Energy eigenvalues and squeezing properties of general systems of coupled quantum anharmonic oscillators

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We have generalized the two-step approach to the solution of systems of N coupled quantum anharmonic oscillators. By using the squeezed vacuum state of each individual oscillator, we construct the tensor product state, and obtain the optimal squeezed vacuum product state through energy minimization. We then employ this optimal state and its associated bosonic operators to define a basis set to construct the Heisenberg matrix. The diagonalization of the matrix enables us to obtain the energy eigenvalues of the coupled oscillators. In particular, we have applied our formalism to determine the eigenenergies of systems of two coupled quantum anharmonic oscillators perturbed by a general polynomial potential, as well as three and four coupled systems. Furthermore, by performing a first-order perturbation analysis about the optimal squeezed vacuum product state, we have also examined into the squeezing properties of two coupled oscillator systems.

I. INTRODUCTION

As an important model in physics, quantum oscillators have been studied extensively in the past [1–9]. Motivated by its application to molecular physics and quantum field theory, researchers begin to investigate the consequences and effects of interaction within a system of coupled oscillators. This leads to studies into pairs of interacting harmonic oscillators [10,11] or quartic anharmonic oscillators [12–15], which serve as useful starting points to understand the many interesting properties typical of quantum systems with many degrees of freedom. For example, Bosco *et al.* first look into a system of two coupled harmonic oscillators to arrive at the correct sequence of variable transformations, before generalizing their treatment to the more complicated N coupled oscillator systems [10].

However, to examine more realistic and complex physical systems, it is necessary to incorporate the anharmonic character of the oscillations. Recently, researchers are able to model the hole transfer problem in *B*-DNA double strands [16] by modeling clusters of nucleobases as a chain of coupled quantum anharmonic oscillators. They have been able to explore a larger size system with this model without incurring too much computational cost. Nonetheless, a system of two coupled quantum anharmonic oscillators is interesting in their own right as realistic models of physical systems. For example, they have been employed to model molecular systems such as the five-membered ring compounds including 2,5-dihydrofuran, cyclopentanone, cyclopentene, silacyclopentane, and germacyclopentane; and have enabled the calculation of the bend and twist eigenvalues and energy transitions of these systems. Furthermore, they are able to describe the nonlinear optical activity of a dimerlike chiral molecule [17], the semiclassical theory of bound state [14], and the interaction of quantum system with classical environment [18].

It is well known that systems of two coupled quantum anharmonic oscillators cannot be treated analytically. Hence, diverse schemes such as mixed diagonalization [15], semiclassical calculation [14], Bargmann representation [12], and

operator method [19] have been developed to approximate their properties. In this paper, we have explored another solution to this problem by extending a two-step diagonalization approach. The perturbing anharmonic potential that we consider has the general form of a polynomial of two variables with degree up to m: $\sum_{v=0}^{m} \sum_{u=0}^{m-v} c_{uv} x_1^u x_2^v$. Our approach first performs a Bogoliubov transform on the original bosonic operators of the coupled oscillator system to produce a set of bosonic operators. These bosonic operators and the associated squeezed vacuum product state are parametrized by a set of independent parameters. We then select the optimal bosonic operators and squeezed vacuum product state through minimizing the energy of the system by a variation of these parameters. We then employ these operators and vacuum product state to determine the energy eigenvalues and squeezing properties of the general systems of two coupled quantum anharmonic oscillators.

The organization of the paper is as follows. In Sec. II, we extend the two-step approach to higher dimension, and construct a multidimensional matrix representation of the Hamiltonian. The energy eigenvalues of the coupled anharmonic oscillators are then obtained by standard diagonalization scheme after transforming the multidimensional matrix to a normal two-dimensional (2D) matrix. In Sec. III, we verify our approach by applying the method to both symmetric and unsymmetric two coupled quartic oscillators, and compare our results with those obtained by other approaches. We find close agreement of our results with these approaches. Subsequently, we apply our method to investigate the energy eigenvalues of a pair of coupled quantum pendulums. In Sec. IV, we continue to validate the applicability of the extended two-step approach by solving the eigenenergies of systems of three and four coupled sextic oscillators. In Sec. V, we first show that the squeezed vacuum state is a coherent state by using the fact that it is an eigenstate of the "two-photon annihilation operator." We then proceed to study the squeezing properties of the ground state of the two coupled anharmonic oscillators perturbatively with respect to the squeezed vacuum state. Finally, we conclude our paper in Sec. VI.

