Circulant states with positive partial transpose

Dariusz Chruściński and Andrzej Kossakowski
Institute of Physics, Nicolaus Copernicus University, Grudziądzka 5/7, 87-100 Toruń, Poland
(Received 14 June 2007; published 10 September 2007)

We construct a large class of quantum $d\otimes d$ states which are positive under partial transposition (so called PPT states). The construction is based on certain direct sum decomposition of the total Hilbert space displaying characteristic circular structure—that is why we call them circulant states. It turns out that partial transposition maps any such decomposition into another one and hence both original density matrix and its partially transposed partner share similar cyclic properties. This class contains many well-known examples of PPT states from the literature and gives rise to a huge family of completely new states.

DOI: 10.1103/PhysRevA.76.032308 PACS number(s): 03.67.Mn, 03.65.Ud, 03.67.−a

I. INTRODUCTION

The interest on quantum entanglement has dramatically increased during the last two decades due to the emerging field of quantum information theory [1]. It turns out that quantum entanglement may be used as basic resources in quantum information processing and communication. The prominent examples are quantum cryptography, quantum teleportation, quantum error correction codes, and quantum computation.

It is well known that it is extremely hard to check whether a given density matrix describing a quantum state of the composite system is separable or entangled. There are several operational criteria which enable one to detect quantum entanglement (see, e.g., Ref. [2] for a recent review). The most famous Peres-Horodecki criterion [3,4] is based on the partial transposition: if a state ρ is separable then its partial transposition $(\mathbb{1}\otimes\tau)\rho$ is positive. States which are positive under partial transposition are called PPT states. Clearly each separable state is necessarily PPT but the converse is not true. It was shown by Horodecki *et al.* [5] that PPT condition is both necessary and sufficient for separability for $2\otimes 2$ and $2\otimes 3$ systems.

Now, since all separable states belong to a set of PPT states, the structure of this set is of primary importance in quantum information theory. Unfortunately, this structure is still unknown, that is, one may check whether a given state is PPT but we do not know how to construct a general quantum state with PPT property. There are only several examples of PPT states which do not show any systematic methods of constructing them [with one exception, i.e., a class of PPT entangled states which is based on a concept of unextendible product bases [6] (see also Ref. [7])]. Other examples of PPT entangled states were constructed in Refs. [4,8–15] and the extreme points of the set of PPT states were recently analyzed in [19]. PPT states play also a crucial role in mathematical theory of positive maps and, as is well known, these maps are very important in the study of quantum entanglement. The mathematical structure of quantum entangled states with positive partial transposition were studied in Refs. [16–18].

Recently in Ref. [20] we proposed a class of PPT states in $d\otimes d$ which are invariant under the maximal commutative subgroup of U(d), i.e., d-dimensional torus $\mathrm{U}(1)\times\cdots\times\mathrm{U}(1)$. In the present paper we propose another class which defines considerable generalization of Ref. [20]. The construction of this new class is based on certain decomposition of the total Hilbert space $\mathbb{C}^d\otimes\mathbb{C}^d$ into direct sum of d-dimensional subspaces. This decomposition is controlled by some cyclic property, that is, knowing one subspace, say Σ_0, the remaining subspaces $\Sigma_1,\ldots,\Sigma_{d-1}$ are uniquely determined by applying a cyclic shift to elements from Σ_0. Now, we call a density matrix ρ a "circulant state" if ρ is a convex combination of density matrices supported on Σ_α. The crucial observation is that a partial transposition of the circulant state has again a circular structure corresponding to another direct sum decomposition $\tilde{\Sigma}_0\oplus\cdots\oplus\tilde{\Sigma}_{d-1}$.

The paper is organized as follows. For pedagogical reason we first illustrate our general method for $d=2$ in Sec. II and for $d=3$ in Sec. III. Interestingly, there is only one circular decomposition for $d=2$ and exactly two different decompositions for $d=3$. In general case presented in Sec. IV there are $(d-1)!$ decompositions labeled by permutations from the symmetric group S_{d-1}. Section V presents several known examples of PPT states that do belong to our class. Final conclusions are collected in the last section.

II. TWO QUBITS

A. An instructive example

Consider a density matrix living in $\mathbb{C}^2\otimes\mathbb{C}^2$ which has the following form:

$$\rho=\left(\begin{array}{cc|cc} a_{00} & \cdot & \cdot & a_{01} \\ \cdot & b_{00} & b_{01} & \cdot \\ \hline \cdot & b_{10} & b_{11} & \cdot \\ a_{10} & \cdot & \cdot & a_{11} \end{array}\right). \tag{1}$$

In order to have more transparent pictures we replaced all vanishing matrix elements by dots (we use this convention throughout this paper). It is clear that Eq. (1) defines a positive operator iff the following 2×2 matrices:

$$a=\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix},\quad b=\begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} \tag{2}$$

are positive. Normalization adds additional condition

$$\mathrm{Tr}\, a + \mathrm{Tr}\, b = 1.$$

Now, the crucial observation is that partially transposed matrix $\rho^\tau = (\mathbb{1} \otimes \tau)\rho$ belongs to the same class as original ρ

$$\rho^\tau = \left(\begin{array}{cc|cc} \tilde{a}_{00} & \cdot & \cdot & \tilde{a}_{01} \\ \cdot & \tilde{b}_{00} & \tilde{b}_{01} & \cdot \\ \hline \cdot & \tilde{b}_{10} & \tilde{b}_{11} & \cdot \\ \tilde{a}_{10} & \cdot & \cdot & \tilde{a}_{11} \end{array}\right), \tag{3}$$

where the matrices $\tilde{a}=[\tilde{a}_{ij}]$ and $\tilde{b}=[\tilde{b}_{ij}]$ read

$$\tilde{a} = \begin{pmatrix} a_{00} & b_{01} \\ b_{10} & a_{11} \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} b_{00} & a_{01} \\ a_{10} & b_{11} \end{pmatrix}. \tag{4}$$

Hence, ρ defined in Eq. (1) is PPT iff

$$\tilde{a} \geqslant 0 \quad \text{and} \quad \tilde{b} \geqslant 0. \tag{5}$$

The above conditions together with $a \geqslant 0$ and $b \geqslant 0$ may be equivalently rewritten as

$$a_{00}a_{11} \geqslant |a_{01}|^2,$$

$$a_{00}a_{11} \geqslant |b_{01}|^2, \tag{6}$$

and

$$b_{00}b_{11} \geqslant |a_{01}|^2,$$

$$b_{00}b_{00} \geqslant |b_{01}|^2, \tag{7}$$

which presents the full characterization of PPT states within a class (1). We stress that for $b_{01}=b_{10}=0$ the above class reduces to the family of PPT states considered in Ref. [20].

B. Cyclic structure

In order to generalize the above example to higher dimensional cases let us observe that there is an interesting property of cyclicity which governs the structure of Eq. (1). For this reason we call Eq. (1) the circulant state. Note that ρ may be written as a direct sum

$$\rho = \rho_0 + \rho_1, \tag{8}$$

where ρ_0 and ρ_1 are supported on two orthogonal subspaces

$$\Sigma_0 = \mathrm{span}\{e_0 \otimes e_0, e_1 \otimes e_1\},$$

$$\Sigma_1 = \mathrm{span}\{e_0 \otimes e_1, e_1 \otimes e_0\}, \tag{9}$$

where $\{e_0, e_1\}$ is a computational base in $\mathbb{C}^2$, and clearly

$$\Sigma_0 \oplus \Sigma_1 = \mathbb{C}^2 \otimes \mathbb{C}^2.$$

One has

$$\rho_0 = \sum_{i,j=0}^{1} a_{ij} e_{ij} \otimes e_{ij}, \tag{10}$$

$$\rho_1 = \sum_{i,j=0}^{1} b_{ij} e_{ij} \otimes e_{i+1,j+1}, \tag{11}$$

where

$$e_{ij} = |e_i\rangle\langle e_j|, \tag{12}$$

and one adds mod 2. Now, let us introduce the shift operator $S: \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$S e_i = e_{i+1} \pmod 2. \tag{13}$$

It is clear that matrix elements S_{ij} define the following circulant matrix [21]:

$$S = \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix}. \tag{14}$$

One finds that

$$\Sigma_1 = (\mathbb{1} \otimes S)\Sigma_0. \tag{15}$$

Moreover, introducing two orthogonal projectors P_0 and $P_1 = (\mathbb{1} \otimes S)P_0(\mathbb{1} \otimes S)^*$ projecting onto Σ_0 and Σ_1, respectively,

$$P_0 = \sum_{i=0}^{1} e_{ii} \otimes e_{ii}, \tag{16}$$

$$P_1 = \sum_{i=0}^{1} e_{ii} \otimes e_{i+1,i+1}, \tag{17}$$

one finds

$$\rho_i = P_i \rho P_i \tag{18}$$

and, hence,

$$\rho = P_0 \rho P_0 + P_1 \rho P_1. \tag{19}$$

Now, it turns out that Eq. (1) may be nicely rewritten in terms of S. Introducing the following diagonal matrices:

$$x_{ij} = \begin{pmatrix} a_{ij} & \cdot \\ \cdot & b_{ij} \end{pmatrix}, \tag{20}$$

one may rewrite Eq. (1) in the following form:

$$\rho = \left(\begin{array}{c|c} S^0 x_{00} S^0 & S^0 x_{01} S^1 \\ \hline S^1 x_{10} S^0 & S^1 x_{11} S^1 \end{array}\right). \tag{21}$$

