Composite geometric phase for multipartite entangled states

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When an entangled state evolves under local unitaries, the entanglement in the state remains fixed. Here we show that the dynamical phase acquired by an entangled state in such a scenario can always be understood as the sum of the dynamical phases of its subsystems. In contrast, the equivalent statement for the geometric phase is not generally true unless the state is separable. For an entangled state an additional term is present, the mutual geometric phase, that measures the change the additional correlations present in the entangled state make to the geometry of the state space. For N qubit states we find that this change can be explained solely by classical correlations for states with a Schmidt decomposition and solely by quantum correlations for W states.

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In a seminal paper [1], Berry recognized that a quantum system undergoing a cyclic, adiabatic evolution records the path of its evolution as a geometrical quantity in the phase of its wave function. This phase, the geometric phase (GP), forms part of the total phase of the wave function in addition to the more familiar dynamical phase. Since Berry's initial discovery, the GP has been found to occur in more general circumstances: nonadiabatic [2] and noncyclic evolutions [3], the non-Abelian form [4], and for states that are mixed [5].

GPs are of interest for many reasons, among them topological effects in many-body systems [6] and their use for quantum-information processing [7], a paradigm where entanglement is known to be key in its advantage over the classical counterpart. GPs may also be used as a tool. Because they depend on the geometry of the space the states traverse, GPs can provide information on this space. As an example, this property can be used to discover the coordinates of quantum phase transitions in a parameter space [8]. On another currently popular front, entanglement has been shown to be present in many-body systems at finite temperatures [9]. Studying GPs of multipartite entangled states may therefore prove to be complementary to both efforts.

GPs have been studied only for entangled bipartite systems so far. These include qubits precessing in magnetic fields [10] and general evolution [11] both without interaction (fixed entanglement), and various specific Hamiltonians for bipartite systems with interactions (changing entanglement) [12,13]. It has been shown [10,11] in bipartite systems that even if there are no interactions during the evolution, fixed entanglement affects the geometric phase.

It is this last relation that we investigate further in this paper for multipartite systems. Of the two terms making up the total phase of a quantum state, the dynamical and geometric phases, we will show that under local evolutions the dynamical phase of a composite state can always be understood as the sum of the dynamical phases that comprise the composite system whether the state is entangled or not. We will also show that the same statement cannot be made for a GP when the state is entangled. When the composite state is entangled, an additional correction term, much like an interference term, $\Delta\gamma$, is present. Explicitly, $\Gamma=\Sigma_{n=1}^{N}\gamma_n^M+\Delta\gamma$ where the composite state GP is Γ and the nth of the N subsystems has GP γ_n^M. The correction term we call the mutual GP. Each subsystem ρ_n is obtained by tracing over the other $N-1$ subsystems in the composite state ρ.

In tracing out the subsystems from the composite state, we have removed both quantum and classical correlations [14]. The extra correlations present in the composite state, inaccessible when one has access to or control of only one of the subsystems, modify the GP but not the dynamical phase when the state is entangled. Should one have access to or control of only one of the subsystems, ρ_n, then a GP of γ_n^M would be observed. As previously noted, GPs are dependent on the underlying geometry of the quantum state space, and here we are using the GP as a tool to sample the change these correlations cause in the geometry of the state space. We characterize which type of correlations, quantum (entanglement) or classical, are responsible for the correction $\Delta\gamma$ for two specific types of state for which we know how to quantify the entanglement using the relative entropy of entanglement E_R [15]. We show that $\Delta\gamma$ can be attributed solely to classical correlations for states with a Schmidt decomposition (labeled hereafter as S states) and solely to entanglement for W states.

Our aim is to study the effect of fixed entanglement on the mutual GP $\Delta\gamma$. We fix entanglement during the evolution by requiring that the evolution on the composite state, the unitary $\mathcal{U}$, be composed of local unitaries U_n acting on each subsystem n. We write this as

$$\mathcal{U}(t)=\bigotimes_{n=1}^{N}U_n(t). \tag{1}$$

With this condition, we can show that the dynamical phase of the composite system, Δ, is always given by the sum of the dynamical phases of its subsystems, δ_n^M. The definition of the dynamical phase [5] is

$$\Delta=-i\int_0^T \mathrm{tr}[\rho\mathcal{U}^{\dagger}(t)\dot{\mathcal{U}}(t)]dt. \tag{2}$$