II. GENERALIZED TWO-STEP APPROACH TO SYSTEMS OF N COUPLED QUANTUM ANHARMONIC OSCILLATORS

A. Extended two-step approach

The two-step approach [8] can be extended to a system of N coupled quantum anharmonic oscillators. In the first step, a Bogoliubov transformation is used to define a set of creation and annihilation operators $b_j^\dagger$ and b_j from the original set $a_j^\dagger$ and a_j, as follows:

$$b_j = \frac{a_j - t_j a_j^\dagger}{\sqrt{1 - t_j^2}}, \tag{1}$$

$$b_j^\dagger = \frac{a_j^\dagger - t_j a_j}{\sqrt{1 - t_j^2}}, \tag{2}$$

where t_j is real and satisfies $|t_j| < 1$. Since the transformation is canonical, these new bosonic operators satisfy the commutation relation $[b_i, b_j^\dagger] = \delta_{ij}$, where δ_{ij} is the Kronecker δ. Note that $i, j = 1, 2, \ldots, N$. Furthermore, the corresponding ground state $|\phi_j, t_j\rangle$, which satisfies $b_j|\phi_j, t_j\rangle = 0$, is a squeezed vacuum state. It is defined through the action of the squeeze operator $S(t_j)$ on the original ground state $|0_j\rangle$:

$$|\phi_j, t_j\rangle = S(t_j)|0_j\rangle = \exp\left(\frac{t_j}{2}a_j^{\dagger 2}\right)|0_j\rangle = \sum_{n_j=0}^{\infty} \frac{1}{n_j!}\left(\frac{t_j}{2}a_j^{\dagger 2}\right)^{n_j}|0_j\rangle$$

$$= \sum_{n_j=0}^{\infty} \frac{t_j^{n_j}\sqrt{(2n_j)!}}{2^{n_j}n_j!}|2n_j\rangle. \tag{3}$$

Note that n_j is the index to the original number state $|n_j\rangle$. The

squeeze operator $S(t_j)$ is, however, different from the general one [20] by a factor of $\exp\left(-\frac{t_j}{2}a_j^2\right)$. Incidentally, the squeezed vacuum state is also the eigenstate $|\lambda, +1\rangle$ of the "two-photon annihilation operator" (TAO) $a_j^{\dagger -1}a_j$ [20] with eigenvalue t_j,

$$a_j^{\dagger -1}a_j|\phi_j, t_j\rangle = t_j|\phi_j, t_j\rangle, \tag{4}$$

where $a_j^{\dagger -1}$ is the inverse creation operator.

Next, let us take the tensor product of the individual oscillator states as follows:

$$|\phi_1, t_1\rangle \otimes |\phi_2, t_2\rangle \otimes \cdots \otimes |\phi_N, t_N\rangle$$

$$= |\phi_1, \phi_2, \ldots, \phi_N, t_1, t_2, \ldots, t_N\rangle$$

$$= \exp\left(\sum_{j=1}^{N} \frac{t_j}{2}a_j^{\dagger 2}\right)|0_1, 0_2, \ldots, 0_N\rangle. \tag{5}$$

The bosonic operators now operate on these states in the following way:

$$b_j|\phi_1, \phi_2, \ldots, \phi_N, t_1, t_2, \ldots, t_N\rangle = 0 \quad \text{for } j = 1, 2, \ldots, N. \tag{6}$$

Thus, we observe that the set of bosonic operators is associated with the new squeezed vacuum product state $|\phi_1, \phi_2, \ldots, \phi_N, t_1, t_2, \ldots, t_N\rangle$, and they are parametrized by N independent parameters $t_1, t_2, \ldots, t_N$. By defining

$$\Omega_j = \frac{1 - t_j}{1 + t_j}, \tag{7}$$

which amounts to a rescaling of the form $x_j \to \sqrt{\Omega_j}x_j$, we select the optimal set of bosonic operators and squeezed vacuum product state by minimizing the ground state energy,

$$E(\Omega_1, \Omega_2, \ldots, \Omega_N) = \frac{\langle \phi_1, \phi_2, \ldots, \phi_N, \Omega_1, \Omega_2, \ldots, \Omega_N|H|\phi_1, \phi_2, \ldots, \phi_N, \Omega_1, \Omega_2, \ldots, \Omega_N\rangle}{\langle \phi_1, \phi_2, \ldots, \phi_N, \Omega_1, \Omega_2, \ldots, \Omega_N|\phi_1, \phi_2, \ldots, \phi_N, \Omega_1, \Omega_2, \ldots, \Omega_N\rangle}, \tag{8}$$

with respect to the set of Ω_j. In other words, we solve for the $\bar{\Omega}_j$ which gives

$$\frac{\partial E(\bar{\Omega}_1, \bar{\Omega}_2, \ldots, \bar{\Omega}_N)}{\partial \Omega_j} = 0. \tag{9}$$