It is therefore clear that partially transposed matrix ρ^τ also possesses a cyclic structure

$$\rho^\tau = \left(\begin{array}{c|c} S^0 x_{00} S^0 & S^1 x_{01} S^0 \\ \hline S^0 x_{10} S^1 & S^1 x_{11} S^1 \end{array}\right), \tag{22}$$

and may be decomposed as the following direct sum:

$$\rho^\tau = \tilde{\rho}_0 + \tilde{\rho}_1, \tag{23}$$

with

$$\tilde{\rho}_0 = \sum_{i,j=0}^{1} \tilde{a}_{ij} e_{ij} \otimes e_{ij}, \tag{24}$$

$$\tilde{\rho}_1 = \sum_{i,j=0}^{1} \tilde{b}_{ij} e_{ij} \otimes e_{i+1,j+1}. \tag{25}$$

In analogy with Eqs. (26) and (27) one has

$$\tilde{\rho}_i = P_i \rho^\tau P_i \tag{26}$$

and

$$\rho^\tau = P_0 \rho^\tau P_0 + P_1 \rho^\tau P_1. \tag{27}$$

Note that partial transposition $\rho \to \rho^\tau$ reduces to the following operations on the level on 2×2 matrices:

$$a \to \tilde{a} \quad \text{and} \quad b \to \tilde{b}.$$

Again these operations are fully controlled by the circulant matrix S

$$\tilde{a} = a \circ \mathbb{I} + b \circ S \tag{28}$$

and, similarly,

$$\tilde{b} = b \circ \mathbb{I} + a \circ S, \tag{29}$$

where $x \circ y$ denotes the Hadamard product of two matrices x and y [22].

III. TWO QUTRITS

A similar construction may be performed in $\mathbb{C}^3 \otimes \mathbb{C}^3$. The basic idea is to decompose the total Hilbert space $\mathbb{C}^3 \otimes \mathbb{C}^3$ into a direct sum of three orthogonal subspaces Σ_i related by a certain cyclic property. In analogy to Eq. (13) let us define a shift operator $S: \mathbb{C}^3 \to \mathbb{C}^3$ via

$$S e_i = e_{i+1} \pmod 3. \tag{30}$$

It is clear that matrix elements S_{ij} define the following 3×3 circulant matrix:

$$S = \begin{pmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix}. \tag{31}$$

Now, let us define three orthogonal three-dimensional subspaces in $\mathbb{C}^3 \otimes \mathbb{C}^3$

$$\Sigma_0 = \text{span}\{e_0 \otimes e_0, e_1 \otimes e_1, e_2 \otimes e_2\} \tag{32}$$

and

$$\Sigma_1 = (\mathbb{1} \otimes S)\Sigma_0, \quad \Sigma_2 = (\mathbb{1} \otimes S^2)\Sigma_0. \tag{33}$$

One easily finds

$$\Sigma_1 = \text{span}\{e_0 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_0\},$$

$$\Sigma_2 = \text{span}\{e_0 \otimes e_2, e_1 \otimes e_0, e_2 \otimes e_1\}, \tag{34}$$

together with

$$\Sigma_0 \oplus \Sigma_1 \oplus \Sigma_2 = \mathbb{C}^3 \otimes \mathbb{C}^3.$$

The construction of a circulant state in $\mathbb{C}^3 \otimes \mathbb{C}^3$ goes as follows: define three positive operators ρ_i which are supported on Σ_i $(i=0,1,2)$:

$$\rho_0 = \sum_{i,j=0}^{2} a_{ij} e_{ij} \otimes e_{ij}, \tag{35}$$

$$\rho_1 = \sum_{i,j=0}^{2} b_{ij} e_{ij} \otimes S e_{ij} S^*,$$

$$= \sum_{i,j=0}^{2} b_{ij} e_{ij} \otimes e_{i+1,j+1}, \tag{36}$$

$$\rho_2 = \sum_{i,j=0}^{2} c_{ij} e_{ij} \otimes S^2 e_{ij} S^{*2},$$

$$= \sum_{i,j=0}^{2} c_{ij} e_{ij} \otimes e_{i+2,j+2}, \tag{37}$$

where a_{ij}, b_{ij}, and c_{ij} give rise to the following 3×3 matrices:

$$a = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}, \quad b = \begin{pmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{pmatrix},$$

$$c = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix}.$$

Positivity of ρ_i is guaranteed by positivity of a, b, and c. Finally, define a circulant $3 \otimes 3$ state by

$$\rho = \rho_0 + \rho_1 + \rho_2. \tag{38}$$

It is clear that

$$\rho_\alpha = P_\alpha \rho P_\alpha, \quad \alpha = 0,1,2, \tag{39}$$

where P_α denotes orthogonal projector onto Σ_α:

$$P_0 = \sum_{i=0}^{2} e_{ii} \otimes e_{ii} \tag{40}$$

and

$$P_\alpha = (\mathbb{1} \otimes S^\alpha) P_0 (\mathbb{1} \otimes S^\alpha)^*, \tag{41}$$

with

$$P_0 + P_1 + P_2 = \mathbb{I} \otimes \mathbb{I}. \tag{42}$$

Using definitions of ρ_i one easily finds

$$\rho = \left(\begin{array}{ccc|ccc|ccc} a_{00} & \cdot & \cdot & \cdot & a_{01} & \cdot & \cdot & \cdot & a_{02} \\ \cdot & b_{00} & \cdot & \cdot & \cdot & b_{01} & b_{02} & \cdot & \cdot \\ \cdot & \cdot & c_{00} & c_{01} & \cdot & \cdot & \cdot & c_{02} & \cdot \\ \hline \cdot & \cdot & c_{10} & c_{11} & \cdot & \cdot & \cdot & c_{12} & \cdot \\ a_{10} & \cdot & \cdot & \cdot & a_{11} & \cdot & \cdot & \cdot & a_{12} \\ \cdot & b_{10} & \cdot & \cdot & \cdot & b_{11} & b_{12} & \cdot & \cdot \\ \hline \cdot & b_{20} & \cdot & \cdot & \cdot & b_{21} & b_{22} & \cdot & \cdot \\ \cdot & \cdot & c_{20} & c_{21} & \cdot & \cdot & \cdot & c_{22} & \cdot \\ a_{20} & \cdot & \cdot & \cdot & a_{21} & \cdot & \cdot & \cdot & a_{22} \end{array}\right). \tag{43}$$

Normalization of ρ implies

$$\mathrm{Tr}(a+b+c)=1.$$

It turns out that Eq. (43) may be nicely rewritten in terms of S. Introducing the following diagonal matrices:

$$x_{ij}=\begin{pmatrix} a_{ij} & \cdot & \cdot \\ \cdot & b_{ij} & \cdot \\ \cdot & \cdot & c_{ij} \end{pmatrix}, \tag{44}$$

one may rewrite Eq. (43) in the following block form:

$$\rho=\left(\begin{array}{c|c|c} S^0x_{00}S^{*0} & S^0x_{01}S^{*1} & S^0x_{02}S^{*2} \\ \hline S^1x_{10}S^{*0} & S^1x_{11}S^{*1} & S^1x_{12}S^{*2} \\ \hline S^2x_{20}S^{*0} & S^2x_{21}S^{*1} & S^2x_{22}S^{*2} \end{array}\right). \tag{45}$$

Partially transposed ρ^τ has the following form:

$$\rho^\tau=\left(\begin{array}{ccc|ccc|ccc} \widetilde{a}_{00} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{01} & \cdot & \widetilde{a}_{02} & \cdot \\ \cdot & \widetilde{b}_{00} & \cdot & \widetilde{b}_{01} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{02} \\ \cdot & \cdot & \widetilde{c}_{00} & \cdot & \widetilde{c}_{01} & \cdot & \widetilde{c}_{02} & \cdot & \cdot \\ \hline \cdot & \widetilde{b}_{10} & \cdot & \widetilde{b}_{11} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{12} \\ \cdot & \cdot & \widetilde{c}_{10} & \cdot & \widetilde{c}_{11} & \cdot & \widetilde{c}_{12} & \cdot & \cdot \\ \widetilde{a}_{10} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{11} & \cdot & \widetilde{a}_{12} & \cdot \\ \hline \cdot & \cdot & \widetilde{c}_{20} & \cdot & \widetilde{c}_{21} & \cdot & \widetilde{c}_{22} & \cdot & \cdot \\ \widetilde{a}_{20} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{21} & \cdot & \widetilde{a}_{22} & \cdot \\ \cdot & \widetilde{b}_{20} & \cdot & \widetilde{b}_{21} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{22} \end{array}\right), \tag{46}$$

where the matrices $\widetilde{a}=[\widetilde{a}_{ij}]$, $\widetilde{b}=[\widetilde{b}_{ij}]$, and $\widetilde{c}=[\widetilde{c}_{ij}]$ read as

$$\widetilde{a}=\begin{pmatrix} a_{00} & c_{01} & b_{02} \\ c_{10} & b_{11} & a_{12} \\ b_{20} & a_{21} & c_{22} \end{pmatrix}, \quad \widetilde{b}=\begin{pmatrix} b_{00} & a_{01} & c_{02} \\ a_{10} & c_{11} & b_{12} \\ c_{20} & b_{21} & a_{22} \end{pmatrix},$$

$$\widetilde{c}=\begin{pmatrix} c_{00} & b_{01} & a_{02} \\ b_{10} & a_{11} & c_{12} \\ a_{20} & c_{21} & b_{22} \end{pmatrix}.$$

Note, that

$$\rho^\tau=\widetilde{\rho}_0+\widetilde{\rho}_1+\widetilde{\rho}_2, \tag{47}$$

where $\widetilde{\rho}_k$ are supported on three orthogonal subspaces of $\mathbb{C}^3\otimes\mathbb{C}^3$:

$$\widetilde{\Sigma}_0=\mathrm{span}\{e_0\otimes e_0,e_1\otimes e_2,e_2\otimes e_1\},$$

$$\widetilde{\Sigma}_1=\mathrm{span}\{e_0\otimes e_1,e_1\otimes e_0,e_2\otimes e_2\},$$

$$\widetilde{\Sigma}_2=\mathrm{span}\{e_0\otimes e_2,e_1\otimes e_1,e_2\otimes e_0\}. \tag{48}$$

Therefore, one has the following.