Substituting Eq. (1) into this definition we see that

$$\Delta = -i\int_0^T \mathrm{tr}\left(\rho\sum_{n=1}^{N} U_n^\dagger(t)\dot{U}_n(t)\bigotimes_{m=1,m\neq n}^{N}\mathbb{I}_m\right)dt$$
$$= \sum_{n=1}^{N} -i\int_0^T \mathrm{tr}[\rho_n U_n^\dagger(t)\dot{U}_n(t)]dt, \quad (3)$$

showing $\Delta=\Sigma_{n=1}^{N}\delta_n^M$. The composite dynamical phase is always equal to the sum of the subsystem dynamical phases whether the state is entangled or not. This statement holds because we have constrained the dynamics to be local. Before demonstrating that this is not the case for GP, we discuss parallel transport as a useful way to isolate the GP.

A quantum state is said to be parallel transported when it acquires no dynamical phase at each point along its evolution. Formally, the mathematical condition that requires the state is in phase with itself at each point is $\mathrm{tr}\rho\mathcal{U}(t)^\dagger\dot{\mathcal{U}}(t)=0\,\forall\, t$. If the state is parallel transported (by $\mathcal{U}^\parallel$) then the total phase obtained by the state will be equal to the GP. Pancharatnam [16] gave a natural prescription to obtain the total phase, generalized to mixed states [5]. As mentioned, under parallel transport conditions this becomes the GP Γ accumulated over the time $t\in[0,T]$,

$$\Gamma = \arg\{\mathrm{tr}\,\rho\mathcal{U}^\parallel(T)\}. \quad (4)$$

For unentangled states of the form $\rho=\otimes_{n=1}^{N}\rho_n$ we see that under local unitaries Eq. (4) becomes $\Gamma=\arg\{\Pi_{n=1}^{N}\mathrm{tr}\rho_n U_n^\parallel(T)\}$ so $\Gamma=\Sigma_{n=1}^{N}\gamma_n^M$. But in general we see that $\Gamma\neq\Sigma_{n=1}^{N}\gamma_n^M$.

It is known that there are many parallel transport conditions for mixed states. In this study the subsystems ρ_n will be the mixed states. Some of these parallel transport conditions produce GPs that are a property of an arbitrary entangled purification of the mixed state in a higher-dimensional space, not just of the mixed state itself. It is also known, however, that a subset of these conditions, those proposed by [5] ("stronger" parallel transport conditions) provide a mixed state GP that is a property of the evolution of the mixed state only [17]. For this reason we use the stronger parallel transport conditions to constrain the local unitaries. These conditions require that every eigenvector $|\phi_i^n\rangle$ of each subsystem n be parallel transported, formally $\langle\phi_i^n|U_n^\dagger(t)\dot{U}_n(t)|\phi_i^n\rangle=0\,\forall\, i,n,t$. We write the unitary acting on subsystem n that satisfies these conditions $U_n^\parallel$.

We now calculate Γ and γ_n^M for arbitrary superpositions of W states which contain, among others, W, S, and Greenberger-Horne-Zeilinger (GHZ) states. We will write these states as

$$|\Psi(0)\rangle = \sum_{k=0}^{N} a_k|N,k\rangle, \quad (5)$$

where a_k are the complex probability amplitudes and the W state $|N,k\rangle$ is defined by

$$|N,k\rangle = \frac{1}{\sqrt{\binom{N}{k}}}\hat{S}|\underbrace{00\ldots0}_{N-k}\underbrace{11\ldots1}_{k}\rangle. \quad (6)$$

$\hat{S}$ is the total symmetrization operator. The time evolution of the state $|\Psi(0)\rangle$ is given by $|\Psi(t)\rangle=\mathcal{U}(t)|\Psi(0)\rangle$. Each individual qubit in the state considered on its own is given by the density matrix $\rho_n(t)=\mathrm{tr}_{1,\ldots,N,\neq n}|\Psi(t)\rangle\langle\Psi(t)|$. In the $|0\rangle$, $|1\rangle$ basis the subsystem state explicitly is

$$\rho_n(0) = \rho_{00}|0\rangle\langle 0| + \rho_{11}|1\rangle\langle 1| + \rho_{01}|0\rangle\langle 1| + \rho_{01}^*|1\rangle\langle 0|, \quad (7)$$