Then, the optimal generalized number product basis $|\bar{n}_1, \bar{n}_2, \ldots, \bar{n}_N\rangle$ can be generated from the optimal squeezed vacuum product state

$$|\bar{\phi}_1, \bar{\phi}_2, \ldots, \bar{\phi}_N\rangle = |\phi_1, \phi_2, \ldots, \phi_N, \bar{\Omega}_1, \bar{\Omega}_2, \ldots, \bar{\Omega}_N\rangle \tag{10}$$

in the following way:

$$|\bar{n}_1, \bar{n}_2, \ldots, \bar{n}_N\rangle = \frac{\bar{b}_1^{\dagger \bar{n}_1}\bar{b}_2^{\dagger \bar{n}_2}\cdots\bar{b}_N^{\dagger \bar{n}_N}}{\sqrt{\bar{n}_1!\,\bar{n}_2!\cdots\bar{n}_N!}}|\bar{\phi}_1, \bar{\phi}_2, \ldots, \bar{\phi}_N\rangle. \tag{11}$$

In the second step, we use the optimal generalized number product basis to obtain the Heisenberg matrix through truncating the basis by limiting $\bar{n}_j \leq M$. The Heisenberg matrix is a multidimensional matrix. We convert it to a $M^N \times M^N$ matrix by means of the following transformation:

$$C(r, s) = \langle i_1, i_2, \ldots, i_N|H|j_1, j_2, \ldots, j_N\rangle, \tag{12}$$

where

$$r = \sum_{k=1}^{N} (i_k - 1)M^{N-k} + 1,$$

$$s = \sum_{k=1}^{N} (j_k - 1)M^{N-k} + 1, \qquad (13)$$

and $i_k, j_k = 1, 2, \ldots, M$. By diagonalizing $C(r,s)$, the eigenenergies of the system of N coupled quantum anharmonic oscillators can be determined.

B. Application to systems of two coupled quantum anharmonic oscillators

Let us apply the extended two-step approach to a pair of coupled quantum anharmonic oscillators, which is described by a Hamiltonian of the following general form:

$$H = \sum_{j=1}^{2} \left(\frac{p_j^2}{2m_j} + \frac{1}{2} m_j \omega_j^2 x_j^2 \right) + \lambda \sum_{v=0}^{m} \sum_{u=0}^{m-v} c_{uv} x_1^u x_2^v, \qquad (14)$$

where λ is the perturbative constant and c_{uv} is the coupling coefficients. Note that the polynomial of the perturbating anharmonic potential is up to degree m. The Hamiltonian can also be written in the second-quantized form based on the creation and annihilation operators of each oscillator as follows:

$$H = \sum_{j=1}^{2} \hbar \omega_j \left(a_j^\dagger a_j + \frac{1}{2} \right) + \lambda \sum_{v=0}^{m} \sum_{u=0}^{m-v} c_{uv} \frac{\beta_1^u \beta_2^v}{2^{(u+v)/2}} (a_1^\dagger + a_1)^u$$
$$\times (a_2^\dagger + a_2)^v, \qquad (15)$$

where $x_j = \beta_j(a_j^\dagger + a_j)/\sqrt{2}$, $p_j = i\hbar(a_j^\dagger - a_j)/\sqrt{2}\beta_j$, and $\beta_j = \sqrt{\hbar/m_j\omega_j}$.