Theorem 1. A circulant $3\otimes 3$ state ρ is PPT iff the matrices $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ are positive. Note, that

$$\widetilde{\Sigma}_0=(\mathbb{1}\otimes\widetilde{\Pi})\Sigma_0,$$

where $\widetilde{\Pi}$ is the following permutation matrix:

$$\widetilde{\Pi}=\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix}. \tag{49}$$

Again, one has a cyclic structure

$$\widetilde{\Sigma}_i=(\mathbb{1}\otimes S)\widetilde{\Sigma}_0. \tag{50}$$

Moreover, it is clear that

$$\widetilde{\rho}_\alpha=\widetilde{P}_\alpha\rho^\tau\widetilde{P}_\alpha, \quad \alpha=0,1,2,$$

where $\widetilde{P}_\alpha$ denotes orthogonal projector onto $\widetilde{\Sigma}_\alpha$:

$$\widetilde{P}_0=(\mathbb{1}\otimes\Pi)P_0(\mathbb{1}\otimes\Pi^*) \tag{51}$$

and

$$\widetilde{P}_\alpha=(\mathbb{1}\otimes S^\alpha)\widetilde{P}_0(\mathbb{1}\otimes S^\alpha)^*, \tag{52}$$

with

$$\widetilde{P}_0+\widetilde{P}_1+\widetilde{P}_2=\mathbb{I}\otimes\mathbb{I}. \tag{53}$$

It is therefore clear that ρ^τ is again a circulant operator and its circular structure is governed by

$$\rho^\tau=\left(\begin{array}{c|c|c} S^0\widetilde{x}_{00}S^{*0} & S^0\widetilde{x}_{01}S^{*2} & S^0\widetilde{x}_{02}S^{*1} \\ \hline S^2\widetilde{x}_{10}S^{*0} & S^2\widetilde{x}_{11}S^{*2} & S^2\widetilde{x}_{12}S^{*1} \\ \hline S^1\widetilde{x}_{20}S^{*0} & S^1\widetilde{x}_{21}S^{*2} & S^1\widetilde{x}_{22}S^{*1} \end{array}\right), \tag{54}$$

where

$$\tilde{x}_{ij} = \begin{pmatrix} \tilde{a}_{ij} & \cdot & \cdot \\ \cdot & \tilde{b}_{ij} & \cdot \\ \cdot & \cdot & \tilde{c}_{ij} \end{pmatrix}. \tag{55}$$

Interestingly, matrices $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ may be nicely defined in terms of $\tilde{\Pi}$ and S. It is not difficult to show that

$$\tilde{a} = a \circ \tilde{\Pi} + b \circ (\tilde{\Pi} S) + c \circ (\tilde{\Pi} S^2),$$

$$\tilde{b} = b \circ \tilde{\Pi} + c \circ (\tilde{\Pi} S) + a \circ (\tilde{\Pi} S^2),$$

$$\tilde{c} = c \circ \tilde{\Pi} + a \circ (\tilde{\Pi} S) + b \circ (\tilde{\Pi} S^2), \tag{56}$$

where "$\circ$" denotes the Hadamard product.

Let us stress that this class in a significant way enlarges the class considered in Ref. [20]. One can reconstruct Ref. [20] by taking as b and c diagonal matrices:

$$b = \begin{pmatrix} b_{00} & \cdot & \cdot \\ \cdot & b_{11} & \cdot \\ \cdot & \cdot & b_{22} \end{pmatrix}, \quad c = \begin{pmatrix} c_{00} & \cdot & \cdot \\ \cdot & c_{11} & \cdot \\ \cdot & \cdot & c_{22} \end{pmatrix}.$$

Then one finds for the partially transposed matrix

$$\tilde{a} = \begin{pmatrix} a_{00} & \cdot & \cdot \\ \cdot & b_{11} & a_{12} \\ \cdot & a_{21} & c_{22} \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} b_{00} & a_{01} & \cdot \\ a_{10} & c_{11} & \cdot \\ \cdot & \cdot & a_{22} \end{pmatrix},$$

and

$$\tilde{c} = \begin{pmatrix} c_{00} & \cdot & a_{02} \\ \cdot & a_{11} & \cdot \\ a_{20} & \cdot & b_{22} \end{pmatrix}.$$

IV. GENERAL $d \otimes d$ CASE

Now we are ready to construct circular states in $d \otimes d$. The basic idea is to decompose the total Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^d$ into a direct sum of d orthogonal d-dimensional subspaces related by a certain cyclic property. It turns out that there are $(d-1)!$ different cyclic decompositions and it is therefore clear that they may be labeled by permutations from the symmetric group S_{d-1}. For $d=2$ one has only one decomposition

$$\Sigma_0 = \mathrm{span}\{e_0 \otimes e_0, e_1 \otimes e_1\},$$

$$\Sigma_1 = \mathrm{span}\{e_0 \otimes e_1, e_1 \otimes e_0\}, \tag{57}$$

whereas for $d=3$ we have found 2 different cyclic decompositions

$$\Sigma_0 = \mathrm{span}\{e_0 \otimes e_0, e_1 \otimes e_1, e_2 \otimes e_2\},$$

$$\Sigma_1 = \mathrm{span}\{e_0 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_0\},$$

$$\Sigma_2 = \mathrm{span}\{e_0 \otimes e_2, e_1 \otimes e_0, e_2 \otimes e_1\}, \tag{58}$$

and

$$\tilde{\Sigma}_0 = \mathrm{span}\{e_0 \otimes e_0, e_1 \otimes e_2, e_2 \otimes e_1\},$$

$$\tilde{\Sigma}_1 = \mathrm{span}\{e_0 \otimes e_1, e_1 \otimes e_0, e_2 \otimes e_2\},$$

$$\tilde{\Sigma}_2 = \mathrm{span}\{e_0 \otimes e_2, e_1 \otimes e_1, e_2 \otimes e_0\}. \tag{59}$$

Let us introduce a basic d-dimensional subspace

$$\Sigma_0 = \mathrm{span}\{e_0 \otimes e_0, e_1 \otimes e_1, \ldots, e_{d-1} \otimes e_{d-1}\}. \tag{60}$$

Now, for any permutation $\pi \in S$ let us define Σ_0^π which is spanned by

$$e_0 \otimes e_{\pi(0)}, e_1 \otimes e_{\pi(1)}, \ldots, e_{d-1} \otimes e_{\pi(d-1)}. \tag{61}$$

Note that introducing a permutation matrix Π corresponding to π one has

$$\Sigma_0^\pi = (\mathbb{1} \otimes \Pi)\Sigma_0. \tag{62}$$

Actually, it is enough to consider only a subset of permutations such that $\pi(0)=0$, it means that vector $e_0 \otimes e_0$ always belongs to the subspace number "0" in each decomposition. Finally, the remaining $(d-1)$ subspaces in the decomposition labeled by π are defined via

$$\Sigma_\alpha^\pi = (\mathbb{1} \otimes S^\alpha)\Sigma_0^\pi,$$
$$= (\mathbb{1} \otimes S^\alpha \Pi)\Sigma_0, \tag{63}$$

where S is a circulant matrix corresponding to shift in $\mathbb{C}^d$:

$$S = \begin{pmatrix} \cdot & \cdot & \ldots & \cdot & 1 \\ 1 & \cdot & \ldots & \cdot & \cdot \\ \cdot & 1 & \ldots & \cdot & \cdot \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdot & \cdot & \ldots & 1 & \cdot \end{pmatrix}. \tag{64}$$

One easily check

$$\Sigma_0^\pi \oplus \Sigma_1^\pi \oplus \cdots \oplus \Sigma_{d-1}^\pi = \mathbb{C}^d \otimes \mathbb{C}^d.$$

To construct a circulant state corresponding to this decomposition let us introduce d positive $d \times d$ matrices $a^{(\alpha)} = [a_{ij}^{(\alpha)}]$; $\alpha = 0, 1, \ldots, d-1$. Now, define d positive operators ρ_α^π supported on Σ_α^π via

$$\rho_\alpha^\pi = \sum_{i,j=0}^{d-1} a_{ij}^{(\alpha)} e_{ij} \otimes S^\alpha e_{\pi(i),\pi(j)} S^{*\alpha} = \sum_{i,j=0}^{d-1} a_{ij}^{(\alpha)} e_{ij} \otimes e_{\pi(i)+\alpha,\pi(j)+\alpha}. \tag{65}$$

Finally,

$$\rho_\pi = \rho_0^\pi + \rho_1^\pi + \cdots + \rho_{d-1}^\pi, \tag{66}$$

defines circulant state (corresponding to π). Normalization

of ρ_π is equivalent to the following condition for matrices $a^{(\alpha)}$:

$$\mathrm{Tr}(a^{(0)} + a^{(1)} + \cdots + a^{(d-1)}) = 1.$$

Interestingly, ρ_π has the following transparent block form: introduce a set of d^2 diagonal matrices

$$x_{ij} = \begin{pmatrix} a_{ij}^{(0)} & \cdot & \cdots & \cdot \\ \cdot & a_{ij}^{(1)} & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdots & a_{ij}^{(d-1)} \end{pmatrix}, \tag{67}$$

then one finds

$$\rho_\pi = \left(\begin{array}{c|c|c|c} S^{\pi(0)}x_{00}S^{\pi(0)*} & S^{\pi(0)}x_{01}S^{\pi(1)*} & \cdots & S^{\pi(0)}x_{0,d-1}S^{\pi(d-1)*} \\ \hline S^{\pi(1)}x_{10}S^{\pi(0)*} & S^{\pi(1)}x_{11}S^{\pi(1)*} & \cdots & S^{\pi(1)}x_{1,d-1}S^{\pi(d-1)*} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline S^{\pi(d-1)}x_{d-1,0}S^{\pi(0)*} & S^{\pi(d-1)}x_{d-1,1}S^{\pi(1)*} & \cdots & S^{\pi(d-1)}x_{d-1,d-1}S^{\pi(d-1)*} \end{array}\right). \tag{68}$$

Having defined a circulant state ρ_π let us look for a partially transposed matrix ρ_π^τ. Now comes the crucial observation.

