ_j a map from $\mathcal{H}_B$ to $\mathcal{H}_A$, which is the reason why its adjoint appears in (22). ∎

B. Example

Let us apply Theorem 2 to see what pure states of full Schmidt rank are deterministically mapped onto pure states by the following separable random unitary channel on two qubits:

$$\Lambda(\rho) = p\rho + (1-p)(X \otimes Z)\rho(X \otimes Z). \tag{29}$$

The Kraus operators are $I \otimes I$ and $X \otimes Z$, so $U_1 = I$ and $U_2 = X$. Thus the condition (21) for a collection of states $\{|\psi_j\rangle\}$ to be deterministically mapped to pure states is

$$X \psi_j \psi_k^\dagger \doteq \psi_j \psi_k^\dagger X. \tag{30}$$

It is easily checked that

$$|\psi_1\rangle = (|+\rangle|0\rangle + |-\rangle|1\rangle)/\sqrt{2} \tag{31}$$

is mapped to itself by (29). If the corresponding

$$\psi_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{32}$$

is inserted in (30) with $k=1$, one can show that (30) is satisfied for any 2×2 matrix

$$\psi_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \tag{33}$$

having $c_j = \pm a_j$ and $d_j = \mp b_j$, and that in turn these satisfy (30) for every j and k. Thus all states of the form

$$|\psi_\pm\rangle = a|00\rangle + b|01\rangle \pm a|10\rangle \mp b|11\rangle, \tag{34}$$

with a and b complex numbers, are mapped by this channel into pure states.

IV. CONCLUSIONS

Our main results are in Theorem 1: if a pure state on a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$ is deterministically mapped to a pure state by a separable operation $\{A_m \otimes B_m\}$, then the product of the Schmidt coefficients can only decrease, and if it remains the same, the two sets of Schmidt coefficients are identical to each other, and the A_m and B_m operators are proportional to unitaries. (See the detailed statement of the theorem for situations in which some of the Schmidt coefficients vanish.) This *product condition* is necessary but not sufficient: i.e., even if it is satisfied there is no guarantee that a separable operation exists which can carry out the specified map. Indeed, we think it is likely that when both $\mathcal{H}_A$ and $\mathcal{H}_B$ have dimension 3 or more there are situations in which the product condition is satisfied but a deterministic map is not possible. The reason is that (5) is consistent with $|\phi\rangle$ having a larger entanglement than $|\psi\rangle$, and we doubt whether a separable operation can increase entanglement. While it is known that LOCC cannot increase the average entanglement [[1], Sec. XV D], there seems to be no similar result for general separable operations. This is an important open question.

It is helpful to compare the product condition (5) with Nielsen's majorization condition, which says that a deterministic separable operation of the LOCC type can map $|\psi\rangle$ to $|\phi\rangle$ if and only if $|\phi\rangle$ majorizes $|\psi\rangle$ [5]. Corollary 2 of Theorem 1 shows that the two are identical if system A or system B is two dimensional. Under this condition a general separable operation can deterministically map $|\psi\rangle$ to $|\phi\rangle$ only if it is possible with LOCC. This observation gives rise to the conjecture that when either A or B is two dimensional any separable operation is actually of the LOCC form. This conjecture is consistent with the fact that the well-known example [2] of a separable operation that is not a LOCC uses the tensor product of two three dimensional spaces. But whether separable operations and LOCC coincide even in the simple case of a 2×2 system is at present an open question (see note added in proof).

When the dimensions of A and B are both 3 or more, the product condition of Theorem 1 is weaker than the majorization condition: if $|\phi\rangle$ majorizes $|\psi\rangle$ then (5) will hold [10], but the converse is in general not true. Thus there might be situations in which a separable operation deterministically maps $|\psi\rangle$ to $|\phi\rangle$ even though $|\phi\rangle$ does not majorize $|\psi\rangle$. If such cases exist, Corollary 1 of Theorem 1 tells us that $|\psi\rangle$ and $|\phi\rangle$ must be incomparable under majorization: neither one majorizes the other. Finding an instance, or demonstrating its impossibility, would help clarify how general separable operations differ from the LOCC subclass.

When a separable operation deterministically maps $|\psi\rangle$ to $|\phi\rangle$ and the products of the two sets of Schmidt coefficients are the same, part (iii) of Theorem 1 tells us that the collections of Schmidt coefficients are in fact identical, and that the A_m and B_m operators (restricted if necessary to the supports of $|\psi\rangle$) are proportional to unitaries. Given this proportionality (and that the map is deterministic), the identity of the collection of Schmidt coefficients is immediately evident, but the converse is not at all obvious. The result just mentioned can be used to simplify part of the proof in some interesting work on local copying, specifically the unitarity of local Kraus operators in [[11], Sec. 3.1]. It might have applications in other cases where one is interested in deterministic nonlocal operations.

Finally, Theorem 2 gives conditions under which a separable random unitary operation can deterministically map a whole collection of pure states to pure states. These conditions [see (21) or (22)] involve both the unitary operators and the states themselves, expressed as operators using map-state duality, in an interesting combination. While these results apply only to a very special category, they raise the question whether simultaneous deterministic maps of several pure states might be of interest for more general separable operations. The nonlocal copying problem, as discussed in [11–14], is one situation where results of this type are relevant, and there may be others.

Note added in proof. Our conjecture on the equivalence of separable operations and LOCC for low dimensions has been shown to be false [17].

ACKNOWLEDGMENTS

We thank Shengjun Wu for useful conversations. The research described here received support from the National Science Foundation through Grant No. PHY-0456951.

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