In applying our extended approach, we first employ Eqs. (1), (2), and (7) to re-express the Hamiltonian in terms of $b_j^\dagger$, b_j, and Ω_j in the following manner:

$$H = \sum_{j=1}^{2} \hbar \omega_j \left(\frac{1+\Omega_j^2}{4\Omega_j} + \frac{1+\Omega_j^2}{2\Omega_j} b_j^\dagger b_j + \frac{1-\Omega_j^2}{4\Omega_j} (b_j^{\dagger 2} + b_j^2) \right)$$
$$+ \lambda \sum_{v=0}^{m} \sum_{u=0}^{m-v} c_{uv} \frac{\beta_1^u \beta_2^v}{2^{(u+v)/2} \Omega_1^{u/2} \Omega_2^{v/2}}$$
$$\times \sum_{p=0}^{[u/2]} \sum_{q=0}^{u-2p} \frac{u \, ! \, b_1^{\dagger(u-2p-q)} b_1^q}{2^p p \, ! \, q \, ! \, (u-2p-q)!}$$

$$\times \sum_{r=0}^{[v/2]} \sum_{s=0}^{v-2r} \frac{v \, ! \, b_2^{\dagger(v-2r-s)} b_2^s}{2^r r \, ! \, s \, ! \, (v-2r-s)!}, \qquad (16)$$

where the symbol $[x]$ means rounding the value of x to the nearest lower integer. Note that we have expressed the Hamiltonian in the normal-ordered form for the convenience of constructing the Heisenberg matrix later. By applying the variational principle through Eqs. (8) and (9) with $N=2$, the following set of simultaneous equations are obtained:

$$\hbar\omega_1 \frac{\bar{\Omega}_1^2 - 1}{\bar{\Omega}_1} - \lambda \sum_{v=0,2}^{m} \sum_{u=0,2}^{m-v} \frac{c_{uv}\beta_1^u \beta_2^v u \, ! \, v \, ! \, (u/2)}{2^{(u+v-2)}(u/2) \, ! \, (v/2) \, ! \, \bar{\Omega}_1^{u/2} \bar{\Omega}_2^{v/2}} = 0, \qquad (17)$$

$$\hbar\omega_2 \frac{\bar{\Omega}_2^2 - 1}{\bar{\Omega}_2} - \lambda \sum_{v=0,2}^{m} \sum_{u=0,2}^{m-v} \frac{c_{uv}\beta_1^u \beta_2^v u \, ! \, v \, ! \, (v/2)}{2^{(u+v-2)}(u/2) \, ! \, (v/2) \, ! \, \bar{\Omega}_1^{u/2} \bar{\Omega}_2^{v/2}} = 0. \qquad (18)$$

Equations (17) and (18) are then solved numerically to yield the optimal values of Ω_1 and Ω_2.

The transformation performed above has rescaled the Hamiltonian such that the frequencies of the normal modes are now given by $\omega_j \bar{\Omega}_j$, while the magnitude of the perturbation has diminished by a factor of the form $1/\sqrt{\bar{\Omega}_1^u \bar{\Omega}_2^v}$, with u and v being integers smaller than m. This has the implication that the rescaled system is amenable to perturbative techniques based on the optimal bosonic operators $\bar{b}_j^\dagger$ and $\bar{b}_j$, the squeezed vacuum product states $|\bar{\phi}_1, \bar{\phi}_2\rangle$, and the generalized number product states

$$|\bar{n}_1, \bar{n}_2\rangle = \frac{\bar{b}_1^{\dagger \bar{n}_1} \bar{b}_2^{\dagger \bar{n}_2}}{\sqrt{\bar{n}_1 \, ! \, \bar{n}_2!}} |\bar{\phi}_1, \bar{\phi}_2\rangle. \qquad (19)$$

Using these generalized number product states as basis, we obtain the multidimension matrix representation of H for the system of two coupled oscillators,

$$H_{i_1 j_1 i_2 j_2} = \langle i_1, i_2 | H | j_1, j_2 \rangle = \left(\hbar\omega_1 \frac{1+\bar{\Omega}_1^2}{4\bar{\Omega}_1} + \hbar\omega_1 j_1 \frac{1+\bar{\Omega}_1^2}{2\bar{\Omega}_1} + \hbar\omega_2 \frac{1+\bar{\Omega}_2^2}{4\bar{\Omega}_2} + \hbar\omega_2 j_2 \frac{1+\bar{\Omega}_2^2}{2\bar{\Omega}_2} \right) \delta_{i_1 j_1} \delta_{i_2 j_2} + \hbar\omega_1 \frac{1-\bar{\Omega}_1^2}{4\bar{\Omega}_1} (\sqrt{j_1(j_1-1)} \delta_{i_1,j_1-2}$$