Theorem 2. If ρ_π is a circulant state corresponding to permutation π such that $\pi(0)=0$, then its partial transposition ρ_π^τ is also circulant with respect to another decomposition corresponding to permutation $\tilde{\pi}$ such that

$$\pi(i) + \tilde{\pi}(i) = d, \tag{69}$$

for $i=1,2,\ldots,d-1$, and $\tilde{\pi}(0)=0$.

Note that for $d=2$ there is only one (trivial) permutation $[\pi(0)=0,\pi(1)=1]$ and hence $\tilde{\pi}=\pi$, that is both π and $\tilde{\pi}$ define the same decomposition (57). For $d=3$ one has two different permutations in S_2: the trivial one $[\pi(0)=0,\pi(1)=1, \pi(2)=2]$ which corresponds to Eq. (58) and "true" permutation $[\tilde{\pi}(0)=0,\tilde{\pi}(1)=2,\tilde{\pi}(2)=1]$ which corresponds to Eq. (59).

Hence, ρ_π^τ may be decomposed as follows:

$$\rho_\pi^\tau = \tilde{\rho}_0^\pi + \tilde{\rho}_1^\pi + \cdots + \tilde{\rho}_{d-1}^\pi, \tag{70}$$

and $\tilde{\rho}_\alpha^\pi$ are defined by

$$\tilde{\rho}_\alpha^\pi = \sum_{i,j=0}^{d-1} \tilde{a}_{ij}^{(\alpha)} e_{ij} \otimes S^\alpha e_{\tilde{\pi}(i),\tilde{\pi}(j)} S^{*\alpha} = \sum_{i,j=0}^{d-1} \tilde{a}_{ij}^{(\alpha)} e_{ij} \otimes e_{\tilde{\pi}(i)+\alpha,\tilde{\pi}(j)+\alpha}, \tag{71}$$

where again we trivially extended $\tilde{\pi}$ from S_{d-1} to S_d by $\tilde{\pi}(0)\equiv 0$.

In analogy to Eq. (68) one finds the following block form of ρ_π^τ:

$$\rho_\pi^\tau = \left(\begin{array}{c|c|c|c} S^{\tilde{\pi}(0)}x_{00}S^{\tilde{\pi}(0)*} & S^{\tilde{\pi}(0)}x_{01}S^{\tilde{\pi}(1)*} & \cdots & S^{\tilde{\pi}(0)}x_{0,d-1}S_\pi^{\tilde{\pi}(d-1)*} \\ \hline S^{\tilde{\pi}(1)}x_{10}S^{\tilde{\pi}(0)*} & S^{\tilde{\pi}(1)}x_{11}S^{\tilde{\pi}(1)*} & \cdots & S^{\tilde{\pi}(1)}x_{1,d-1}S^{\tilde{\pi}(d-1)*} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline S^{\tilde{\pi}(d-1)}x_{d-1,0}S^{\tilde{\pi}(0)*} & S^{\tilde{\pi}(d-1)}x_{d-1,1}S^{\tilde{\pi}(1)*} & \cdots & S^{\tilde{\pi}(d-1)}x_{d-1,d-1}S^{\tilde{\pi}(d-1)*} \end{array}\right), \tag{72}$$

where

$$\tilde{x}_{ij} = \begin{pmatrix} \tilde{a}_{ij}^{(0)} & \cdot & \cdots & \cdot \\ \cdot & \tilde{a}_{ij}^{(1)} & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdots & \tilde{a}_{ij}^{(d-1)} \end{pmatrix}. \tag{73}$$

Hence a partial transposition applied to a circulant state ρ_π reduces to (1) introducing "complementary" permutation $\tilde{\pi}$ and (2) defining a new set of $d\times d$ matrices $\tilde{a}^{(\alpha)}=[\tilde{a}_{ij}^{(\alpha)}]$. Now, "complementary" permutation $\tilde{\pi}$ is fully characterized by Eq. (69). Finally, one finds the following intricate formula for $\tilde{a}^{(\alpha)}$:

$$\tilde{a}^{(\alpha)} = \sum_{\beta=0}^{d-1} a^{(\alpha+\beta)} \circ (\tilde{\Pi} S_\pi^\beta) \pmod d, \tag{74}$$

where "$\circ$" denotes the Hadamard product and

$$S_\pi = \Pi^* S \Pi. \tag{75}$$

Therefore, we arrive at our main result.

Theorem 3. A circulant state ρ_π is PPT iff the matrices $\tilde{a}^{(\alpha)}$ defined in Eq. (74) are positive.

V. EXAMPLES

A. PPT class from Ref. [20]

One reconstructs a class of PPT states from Ref. [20] taking the circular decomposition corresponding to trivial permutation with arbitrary (but positive) $a^{(0)}$ and positive diagonal $a^{(k)}$ $(k=1,\ldots,d-1)$. Note, however, that there are new classes defined by the same matrices $a^{(\alpha)}$ but corresponding to different permutations, that is, apart from the state defined by

$$\rho = \sum_{i,j=0}^{d-1} a_{ij}^{(0)} e_{ij} \otimes e_{ij} + \sum_{k=1}^{d-1} \sum_{i=0}^{d-1} a_{ii}^{(k)} e_{ii} \otimes e_{i+k,i+k}, \tag{76}$$

one has its π partner

$$\rho_\pi = (\mathbb{1} \otimes \Pi)\rho(\mathbb{1} \otimes \Pi^*) = \sum_{i,j=0}^{d-1} a_{ij}^{(0)} e_{ij} \otimes e_{\pi(i),\pi(j)} + \sum_{k=1}^{d-1} \sum_{i=0}^{d-1} a_{ii}^{(k)} e_{ii} \otimes e_{\pi(i)+k,\pi(i)+k}. \tag{77}$$

It is, therefore, clear that all examples discussed in Ref. [20] (together with the corresponding "π partners") belong to our new class.

B. π isotropic state

The standard isotropic state [23] in $d \otimes d$

$$\mathcal{I} = \frac{1-\lambda}{d^2} \mathbb{I} \otimes \mathbb{I} + \frac{\lambda}{d} \sum_{i,j=0}^{d-1} e_{ij} \otimes e_{ij}, \tag{78}$$

corresponds to trivial permutation and it is defined by the following set of $d \times d$ positive matrices:

$$a_{ij}^{(0)} = \begin{cases} \lambda/d, & i \neq j, \\ \lambda/d + (1-\lambda)/d^2, & i = j \end{cases} \tag{79}$$

and diagonal

$$a_{ij}^{(k)} = \begin{cases} 0, & i \neq j, \\ \lambda/d + (1-\lambda)/d^2, & i = j \end{cases} \tag{80}$$

for $k=1,\ldots,d-1$. Again, for each permutation π we may define π isotropic state

$$\mathcal{I} = (\mathbb{1} \otimes \Pi)\mathcal{I}(\mathbb{1} \otimes \Pi^*) = \frac{1-\lambda}{d^2} \mathbb{I} \otimes \mathbb{I} + \frac{\lambda}{d} \sum_{i,j=0}^{d-1} e_{ij} \otimes e_{\pi(i),\pi(j)}, \tag{81}$$

which is defined by the same set of matrices $a^{(\alpha)}$ but corresponds to π decomposition.