where $\rho_{00}=\Sigma_{k=0}^{N}|a_k|^2(N-k)/N$, $\rho_{11}=\Sigma_{k=0}^{N}|a_k|^2k/N$, and $\rho_{01}=\Sigma_{k=1}^{N}a_{k-1}^*a_k\sqrt{k(N-k+1)}/N$. The GP for both the composite state $|\Psi(T)\rangle$ and each of its subsystems $\rho_n(T)$ is made by substitution into Eq. (4). We have $\Gamma=\arg\{\Sigma_{k,l=0}^{N}a_l^*a_k\langle N,l|\mathcal{U}^\parallel|(T)|N,k\rangle\}$. Cross terms disappear from this equation when each local unitary $U_n^\parallel$ brings each subsystem back to the same ray $[U_n^\parallel(T)|0\rangle=U_n^\parallel(0)|0\rangle$ up to a global phase]. Under this evolution the global GP Γ becomes

$$\Gamma = \arg\left\{\sum_{k=0}^{N}\frac{|a_k|^2}{\binom{N}{k}}\sum_{m=1}^{\binom{N}{k}}\exp^{(i\Sigma_{n=1}^{N}A_{mn}^k\gamma_n)}\right\}, \quad (8)$$

expressing the composite system GP in terms of pure state qubit phases γ_n. We have used the polar representation to define $c_ne^{i\gamma_n}:=\langle 0|U_n(T)|0\rangle$ and its complex conjugate $c_ne^{-i\gamma_n}:=\langle 1|U_n(T)|1\rangle$. γ_n $(-\gamma_n)$ are the GPs that the pure qubit states $|0\rangle$ $(|1\rangle)$ acquire over the evolution $U_n(T)$ and $c_n=1$ when each subsystem unitary is cyclic. The matrix A^k has elements 1 (-1) to capture the sign of γ_n $(-\gamma_n)$ in the exponent. Each ordered row in A_k has $N-k$ elements with value 1 and k elements of -1. There are $\binom{N}{k}$ ordered rows in A^k given by all permutations of the first-row elements.

Equation (8) can be understood as follows. Each W state $|N,k\rangle$ is a superposition of $\binom{N}{k}$ kets, each with phase factor $\exp(i\Sigma_{n=1}^{N}A_{mn}^k\gamma_n)$. The phase factors for each ket are averaged in the m summation, resulting in an overall phase factor for each W state (k). The phase factors for each W state are then averaged (the k summation) to give the total overall GP factor of the state.

The states of the subsystems are given by ρ_n and the GP associated with them is also obtained by substitution into Eq. (4) with $U_n^\parallel|(T)$. We substitute the general form of the eigenvectors and eigenvalues into this equation to obtain $\gamma_n^M=\arg\{\frac{1}{2}(1+r)\langle\phi_1^n|U_n^\parallel(T)|\phi_1^n\rangle+\frac{1}{2}(1-r)\langle\phi_2^n|U_n^\parallel(T)|\phi_2^n\rangle\}$, where the general eigenvectors of a two-level system are $|\phi_1^n\rangle=e^{-i\phi_n/2}\cos(\theta_n/2)|0\rangle+e^{i\phi_n/2}\sin(\theta_n/2)|1\rangle$ and $|\phi_2^n\rangle=-e^{-i\phi_n/2}\sin(\theta_n/2)|0\rangle+e^{i\phi_n/2}\cos(\theta_n/2)|1\rangle$. Rewriting in the $|0\rangle$, $|1\rangle$ basis and substitution of the γ_n identities results in the cyclic GPs of the subsystems being $\gamma_n^M=\arg\{\cos\gamma_n+ir\cos\theta_n\sin\gamma_n\}$. This can also be written as

$$\gamma_n^M = \arctan\left\{\left(\sum_{k=0}^{N}|a_k|^2\frac{N-2k}{N}\right)\tan\gamma_n\right\} \quad (9)$$

for nondegenerate eigenvalues of ρ_n. We can interpret this result in a similar manner to the pure state result. Writing Eq. (9) as $\gamma_n^M=\arg\{\Sigma_{k=0}^{N}|a_k|^2[(N-k)/N]e^{i\gamma_n}+\Sigma_{k=0}^{N}|a_k|^2(k/N)e^{-i\gamma_n}\}$, we can see that γ_n^M is an average of the phase factors of the

states $|0\rangle$ and $|1\rangle$ weighted by their relative frequencies in the composite state. For the remainder of this paper we will discuss specific cases of Eqs. (8) and (9).

To simplify things, each local unitary will now be identical, i.e., $\mathcal{U}^{\|}=\otimes_{n=1}^{N}U^{\|}$ in all the following. All pure state qubit GPs γ_n also become identical and will be labeled γ.