$$+ \sqrt{(j_1+1)(j_1+2)} \delta_{i_1,j_1+2}) \delta_{i_2 j_2} + \hbar\omega_2 \frac{1-\bar{\Omega}_2^2}{4\bar{\Omega}_2} (\sqrt{j_2(j_2-1)} \delta_{i_2,j_2-2} + \sqrt{(j_2+1)(j_2+2)} \delta_{i_2,j_2+2}) \delta_{i_1 j_1}$$

$$+ \lambda \sum_{v=0}^{m} \sum_{u=0}^{m-v} \frac{c_{uv}\beta_1^u \beta_2^v}{2^{(u+v)/2}\bar{\Omega}_1^{u/2}\bar{\Omega}_2^{v/2}} \sum_{p=0}^{[u/2]} \sum_{q=0}^{u-2p} \frac{u \, ! \, \sqrt{j_1 \, ! \, (u-2p-2q+j_1)!}}{2^p p \, ! \, q \, ! \, (j_1-q) \, ! \, (u-2p-q)!}$$

$$\times \sum_{r=0}^{[v/2]} \sum_{s=0}^{v-2r} \frac{v \, ! \, \sqrt{j_2 \, ! \, (v-2r-2s+j_2)!}}{2^r r \, ! \, s \, ! \, (j_2-s) \, ! \, (v-2r-s)!} \delta_{i_1,j_1+u-2p-2q} \delta_{i_2,j_2+v-2r-2s}, \qquad (20)$$

where $i_1, i_2, j_1, j_2 = 1, 2, \ldots, M$. This is a $M \times M \times M \times M$ 4-D matrix. Next, we convert this multidimension matrix into a $M^2 \times M^2$ matrix through the following transformation:

$$C[M(i_1 - 1) + i_2, M(j_1 - 1) + j_2] = H(i_1, j_1, i_2, j_2). \quad (21)$$

By applying the standard diagonalization scheme to the $C(r, s)$ matrix, we obtain the eigenenergies of the coupled system.

Finally, it is interesting to note that our approach for the systems of two coupled anharmonic oscillators is a generalization of that for a single oscillator with a polynomial potential [2]. This is because the latter system can be treated by simply setting the parameter of the second oscillator in the two coupled oscillator system to zero.

III. ENERGY LEVELS OF A PAIR OF COUPLED ANHARMONIC OSCILLATORS

A. Coupled quartic anharmonic oscillators

We first apply our method to a pair of coupled quartic anharmonic oscillators with the Hamiltonian

$$H = \frac{p_1^2}{2m_1} + \frac{1}{2}m_1\omega_1^2 x_1^2 + \frac{p_2^2}{2m_2} + \frac{1}{2}m_2\omega_2^2 x_2^2 + \lambda(c_{40}x_1^4 + c_{22}x_1^2 x_2^2 + c_{04}x_2^4), \quad (22)$$

which models the dynamics of the five-membered ring compounds [13], and compare our results to those in [12].

We truncate the Hamiltonian matrix at $M = 5, 9, 15,$ and 19. Table I shows that the low-lying energy eigenstates of the symmetrically coupled quartic anharmonic oscillators converge throughout the whole range of λ. We then choose to truncate the basis at $M = 9$ and compare our results to those in [12] (see Table II). Note that E_n is the result of our calculation, while E_n^H is that due to Hioe *et al.* The results for unsymmetrically coupling, i.e., $c_{40} \neq c_{04}$, is shown in Table III. Note that in this paper, all energy eigenvalues are in atomic units such that $\hbar = m_i = 1$ $(i = 1, 2, \ldots, N)$.

B. Coupled cubic anharmonic oscillators

Next, we apply our method to a system of harmonic oscillators coupled by cubic potential

$$H = \frac{p_1^2}{2m_1} + \frac{1}{2}m_1\omega_1^2 x_1^2 + \frac{p_2^2}{2m_2} + \frac{1}{2}m_2\omega_2^2 x_2^2 + \lambda(c_{12}x_1 x_2^2 + c_{30}x_1^3), \quad (23)$$

which has been employed to approximate the normal mode of an anharmonic triatomic oscillator. This system has in fact been used in the study of the semiclassical theory of bound states by Eastes *et al.* [14].

Low-lying energy levels are computed by truncating the Hamiltonian matrix at $M = 9$. Our results $E_{n_1 n_2}$ are found to compare favorably to the numerically computed exact results $E_{n_1 n_2}^W$ of [14] (see Table IV).