C. π Werner state

The celebrated Werner state [24] is defined by the following well known formula:

$$\mathcal{W} = (1-p)Q^+ + pQ^-, \tag{82}$$

where

$$Q^\pm = \frac{1}{d(d \pm 1)} (\mathbb{I} \otimes \mathbb{I} \pm \mathbb{F}), \tag{83}$$

and $\mathbb{F}$ denotes a flip operator defined by

$$\mathbb{F} = \sum_{i,j=0}^{d-1} e_{ij} \otimes e_{ji}.$$

It is clear that $\mathcal{W}$ belongs to a class of bipartite operators obtained from the class of isotropic states by applying a partial transposition. One easily finds

$$\tilde{a}_{ij}^{(0)} = \begin{cases} x_-, & i \neq j, \\ x_- + x_+, & i = j \end{cases} \tag{84}$$

and

$$\tilde{a}^{(k)} = x_+ \mathbb{I}, \quad k = 1, \ldots, d-1, \tag{85}$$

where

$$x_\pm = \frac{1-p}{d^2+d} \pm \frac{p}{d^2-d}. \tag{86}$$

It is clear that for any permutation π one may define π Werner state

$$\mathcal{W}_\pi = (\mathbb{1} \otimes \Pi)\mathcal{W}(\mathbb{1} \otimes \Pi^*) = (1-p)Q_\pi^+ + pQ_\pi^-, \tag{87}$$

where

$$Q_\pi^\pm = \frac{1}{d(d \pm 1)} (\mathbb{I} \otimes \mathbb{I} \pm \mathbb{F}_\pi), \tag{88}$$

and $\mathbb{F}_\pi$ denotes a "π flip operator" defined by

$$\mathbb{F}_\pi = (\mathbb{1} \otimes \Pi)\mathbb{F}(\mathbb{1} \otimes \Pi^*) = \sum_{i,j=0}^{d-1} e_{ij} \otimes e_{\pi(j),\pi(j)}.$$

D. Ha example in $4 \otimes 4$

Ha [16] constructed a $4 \otimes 4$ PPT state which was used to check that the seminal Robertson positive map $\Lambda: M_4(\mathbb{C}) \to M_4(\mathbb{C})$ [25] is indecomposable. Ha's state belongs to our class labeled by a trivial permutation π_1 (see the Appendix) with four positive matrices defined as follows:

$$a = \begin{pmatrix} 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad b = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$c = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad d = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

The partially transposed state defines a circulant operator corresponding to decomposition labeled by $\tilde{\pi}_1$ and defined by

$$\tilde{a}=\begin{pmatrix}1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad \tilde{b}=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot\\ \cdot&1&1&\cdot\\ \cdot&1&1&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{pmatrix},$$

$$\tilde{c}=\begin{pmatrix}1&\cdot&-1&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ -1&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad \tilde{d}=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{pmatrix}.$$

Evidently $\tilde{a},\tilde{b},\tilde{c},\tilde{d}\geqslant 0$. Interestingly, as was shown by Ha [16] both ρ and ρ^{τ} are of Schmidt rank two (it proves that Robertson map is not only indecomposable but even atomic, i.e., it cannot be written as a sum of two-positive and two-copositive maps). We stress that this example does not belong to the previous class defined in Ref. [20].

E. Fei *et al.* bound entangled state in $4\otimes 4$

Fei *et al.* [27] constructed $4\otimes 4$ bound entangled state which correspond to $\tilde{\pi}_1$ decomposition (see the Appendix). It is defined by the following set of $\tilde{a},\tilde{b},\tilde{c},\tilde{d}$:

$$\begin{pmatrix}x_1&\cdot&\cdot&\cdot\\ \cdot&x_5&\cdot&-x_5\\ \cdot&\cdot&x_1&\cdot\\ \cdot&-x_5&\cdot&x_5\end{pmatrix},\quad \begin{pmatrix}x_3&-x_3&\cdot&\cdot\\ -x_3&x_3&\cdot&\cdot\\ \cdot&\cdot&x_4&-x_4\\ \cdot&\cdot&-x_4&x_4\end{pmatrix},$$

$$\begin{pmatrix}x_2&\cdot&-x_2&\cdot\\ \cdot&x_1&\cdot&\cdot\\ -x_2&\cdot&x_2&\cdot\\ \cdot&\cdot&\cdot&x_1\end{pmatrix},\quad \begin{pmatrix}\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{pmatrix}.$$

Evidently $\tilde{a},\tilde{b},\tilde{c},\tilde{d}\geqslant 0$, for $x_i\geqslant 0$. Now, the partially transposed state is circular with respect to π_1 decomposition (see the Appendix) and it is defined by the following set of 4×4 matrices a,b,c,d:

$$\begin{pmatrix}x_1&-x_3&-x_2&\cdot\\ -x_3&x_1&\cdot&-x_5\\ -x_2&\cdot&x_1&-x_4\\ \cdot&-x_5&-x_4&x_1\end{pmatrix},\quad \begin{pmatrix}x_3&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&x_4&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{pmatrix},$$

$$\begin{pmatrix}x_2&\cdot&\cdot&\cdot\\ \cdot&x_5&\cdot&\cdot\\ \cdot&\cdot&x_2&\cdot\\ \cdot&\cdot&\cdot&x_5\end{pmatrix},\quad \begin{pmatrix}\cdot&\cdot&\cdot&\cdot\\ \cdot&x_3&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&x_4\end{pmatrix}.$$

It is clear that in general a is not a positive matrix. However, for $x_1=(1-\varepsilon)/4$ and $x_2=x_3=x_4=x_5=\varepsilon/8$ it has three different eigenvalues

$$1/4,\quad (1-2\varepsilon)/4,\quad (1-\varepsilon)/4,$$

and hence ρ is PPT for $0\leqslant\varepsilon\leqslant 1/2$. It was shown [27] that ρ being PPT is entangled.

VI. CONCLUSIONS

We have constructed a large class of PPT states in $d\otimes d$ which correspond to circular decompositions of $\mathbb{C}^d\otimes\mathbb{C}^d$ into direct sums of d-dimensional subspaces. This class significantly enlarges the previous class defined in Ref. [20]. It contains several known examples from the literature and produces a highly nontrivial family of new states.

There are many open problems: the basic question is how to detect entanglement within this class of PPT states. One may expect that there is a special class of entanglement witnesses which are sensitive to entanglement encoded into circular decompositions. The related mathematical problem is the construction of linear indecomposable positive maps Λ: $M_d(\mathbb{C})\rightarrow M_d(\mathbb{C})$ satisfying

$$(\mathbb{1}\otimes\Lambda)\rho\not\geqslant 0, \tag{89}$$

for some circulant PPT state ρ. A corresponding class of such maps correlated with the previous class of PPT states [20] was recently proposed in Ref. [26]. It would be interesting to establish a structure of edge states [28,29] within circulant PPT states since the knowledge of edge states is sufficient to characterize all PPT states. Finally, it is interesting to explore the possibility of other decompositions leading to new classes of PPT states. We stress that the seminal Horodecki $3\otimes 3$ entangled PPT state [4] does not belong to our class. In a forthcoming paper we show that this state belongs to a new class of PPT states which is governed by another type of decompositions of $\mathbb{C}^d\otimes\mathbb{C}^d$.

ACKNOWLEDGMENT

This work was partially supported by the Polish State Committee for Scientific Research.

APPENDIX

For $d=4$ one has six different decompositions of $\mathbb{C}^4\otimes\mathbb{C}^4$ into the direct sum of four four-dimensional subspaces. These are labeled by permutations from the symmetric group S_3. One finds

$$(\pi_1(0)=0,\pi_1(1)=1,\pi_1(2)=2,\pi_1(3)=3),$$

$$(\tilde{\pi}_1(0)=0,\tilde{\pi}_1(1)=3,\tilde{\pi}_1(2)=2,\tilde{\pi}_1(3)=1),$$

$$(\pi_2(0)=0,\pi_2(1)=2,\pi_2(2)=3,\pi_3(3)=1),$$

$$(\tilde{\pi}_2(0)=0,\tilde{\pi}_2(1)=2,\tilde{\pi}_2(2)=1,\tilde{\pi}_2(3)=3),$$

$$(\pi_3(0)=0,\pi_3(1)=3,\pi_3(2)=1,\pi_3(3)=2),$$

$$(\tilde{\pi}_3(0)=0,\tilde{\pi}_3(1)=1,\tilde{\pi}_3(2)=3,\tilde{\pi}_3(3)=2).$$

The corresponding permutation matrices read as

$$\Pi_1 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad \tilde{\Pi}_1 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix}, \quad \text{(A1)}$$

$$\Pi_2 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}, \quad \tilde{\Pi}_2 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}, \quad \text{(A2)}$$

$$\Pi_3 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \end{pmatrix}, \quad \tilde{\Pi}_3 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}. \quad \text{(A3)}$$

Moreover, one finds for (a) $\tilde{\Pi}_1 S_{\pi_1}$, $\tilde{\Pi}_1 S_{\pi_1}^2$, and $\tilde{\Pi}_1 S_{\pi_1}^3$:

$$\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}, \quad \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix},$$

(b) $\tilde{\Pi}_2 S_{\pi_2}$, $\tilde{\Pi}_2 S_{\pi_2}^2$, and $\tilde{\Pi}_2 S_{\pi_2}^3$:

$$\begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix}, \quad \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix},$$

(c) and for $\tilde{\Pi}_3 S_{\pi_3}$, $\tilde{\Pi}_3 S_{\pi_3}^2$, and $\tilde{\Pi}_3 S_{\pi_3}^3$:

$$\begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}, \quad \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}, \quad \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix},$$

respectively.