The first specific example is that of S states. For these states only two amplitudes in Eq. (5) are nonzero, a_0 and a_N. The S state is

$$|\Psi(0)\rangle = \sqrt{\frac{1}{2}(1+r)}|0\rangle^{\otimes N} + \sqrt{\frac{1}{2}(1-r)}|1\rangle^{\otimes N}. \quad (10)$$

We have written a_0 and a_N as the square root of the eigenvalues of a bipartite state. Substitution into Eq. (8) yields

$$\Gamma = \arg\left\{\frac{1}{2}(1+r)e^{iN\gamma} + \frac{1}{2}(1-r)e^{-iN\gamma}\right\} = \arctan\{r\tan N\gamma\}. \quad (11)$$

In this formula pure state phases rather than phase factors add up. Each of the local GPs for the S state is

$$\gamma^M = \arctan\{r\tan\gamma\}. \quad (12)$$

To quantify the amount of entanglement in the composite states we use the relative entropy of entanglement E_R [15,18]. For S states E_R is given by

$$E_R = 1 - \frac{1}{2}[(1+r)\log_2(1+r) + (1-r)\log_2(1-r)]. \quad (13)$$

E_R takes a maximum value of 1 when $r=0$ and a minimum value 0 when $r=1$. When a S state is maximally entangled ($r=0$) Γ can take only the values 0 and π; however, Eq. (12) for γ^M is no longer valid as ρ_n is degenerate and must be evaluated via path ordering (see [19]). At the other extreme, separable states ($r=1$), we see $\Gamma=N\gamma^M$ as expected.

The second specific example is a W state $|\Psi(0)\rangle=|N,k\rangle$. The identical local evolution in this case is particularly appropriate as these states often occur when subsystems are indistinguishable. For this state $|a_k|^2=1$ and the composite GP takes a particularly simple form,

$$\Gamma = (N-2k)\gamma. \quad (14)$$

For a W state the GP is the sum of the pure state qubit phases much as in a pure separable state. This result in fact holds for any arbitrary superposition of W kets (for example, $a_1|001\rangle + a_2|010\rangle + a_3|100\rangle$). However, the subsystems are not generally pure and their GPs are generally not γ and $-\gamma$. They are

$$\gamma^M = \arctan\left\{\left(\frac{N-2k}{N}\right)\tan\gamma\right\}. \quad (15)$$

The relative entropy of entanglement for W states is known to be [20,21]

$$E_R = -\log_2\binom{N}{k} - (N-k)\log_2\left(\frac{N-k}{N}\right) - k\log_2\left(\frac{k}{N}\right). \quad (16)$$

Entanglement is maximal when $N=2k$ in which case $E_R=N-\log_2\binom{N}{N/2}$ and is minimal ($E_R=0$) when $k=0,N$ (the state is separable). When the W state is maximally entangled $\Gamma=0$. This limit is similar to the maximally entangled S state, except it could also take the value π. We also note, for the same reason as for maximally entangled S states, Eq. (15) is not valid for calculation of γ^M when $N=2k$. Again, $\Gamma=N\gamma^M$ for separable states (when $k=0,N$).

We now have the ingredients to calculate the mutual GP $\Delta\gamma$, written explicitly as

$$\Delta\gamma = \Gamma - \sum_{n=1}^{N}\gamma_n^M. \quad (17)$$

As previously mentioned, we have removed the quantum (entanglement) but also the classical correlations in tracing out each subsystem from the composite. We can characterize which missing correlations are responsible for the correction term $\Delta\gamma$ using ideas from entanglement distance measures, specifically, here, the relative entropy of entanglement E_R.

To calculate E_R, the closest separable state ρ_S to the entangled state needs to be found. Here, closest state means the separable state from the set of all separable states that minimizes the quantum relative entropy $S(\rho\|\rho_S)=\mathrm{tr}(\rho\log\rho-\rho\log\rho_S)$ between it and the entangled state ρ. The minimum value of $S(\rho\|\rho_S)$ is equal to E_R. The procedure maximizes classical correlations between the entangled and the separable state; any correlations left over are quantum. The closest separable states are given in [18] for S states and [20,21] for W states. The GPs Γ_S for these closest separable states are

$$\Gamma_S = \arg\left\{\frac{1}{2}(1+r)e^{iN\gamma} + \frac{1}{2}(1-r)e^{-iN\gamma}\right\} \quad (18)$$

for the S state and

$$\Gamma_S = N\arctan\left\{\left(\frac{N-2k}{N}\right)\tan\gamma\right\} \quad (19)$$

for the W state. In the latter, we have used the binomial theorem and the property $N\arg\{a\}=\arg\{a^N\}$.