C. Coupled oscillators with cubic and quartic anharmonicity

We have also explored the applicability of our approach to a more complicated system of harmonic oscillators coupled

TABLE I. Convergence of the first three states of the system of coupled quartic anharmonic oscillators for the case of $\hbar = m_1 = m_2 = \omega_1 = \omega_2 = c_{40} = c_{04} = 1$, $c_{22} = 2$.

λ	Size of matrix	E_0	E_1	E_2
0.1	$5^2 \times 5^2$	1.150238	2.417194	3.775260
	$9^2 \times 9^2$	1.150188	2.414344	3.772339
	$15^2 \times 15^2$	1.150188	2.414340	3.772322
	$19^2 \times 19^2$	1.150188	2.414340	3.772322
0.5	$5^2 \times 5^2$	1.476477	3.241042	5.203985
	$9^2 \times 9^2$	1.476036	3.231518	5.195597
	$15^2 \times 15^2$	1.476025	3.231453	5.195314
	$19^2 \times 19^2$	1.476025	3.231453	5.195313
1.0	$5^2 \times 5^2$	1.724994	3.844320	6.226393
	$9^2 \times 9^2$	1.724206	3.830464	6.214372
	$15^2 \times 15^2$	1.724184	3.830325	6.213817
	$19^2 \times 19^2$	1.724184	3.830323	6.213815
10.0	$5^2 \times 5^2$	3.304048	7.564173	12.430486
	$9^2 \times 9^2$	3.301300	7.527662	12.398972
	$15^2 \times 15^2$	3.301210	7.527050	12.396828
	$19^2 \times 19^2$	3.301210	7.527043	12.396815
100.0	$5^2 \times 5^2$	6.918703	15.952793	26.312626
	$9^2 \times 9^2$	6.912122	15.870528	26.241507
	$15^2 \times 15^2$	6.911900	15.868991	26.236273
	$19^2 \times 19^2$	6.911899	15.868972	26.236240
5000.0	$5^2 \times 5^2$	25.299783	58.446221	96.495492
	$9^2 \times 9^2$	25.274871	58.139655	96.230311
	$15^2 \times 15^2$	25.274025	58.133767	96.210410
	$19^2 \times 19^2$	25.274022	58.133692	96.210282

by mixed cubic and quartic potential. The Hamiltonian of this system is given below:

$$H = \frac{p_1^2}{2m_1} + \frac{1}{2}m_1\omega_1^2 x_1^2 + \frac{p_2^2}{2m_2} + \frac{1}{2}m_2\omega_2^2 x_2^2 + \lambda(c_{40}x_1^4 + c_{30}x_1^3 + c_{04}x_2^4 + c_{03}x_2^3 + c_{21}x_1^2 x_2 + c_{12}x_1 x_2^2 + c_{22}x_1^2 x_2^2). \quad (24)$$

TABLE II. Energy eigenvalues for the first three states of the system of coupled quartic anharmonic oscillators for the case of $\hbar = m_1 = m_2 = \omega_1 = \omega_2 = c_{40} = c_{04} = 1$, $c_{22} = 2$.

λ	E_0	E_0^H	E_1	E_1^H	E_2	E_2^H
0.05	1.0843	1.0843	2.2388	2.2388	3.4542	3.4542
0.10	1.1502	1.1502	2.4143	2.4143	3.7723	3.7723
0.50	1.4760	1.4760	3.2315	3.2315	5.1956	5.1953
1.00	1.7242	1.7242	3.8305	3.8304	6.2144	6.2140
10.0	3.3013	3.3012	7.5277	7.5271	12.399	12.397
100	6.9121	6.9119	15.871	15.869	26.242	26.237
5000	25.275	25.274	58.140	58.134	96.230	96.213

TABLE III. Energy eigenvalues for the first three states of the system of coupled quartic anharmonic oscillators for the case of $\hbar = m_1 = m_2 = \omega_1 = \omega_2 = 1$, $c_{40} = 0.4, c_{04} = 1, c_{22} = 2$.

λ	E_0	E_0^H	E_1	E_1^H	E_2	E_2^H
0.05	1.0669	1.0669	2.1580		2.2231	
0.10	1.1206	1.1206	2.2818		2.3889	
0.50	1.3965	1.3965	2.8989		3.1691	
1.00	1.6123	1.6123	3.3725		3.7451	
10.0	3.0176	3.0175	6.4059		7.3205	
100	6.2773	6.2772	13.3821		15.4119	
5000	22.913	22.913	48.903		56.438	

Energy eigenvalues are computed by truncating the Hamiltonian matrix at $M=11$. The energy levels of the ground state and the 50th excited state are shown in Fig. 1. The results correspond very well to those computed by Hernandez [15].