1. π_1 and $\tilde{\pi}_1$ circulant states

$$\Sigma_0^{\pi_1} = \mathrm{span}\{e_0 \otimes e_0, e_1 \otimes e_1, e_2 \otimes e_2, e_3 \otimes e_3\},$$

$$\Sigma_1^{\pi_1} = \mathrm{span}\{e_0 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_3, e_3 \otimes e_0\},$$

$$\Sigma_2^{\pi_1} = \mathrm{span}\{e_0 \otimes e_2, e_1 \otimes e_3, e_2 \otimes e_1, e_3 \otimes e_2\},$$

$$\Sigma_3^{\pi_1} = \mathrm{span}\{e_0 \otimes e_3, e_1 \otimes e_0, e_2 \otimes e_1, e_3 \otimes e_2\},$$

$$\tilde{\Sigma}_0^{\pi_1} = \mathrm{span}\{e_0 \otimes e_0, e_1 \otimes e_3, e_2 \otimes e_2, e_3 \otimes e_1\},$$

$$\tilde{\Sigma}_1^{\pi_1} = \mathrm{span}\{e_0 \otimes e_1, e_1 \otimes e_0, e_2 \otimes e_3, e_3 \otimes e_2\},$$

$$\tilde{\Sigma}_2^{\pi_1} = \mathrm{span}\{e_0 \otimes e_2, e_1 \otimes e_1, e_2 \otimes e_0, e_3 \otimes e_3\},$$

$$\tilde{\Sigma}_3^{\pi_1} = \mathrm{span}\{e_0 \otimes e_3, e_1 \otimes e_2, e_2 \otimes e_1, e_3 \otimes e_0\}.$$

$$\rho_{\pi_1} = \left(\begin{array}{cccc|cccc|cccc|cccc} a_{00} & \cdot & \cdot & \cdot & \cdot & a_{01} & \cdot & \cdot & \cdot & \cdot & a_{02} & \cdot & \cdot & \cdot & \cdot & a_{03} \\ \cdot & b_{00} & \cdot & \cdot & \cdot & \cdot & b_{01} & \cdot & \cdot & \cdot & \cdot & b_{02} & b_{03} & \cdot & \cdot & \cdot \\ \cdot & \cdot & c_{00} & \cdot & \cdot & \cdot & \cdot & c_{01} & c_{02} & \cdot & \cdot & \cdot & \cdot & c_{03} & \cdot & \cdot \\ \cdot & \cdot & \cdot & d_{00} & d_{01} & \cdot & \cdot & \cdot & \cdot & d_{02} & \cdot & \cdot & \cdot & \cdot & d_{03} & \cdot \\ \hline \cdot & \cdot & \cdot & d_{10} & d_{11} & \cdot & \cdot & \cdot & \cdot & d_{12} & \cdot & \cdot & \cdot & \cdot & d_{13} & \cdot \\ a_{10} & \cdot & \cdot & \cdot & \cdot & a_{11} & \cdot & \cdot & \cdot & \cdot & a_{12} & \cdot & \cdot & \cdot & \cdot & a_{13} \\ \cdot & b_{10} & \cdot & \cdot & \cdot & \cdot & b_{11} & \cdot & \cdot & \cdot & \cdot & b_{12} & b_{13} & \cdot & \cdot & \cdot \\ \cdot & \cdot & c_{10} & \cdot & \cdot & \cdot & \cdot & c_{11} & c_{12} & \cdot & \cdot & \cdot & \cdot & c_{13} & \cdot & \cdot \\ \hline \cdot & \cdot & c_{20} & \cdot & \cdot & \cdot & \cdot & c_{21} & c_{22} & \cdot & \cdot & \cdot & \cdot & c_{23} & \cdot & \cdot \\ \cdot & \cdot & \cdot & d_{20} & d_{21} & \cdot & \cdot & \cdot & \cdot & d_{22} & \cdot & \cdot & \cdot & \cdot & d_{23} & \cdot \\ a_{20} & \cdot & \cdot & \cdot & \cdot & a_{21} & \cdot & \cdot & \cdot & \cdot & a_{22} & \cdot & \cdot & \cdot & \cdot & a_{23} \\ \cdot & b_{20} & \cdot & \cdot & \cdot & \cdot & b_{21} & \cdot & \cdot & \cdot & \cdot & b_{22} & b_{23} & \cdot & \cdot & \cdot \\ \hline \cdot & b_{30} & \cdot & \cdot & \cdot & \cdot & b_{31} & \cdot & \cdot & \cdot & \cdot & b_{32} & b_{33} & \cdot & \cdot & \cdot \\ \cdot & \cdot & c_{30} & \cdot & \cdot & \cdot & \cdot & c_{31} & c_{32} & \cdot & \cdot & \cdot & \cdot & c_{33} & \cdot & \cdot \\ \cdot & \cdot & \cdot & d_{30} & d_{31} & \cdot & \cdot & \cdot & \cdot & d_{32} & \cdot & \cdot & \cdot & \cdot & d_{33} & \cdot \\ a_{30} & \cdot & \cdot & \cdot & \cdot & a_{31} & \cdot & \cdot & \cdot & \cdot & a_{32} & \cdot & \cdot & \cdot & \cdot & a_{33} \end{array}\right). \quad \text{(A4)}$$

$$\rho_{\pi_1}^{\tau} = \left(\begin{array}{cccc|cccc|cccc|cccc}
\widetilde{a}_{00} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{01} & \cdot & \cdot & \widetilde{a}_{02} & \cdot & \cdot & \widetilde{a}_{03} & \cdot & \cdot \\
\cdot & \widetilde{b}_{00} & \cdot & \cdot & \widetilde{b}_{01} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{02} & \cdot & \cdot & \widetilde{b}_{03} & \cdot \\
\cdot & \cdot & \widetilde{c}_{00} & \cdot & \cdot & \widetilde{c}_{01} & \cdot & \cdot & \widetilde{c}_{02} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{c}_{03} \\
\cdot & \cdot & \cdot & \widetilde{d}_{00} & \cdot & \cdot & \widetilde{d}_{01} & \cdot & \cdot & \widetilde{d}_{02} & \cdot & \cdot & \widetilde{d}_{03} & \cdot & \cdot & \cdot \\
\hline
\cdot & \widetilde{b}_{10} & \cdot & \cdot & \widetilde{b}_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{12} & \cdot & \cdot & \widetilde{b}_{13} & \cdot \\
\cdot & \cdot & \widetilde{c}_{10} & \cdot & \cdot & \widetilde{c}_{11} & \cdot & \cdot & \widetilde{c}_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{c}_{13} \\
\cdot & \cdot & \cdot & \widetilde{d}_{10} & \cdot & \cdot & \widetilde{d}_{11} & \cdot & \cdot & \widetilde{d}_{12} & \cdot & \cdot & \widetilde{d}_{13} & \cdot & \cdot & \cdot \\
\widetilde{a}_{10} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{11} & \cdot & \cdot & \widetilde{a}_{12} & \cdot & \cdot & \widetilde{a}_{13} & \cdot & \cdot \\
\hline
\cdot & \cdot & \widetilde{c}_{20} & \cdot & \cdot & \widetilde{c}_{21} & \cdot & \cdot & \widetilde{c}_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{c}_{23} \\
\cdot & \cdot & \cdot & \widetilde{d}_{20} & \cdot & \cdot & \widetilde{d}_{21} & \cdot & \cdot & \widetilde{d}_{22} & \cdot & \cdot & \widetilde{d}_{23} & \cdot & \cdot & \cdot \\
\widetilde{a}_{20} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{21} & \cdot & \cdot & \widetilde{a}_{22} & \cdot & \cdot & \widetilde{a}_{23} & \cdot & \cdot \\
\cdot & \widetilde{b}_{20} & \cdot & \cdot & \widetilde{b}_{21} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{22} & \cdot & \cdot & \widetilde{b}_{23} & \cdot \\
\hline
\cdot & \cdot & \cdot & \widetilde{d}_{30} & \cdot & \cdot & \widetilde{d}_{31} & \cdot & \cdot & \widetilde{d}_{32} & \cdot & \cdot & \widetilde{d}_{33} & \cdot & \cdot & \cdot \\
\widetilde{a}_{30} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{31} & \cdot & \cdot & \widetilde{a}_{32} & \cdot & \cdot & \widetilde{a}_{33} & \cdot & \cdot \\
\cdot & \widetilde{b}_{30} & \cdot & \cdot & \widetilde{b}_{31} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{32} & \cdot & \cdot & \widetilde{b}_{33} & \cdot \\
\cdot & \cdot & \widetilde{c}_{30} & \cdot & \cdot & \widetilde{c}_{31} & \cdot & \cdot & \widetilde{c}_{32} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{c}_{33}
\end{array}\right), \tag{A5}$$

where the matrices $\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d}$ are given by

$$\widetilde{a} = \begin{pmatrix} a_{00} & d_{01} & c_{02} & b_{03} \\ d_{10} & c_{11} & b_{12} & a_{13} \\ c_{20} & b_{21} & a_{22} & d_{23} \\ b_{30} & a_{31} & d_{32} & c_{33} \end{pmatrix}, \quad \widetilde{b} = \begin{pmatrix} b_{00} & a_{01} & d_{02} & c_{03} \\ a_{10} & d_{11} & c_{12} & b_{13} \\ d_{20} & c_{21} & b_{22} & a_{23} \\ c_{30} & b_{31} & a_{32} & d_{33} \end{pmatrix}, \quad \widetilde{c} = \begin{pmatrix} c_{00} & b_{01} & a_{02} & d_{03} \\ b_{10} & a_{11} & d_{12} & c_{13} \\ a_{20} & d_{21} & c_{22} & b_{23} \\ d_{30} & c_{31} & b_{32} & a_{33} \end{pmatrix}, \quad \widetilde{d} = \begin{pmatrix} d_{00} & c_{01} & b_{02} & a_{03} \\ c_{10} & b_{11} & a_{12} & d_{13} \\ b_{20} & a_{21} & d_{22} & c_{23} \\ a_{30} & d_{31} & c_{32} & b_{33} \end{pmatrix}.$$