By taking the difference between the GPs of the composite state and the closest separable state ($\Gamma-\Gamma_S$), we can see exclusively what difference the quantum correlations make to the GP. For S states this difference is zero as $\Gamma=\Gamma_S$. We can therefore state that quantum correlations have no effect on the GP and the state space geometry. We can also state that the correction term $\Delta\gamma$ is the sole result of classical correlations for S states. In contrast, $\Gamma-\Gamma_S=\Delta\gamma$ for W states because $\Gamma_S=\Sigma_{n=1}^{N}\gamma^M$. Here we can state that $\Delta\gamma$ results solely from quantum correlations. One common feature the states share is that entanglement does not affect curvature when $E_R\leqslant 1$. Note, however, that S states can have a continuum of values $0\leqslant E_R\leqslant 1$ independent of N. E_R of W states does have N dependence but discrete values. The smallest of these val-

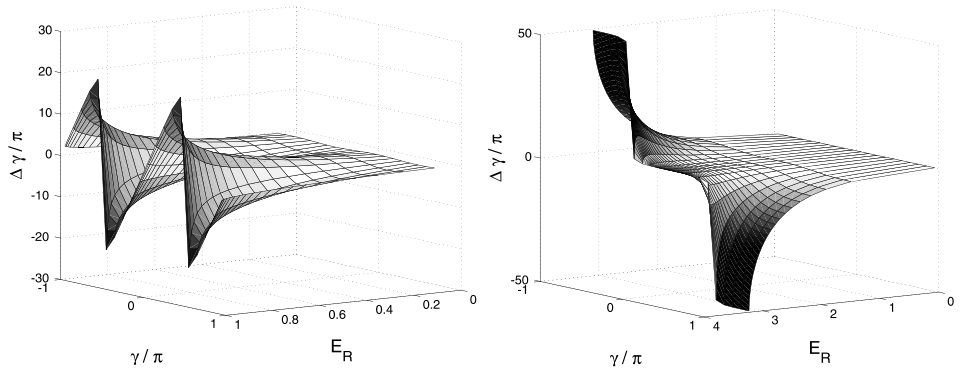

FIG. 1. $\Delta\gamma$ plotted against entanglement E_R and γ for $N=51$. Left panel: S state (classical correlations only). Right panel: W state (quantum correlations only). $\Delta\gamma$ is nonmodular and plotted in the large-N limit with varying path γ and entanglement in the composite state, E_R. For constant γ, S states are monotonic with respect to E_R. W states are monotonic except for the region $|\gamma|/\pi < 1/2$. Periodicities with respect to γ are π and 2π for S and W states, respectively, when E_R is fixed. For both states N controls the magnitude of $\Delta\gamma$.

ues are $E_R=0$ (separable, $k=0,N$), $E_R=1$ (singlet state, $N=2$, $k=1$), or $E_R \geqslant \log_2 e$ for $k=1$, $N-1$. According to the relative entropy of entanglement, perhaps counterintuitively, W states have more quantum character than S states, in the sense that classical correlations cannot approximate these states as closely. This is related to the robustness of the entanglement. Almost all correlations in S states are classical; tracing out just one subsystem results in a separable state. In contrast, W states remain entangled to the last pair of subsystems. Plotted in Fig. 1 is $\Delta\gamma$ for S and W states, the difference classical and quantum correlations make to the GP, respectively. Although we do not know how to quantify the entanglement of states as in Eq. (5), we expect that $\Delta\gamma$ for these states can be attributed to a mixture of classical and quantum correlations, each reducing to what are probably the two special limiting cases, S and W states.

In this paper we have demonstrated that, under local evolutions, entangled states gain a correction term to their GP not present in the dynamical phase. This correction term $\Delta\gamma$ has its origins in the change the extra correlations present in entangled states make to the state space geometry, and we have showed that it can be attributed solely to classical correlations for S states and quantum correlations for W states. Here we have used the relative entropy of entanglement definition of quantum and classical correlations, and it would be interesting to see what statements can be made for other entanglement measures. GPs are path-dependent quantities, but by determining this path dependence one can learn something of the geometry of the underlying space. We hope that GPs may prove useful in determining how classical and quantum correlations modify this geometry.

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