2. π_2 and $\widetilde{\pi}_2$ circulant states

$$\Sigma_0^{\pi_2} = \mathrm{span}\{e_0 \otimes e_0, e_1 \otimes e_2, e_2 \otimes e_3, e_3 \otimes e_1\},$$

$$\Sigma_1^{\pi_2} = \mathrm{span}\{e_0 \otimes e_1, e_1 \otimes e_3, e_2 \otimes e_0, e_3 \otimes e_2\},$$

$$\Sigma_2^{\pi_2} = \mathrm{span}\{e_0 \otimes e_2, e_1 \otimes e_0, e_2 \otimes e_1, e_3 \otimes e_3\},$$

$$\Sigma_3^{\pi_2} = \mathrm{span}\{e_0 \otimes e_3, e_1 \otimes e_1, e_2 \otimes e_2, e_3 \otimes e_0\},$$

$$\widetilde{\Sigma}_0^{\pi_2} = \mathrm{span}\{e_0 \otimes e_0, e_1 \otimes e_2, e_2 \otimes e_1, e_3 \otimes e_3\},$$

$$\widetilde{\Sigma}_1^{\pi_2} = \mathrm{span}\{e_0 \otimes e_1, e_1 \otimes e_3, e_2 \otimes e_2, e_3 \otimes e_0\},$$

$$\widetilde{\Sigma}_2^{\pi_2} = \mathrm{span}\{e_0 \otimes e_2, e_1 \otimes e_0, e_2 \otimes e_3, e_3 \otimes e_1\},$$

$$\widetilde{\Sigma}_3^{\pi_2} = \mathrm{span}\{e_0 \otimes e_3, e_1 \otimes e_1, e_2 \otimes e_0, e_3 \otimes e_2\}.$$

$$\rho_{\pi_2}=\left(\begin{array}{cccc|cccc|cccc|cccc}
a_{00} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{01} & \cdot & \cdot & \cdot & \cdot & a_{02} & \cdot & a_{03} & \cdot & \cdot \\
\cdot & b_{00} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{01} & b_{02} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{03} & \cdot \\
\cdot & \cdot & c_{00} & \cdot & c_{01} & \cdot & \cdot & \cdot & \cdot & c_{02} & \cdot & \cdot & \cdot & \cdot & \cdot & c_{03} \\
\cdot & \cdot & \cdot & d_{00} & \cdot & d_{01} & \cdot & \cdot & \cdot & \cdot & d_{02} & \cdot & d_{03} & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & c_{10} & \cdot & c_{11} & \cdot & \cdot & \cdot & \cdot & c_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & c_{13} \\
\cdot & \cdot & \cdot & d_{10} & \cdot & d_{11} & \cdot & \cdot & \cdot & \cdot & d_{12} & \cdot & d_{13} & \cdot & \cdot & \cdot \\
a_{10} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{11} & \cdot & \cdot & \cdot & \cdot & a_{12} & \cdot & a_{13} & \cdot & \cdot \\
\cdot & b_{10} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{11} & b_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{13} & \cdot \\
\hline
\cdot & b_{20} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{21} & b_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{23} & \cdot \\
\cdot & \cdot & c_{20} & \cdot & c_{21} & \cdot & \cdot & \cdot & \cdot & c_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & c_{23} \\
\cdot & \cdot & \cdot & d_{20} & \cdot & d_{21} & \cdot & \cdot & \cdot & \cdot & d_{22} & \cdot & d_{23} & \cdot & \cdot & \cdot \\
a_{20} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{21} & \cdot & \cdot & \cdot & \cdot & a_{22} & \cdot & a_{23} & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & d_{30} & \cdot & d_{31} & \cdot & \cdot & \cdot & \cdot & d_{32} & \cdot & d_{33} & \cdot & \cdot & \cdot \\
a_{30} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{31} & \cdot & \cdot & \cdot & \cdot & a_{32} & \cdot & a_{33} & \cdot & \cdot \\
\cdot & b_{30} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{31} & b_{32} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{33} & \cdot \\
\cdot & \cdot & c_{30} & \cdot & c_{31} & \cdot & \cdot & \cdot & \cdot & c_{32} & \cdot & \cdot & \cdot & \cdot & \cdot & c_{33}
\end{array}\right). \tag{A6}$$

$$\rho_{\pi_2}^{\tau}=\left(\begin{array}{cccc|cccc|cccc|cccc}
\tilde{a}_{00} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{a}_{01} & \cdot & \cdot & \tilde{a}_{02} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{a}_{03} \\
\cdot & \tilde{b}_{00} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{b}_{01} & \cdot & \cdot & \tilde{b}_{02} & \cdot & \tilde{b}_{03} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \tilde{c}_{00} & \cdot & \tilde{c}_{01} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{c}_{02} & \cdot & \tilde{c}_{03} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \tilde{d}_{00} & \cdot & \tilde{d}_{01} & \cdot & \cdot & \tilde{d}_{02} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{d}_{03} & \cdot \\
\hline
\cdot & \cdot & \tilde{c}_{10} & \cdot & \tilde{c}_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{c}_{12} & \cdot & \tilde{c}_{13} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \tilde{d}_{10} & \cdot & \tilde{d}_{11} & \cdot & \cdot & \tilde{d}_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{d}_{13} & \cdot \\
\tilde{a}_{10} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{a}_{11} & \cdot & \cdot & \tilde{a}_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{a}_{13} \\
\cdot & \tilde{b}_{10} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{b}_{11} & \cdot & \cdot & \tilde{b}_{12} & \cdot & \tilde{b}_{13} & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \tilde{d}_{20} & \cdot & \tilde{d}_{21} & \cdot & \cdot & \tilde{d}_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{d}_{23} & \cdot \\
\tilde{a}_{20} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{a}_{21} & \cdot & \cdot & \tilde{a}_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{a}_{23} \\
\cdot & \tilde{b}_{20} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{b}_{21} & \cdot & \cdot & \tilde{b}_{22} & \cdot & \tilde{b}_{23} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \tilde{c}_{20} & \cdot & \tilde{c}_{21} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{c}_{22} & \cdot & \tilde{c}_{23} & \cdot & \cdot \\
\hline
\cdot & \tilde{b}_{30} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{b}_{31} & \cdot & \cdot & \tilde{b}_{32} & \cdot & \tilde{b}_{33} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \tilde{c}_{30} & \cdot & \tilde{c}_{31} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{c}_{32} & \cdot & \tilde{c}_{33} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \tilde{d}_{30} & \cdot & \tilde{d}_{31} & \cdot & \cdot & \tilde{d}_{32} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{d}_{33} & \cdot \\
\tilde{a}_{30} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{a}_{31} & \cdot & \cdot & \tilde{a}_{32} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{a}_{33}
\end{array}\right), \tag{A7}$$

where the matrices $\tilde{a},\tilde{b},\tilde{c},\tilde{d}$ are given by

$$\tilde{a}=\begin{pmatrix} a_{00} & c_{01} & b_{02} & d_{03} \\ c_{10} & a_{11} & d_{12} & b_{13} \\ b_{20} & d_{21} & c_{22} & a_{23} \\ d_{30} & b_{31} & a_{32} & c_{33} \end{pmatrix}, \quad \tilde{b}=\begin{pmatrix} b_{00} & d_{01} & c_{02} & a_{03} \\ d_{10} & b_{11} & a_{12} & c_{13} \\ c_{20} & a_{21} & d_{22} & b_{23} \\ a_{30} & c_{31} & b_{32} & d_{33} \end{pmatrix}, \quad \tilde{c}=\begin{pmatrix} c_{00} & a_{01} & d_{02} & b_{03} \\ a_{10} & c_{11} & b_{12} & d_{13} \\ d_{20} & b_{21} & a_{22} & c_{23} \\ b_{30} & d_{31} & c_{32} & a_{33} \end{pmatrix}, \quad \tilde{d}=\begin{pmatrix} d_{00} & b_{01} & a_{02} & c_{03} \\ b_{10} & d_{11} & c_{12} & a_{13} \\ a_{20} & c_{21} & b_{22} & d_{23} \\ c_{30} & a_{31} & d_{32} & b_{33} \end{pmatrix}.$$

3. π_3 and $\widetilde{\pi}_3$ circulant states

$$\Sigma_0^{\pi_3} = \mathrm{span}\{e_0 \otimes e_0, e_1 \otimes e_3, e_2 \otimes e_1, e_3 \otimes e_2\},$$

$$\Sigma_1^{\pi_3} = \mathrm{span}\{e_0 \otimes e_1, e_1 \otimes e_0, e_2 \otimes e_2, e_3 \otimes e_3\},$$

$$\Sigma_2^{\pi_3} = \mathrm{span}\{e_0 \otimes e_2, e_1 \otimes e_1, e_2 \otimes e_3, e_3 \otimes e_0\},$$

$$\Sigma_3^{\pi_3} = \mathrm{span}\{e_0 \otimes e_3, e_1 \otimes e_2, e_2 \otimes e_0, e_3 \otimes e_1\},$$

$$\widetilde{\Sigma}_0^{\pi_3} = \mathrm{span}\{e_0 \otimes e_0, e_1 \otimes e_1, e_2 \otimes e_3, e_3 \otimes e_2\},$$

$$\widetilde{\Sigma}_1^{\pi_3} = \mathrm{span}\{e_0 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_0, e_3 \otimes e_3\},$$

$$\widetilde{\Sigma}_2^{\pi_3} = \mathrm{span}\{e_0 \otimes e_2, e_1 \otimes e_3, e_2 \otimes e_1, e_3 \otimes e_0\},$$

$$\widetilde{\Sigma}_3^{\pi_3} = \mathrm{span}\{e_0 \otimes e_3, e_1 \otimes e_0, e_2 \otimes e_2, e_3 \otimes e_1\}.$$

$$\rho_{\pi_3} = \left(\begin{array}{cccc|cccc|cccc|cccc}
a_{00} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{01} & \cdot & a_{02} & \cdot & \cdot & \cdot & \cdot & a_{03} & \cdot \\
\cdot & b_{00} & \cdot & \cdot & b_{01} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{02} & \cdot & \cdot & \cdot & \cdot & b_{03} \\
\cdot & \cdot & c_{00} & \cdot & \cdot & c_{01} & \cdot & \cdot & \cdot & \cdot & \cdot & c_{02} & c_{03} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & d_{00} & \cdot & \cdot & d_{01} & \cdot & d_{02} & \cdot & \cdot & \cdot & \cdot & d_{03} & \cdot & \cdot \\
\hline
\cdot & b_{10} & \cdot & \cdot & b_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{12} & \cdot & \cdot & \cdot & \cdot & b_{13} \\
\cdot & \cdot & c_{10} & \cdot & \cdot & c_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & c_{12} & c_{13} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & d_{10} & \cdot & \cdot & d_{11} & \cdot & d_{12} & \cdot & \cdot & \cdot & \cdot & d_{13} & \cdot & \cdot \\
a_{10} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{11} & \cdot & a_{12} & \cdot & \cdot & \cdot & \cdot & a_{13} & \cdot \\
\hline
\cdot & \cdot & \cdot & d_{20} & \cdot & \cdot & d_{21} & \cdot & d_{22} & \cdot & \cdot & \cdot & \cdot & d_{23} & \cdot & \cdot \\
a_{20} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{21} & \cdot & a_{22} & \cdot & \cdot & \cdot & \cdot & a_{23} & \cdot \\
\cdot & b_{20} & \cdot & \cdot & b_{21} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{22} & \cdot & \cdot & \cdot & \cdot & b_{23} \\
\cdot & \cdot & c_{20} & \cdot & \cdot & c_{21} & \cdot & \cdot & \cdot & \cdot & \cdot & c_{22} & c_{23} & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & c_{30} & \cdot & \cdot & c_{31} & \cdot & \cdot & \cdot & \cdot & \cdot & c_{32} & c_{33} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & d_{30} & \cdot & \cdot & d_{31} & \cdot & d_{32} & \cdot & \cdot & \cdot & \cdot & d_{33} & \cdot & \cdot \\
a_{30} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{31} & \cdot & a_{32} & \cdot & \cdot & \cdot & \cdot & a_{33} & \cdot \\
\cdot & b_{30} & \cdot & \cdot & b_{31} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{32} & \cdot & \cdot & \cdot & \cdot & b_{33}
\end{array}\right). \tag{A8}$$

$$\rho_{\pi_3}^{\tau} = \left(\begin{array}{cccc|cccc|cccc|cccc}
\tilde{a}_{00} & \cdot & \cdot & \cdot & \cdot & \tilde{a}_{01} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{a}_{02} & \cdot & \cdot & \tilde{a}_{03} & \cdot \\
\cdot & \tilde{b}_{00} & \cdot & \cdot & \cdot & \cdot & \tilde{b}_{01} & \cdot & \tilde{b}_{02} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{b}_{03} \\
\cdot & \cdot & \tilde{c}_{00} & \cdot & \cdot & \cdot & \cdot & \tilde{c}_{01} & \cdot & \tilde{c}_{02} & \cdot & \cdot & \tilde{c}_{03} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \tilde{d}_{00} & \tilde{d}_{01} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{d}_{02} & \cdot & \cdot & \tilde{d}_{02} & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \tilde{d}_{10} & \tilde{d}_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{d}_{12} & \cdot & \cdot & \tilde{d}_{13} & \cdot & \cdot \\
\tilde{a}_{10} & \cdot & \cdot & \cdot & \cdot & \tilde{a}_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{a}_{12} & \cdot & \cdot & \tilde{a}_{13} & \cdot \\
\cdot & \tilde{b}_{10} & \cdot & \cdot & \cdot & \cdot & \tilde{b}_{11} & \cdot & \tilde{b}_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{b}_{13} \\
\cdot & \cdot & \tilde{c}_{10} & \cdot & \cdot & \cdot & \cdot & \tilde{c}_{11} & \cdot & \tilde{c}_{12} & \cdot & \cdot & \tilde{c}_{13} & \cdot & \cdot & \cdot \\
\hline
\cdot & \tilde{b}_{20} & \cdot & \cdot & \cdot & \cdot & \tilde{b}_{21} & \cdot & \tilde{b}_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{b}_{23} \\
\cdot & \cdot & \tilde{c}_{20} & \cdot & \cdot & \cdot & \cdot & \tilde{c}_{21} & \cdot & \tilde{c}_{22} & \cdot & \cdot & \tilde{c}_{23} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \tilde{d}_{20} & \tilde{d}_{21} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{d}_{22} & \cdot & \cdot & \tilde{d}_{22} & \cdot & \cdot \\
\tilde{a}_{20} & \cdot & \cdot & \cdot & \cdot & \tilde{a}_{21} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{a}_{22} & \cdot & \cdot & \tilde{a}_{23} & \cdot \\
\hline
\cdot & \cdot & \tilde{c}_{30} & \cdot & \cdot & \cdot & \cdot & \tilde{c}_{31} & \cdot & \tilde{c}_{32} & \cdot & \cdot & \tilde{c}_{33} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \tilde{d}_{30} & \tilde{d}_{31} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{d}_{32} & \cdot & \cdot & \tilde{d}_{32} & \cdot & \cdot \\
\tilde{a}_{30} & \cdot & \cdot & \cdot & \cdot & \tilde{a}_{31} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{a}_{32} & \cdot & \cdot & \tilde{a}_{33} & \cdot \\
\cdot & \tilde{b}_{30} & \cdot & \cdot & \cdot & \cdot & \tilde{b}_{31} & \cdot & \tilde{b}_{32} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{b}_{33}
\end{array}\right), \tag{A9}$$

where the matrices $\tilde{a},\tilde{b},\tilde{c},\tilde{d}$ are given by

$$\tilde{a} = \begin{pmatrix} a_{00} & b_{01} & d_{02} & c_{03} \\ b_{10} & c_{11} & a_{12} & d_{13} \\ d_{20} & a_{21} & c_{22} & b_{23} \\ c_{30} & d_{31} & b_{32} & a_{33} \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} b_{00} & c_{01} & a_{02} & d_{03} \\ c_{10} & d_{11} & b_{12} & a_{13} \\ a_{20} & b_{21} & d_{22} & c_{23} \\ d_{30} & a_{31} & c_{32} & b_{33} \end{pmatrix}, \quad \tilde{c} = \begin{pmatrix} c_{00} & d_{01} & b_{02} & a_{03} \\ d_{10} & a_{11} & c_{12} & b_{13} \\ b_{20} & c_{21} & a_{22} & d_{23} \\ a_{30} & b_{31} & d_{32} & c_{33} \end{pmatrix}, \quad \tilde{d} = \begin{pmatrix} d_{00} & a_{01} & c_{02} & b_{03} \\ a_{10} & b_{11} & d_{12} & c_{13} \\ c_{20} & d_{21} & b_{22} & a_{23} \\ b_{30} & c_{31} & a_{32} & d_{33} \end{pmatrix}.$$

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, e-print arXiv:quant-ph/0702225.

[3] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).

[4] P. Horodecki, Phys. Lett. A **232**, 333 (1997).

[5] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[6] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Phys. Rev. Lett. **82**, 5385 (1999).

[7] D. P. DiVincenco, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Commun. Math. Phys. **238**, 379 (2003); A. O. Pittenger, Linear Algebr. Appl. **359**, 235 (2003).

[8] P. Horodecki, M. Horodecki, and R. Horodecki, Phys. Rev. Lett. **82**, 1056 (1999).

[9] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Phys. Rev. Lett. **88**, 187904 (2002).

[10] D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and A. V. Thapliyal, Phys. Rev. A **61**, 062312 (2000).

[11] P. Horodecki and M. Lewenstein, Phys. Rev. Lett. **85**, 2657 (2000).

[12] D. Bruss and A. Peres, Phys. Rev. A **61**, 030301(R) (2000).

[13] S. Yu and N. L. Liu, Phys. Rev. Lett. **95**, 150504 (2005).

[14] M. Piani and C. Mora, Phys. Rev. A **75**, 012305 (2007).

[15] L. Clarisse, Phys. Lett. A **359**, 603 (2006).

[16] K.-C. Ha (unpublished).

[17] K. Ha, S.-H Kye, and Y. Park, Phys. Lett. A **313**, 163 (2003).

[18] K. Ha and S.-H. Kye, Phys. Lett. A **325**, 315 (2004); , J. Phys. A **38**, 9039 (2005).

[19] J. M. Leinaas, J. Myrheim, and E. Ovrum, e-print arXiv:quant-ph/0704.3348.

[20] D. Chruściński and A. Kossakowski, Phys. Rev. A **74**, 022308 (2006).

[21] A matrix $X \in M_n$ is circulant if

$$X = \begin{pmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{n-1} \\ \alpha_{n-1} & \alpha_0 & \cdots & \alpha_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \cdots & \alpha_0 \end{pmatrix}.$$

[22] A Hadamard (or Schur) product of two $n \times n$ matrices $A=[A_{ij}]$ and $B=[B_{ij}]$ is defined by

$$(A \circ B)_{ij} = A_{ij}B_{ij}.$$

[23] M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4206

(1999).
[24] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[25] A. G. Robertson, Quart. J. Math. Oxford (2) **34**, 87 (1983).
[26] D. Chruściński and A. Kossakowski, e-print arXiv:quant-ph/0606211.
[27] S.-M. Fei, X. Li-Jost, and B.-Z. Sun, Phys. Lett. A **352**, 321 (2006).
[28] M. Lewenstein, B. Kraus, J. I. Cirac, and P. Horodecki, Phys. Rev. A **62**, 052310 (2000).
[29] A. Sanpera, D. Bruss, and M. Lewenstein, Phys. Rev. A **63**, 050301(R) (2001).