D. Coupled quantum pendulums

Finally, we have also applied our approach to a system of two pendulums coupled through the parameter ε. The Hamiltonian of this system is given by [21]

$$H = H_1 + H_2 + H_{int}$$

$$= \sum_{j=1}^{2} \left(\frac{p_j^2}{2} + \alpha(1 - \cos x_j) \right) + \varepsilon[1 - \cos(x_1 - x_2)]$$

$$= \sum_{j=1}^{2} \left(\frac{p_j^2}{2} + \frac{\alpha x_j^2}{2} \right) + \alpha \sum_{m=4,6}^{\infty} (-1)^{m/2-1} \frac{x_j^m}{m!}$$

$$+ \varepsilon \sum_{m=2,4}^{\infty} \sum_{r=0}^{m} (-1)^{m/2-1} \frac{x_1^{m-r} x_2^r}{(m-r)! \, r!}. \tag{25}$$

Notice that we have expressed the cosine potential as an infinite polynomial series about $x_j=0$. In order to apply our approach, we have to approximate the cosine potential by truncating the series at $m=8$. With $M=7$, we have determined the low-lying energy eigenvalues (less than the energy of the separatix $E_s=2\alpha$) from the Heisenberg matrix with $\alpha=50$. The results are displayed in Table V.

IV. APPLICATION OF THE EXTENDED TWO-STEP APPROACH TO SYSTEMS OF THREE AND FOUR COUPLED ANHARMONIC OSCILLATORS

By extending the analysis of Sec. II B, we have determined in this section the eigenenergies of systems of three and four coupled anharmonic oscillators. This has enabled us to further verify the validity of the extended two-step approach to problems of higher dimension.

A. System of three coupled sextic oscillators

Applying the extended two-step approach to a system of three coupled sextic anharmonic oscillators with the following Hamiltonian:

$$H = \frac{p_1^2}{2} + \frac{1}{2}x_1^2 + 2x_1^4 + \frac{1}{2}x_1^6 + \frac{p_2^2}{2} + \frac{1}{2}x_2^2 + 2x_2^4 + \frac{1}{2}x_2^6 + \frac{p_3^2}{2} + \frac{1}{2}x_3^2$$

$$+ 2x_3^4 + \frac{1}{2}x_3^6 + x_1 x_2 + x_1 x_3 + x_2 x_3, \tag{26}$$

we found close convergence of the energy eigenvalues to the results of [22] as shown in Table VI, where we have truncated the Hamiltonian matrix at $M=5$, 9, and 17.

B. System of four coupled sextic oscillators

Next, we apply the approach to a system of four coupled sextic anharmonic oscillators with the following Hamiltonian:

$$H = \frac{p_1^2}{2} + \frac{1}{2}x_1^2 + 2x_1^4 + \frac{1}{2}x_1^6 + \frac{p_2^2}{2} + \frac{1}{2}x_2^2 + 2x_2^4 + \frac{1}{2}x_2^6 + \frac{p_3^2}{2} + \frac{1}{2}x_3^2$$

$$+ 2x_3^4 + \frac{1}{2}x_3^6 + \frac{p_4^2}{2} + \frac{1}{2}x_4^2 + 2x_4^4 + \frac{1}{2}x_4^6 + x_1 x_2 + x_1 x_3 + x_1 x_4$$

$$+ x_2 x_3 + x_2 x_4 + x_3 x_4. \tag{27}$$

We have computed the low-lying energy levels by truncating the Hamiltonian matrix at $M=3$, 5, and 9. Our results are found to compare favorably to those of [23] (see Table VII).

V. SQUEEZING OF THE PRODUCT STATE

In this section, we shall investigate into the effects of nonlinearity and coupling on the squeezing properties of the vacuum product state of the coupled anharmonic oscillators. This shall be performed with respect to the transformed optimal bosonic operators $\bar{b}_j^\dagger$ and $\bar{b}_j$, and its associated squeezed vacuum product state. In Sec. II, we have observed that the transformation to the set of bosonic operators has scaled the angular frequencies of the harmonic potential and

TABLE IV. Energy eigenvalues of the system of coupled cubic anharmonic oscillators for the case of $\hbar = m_1 = m_2 = 1$, $c_{12} = 1$, $c_{30} = 0.1$.

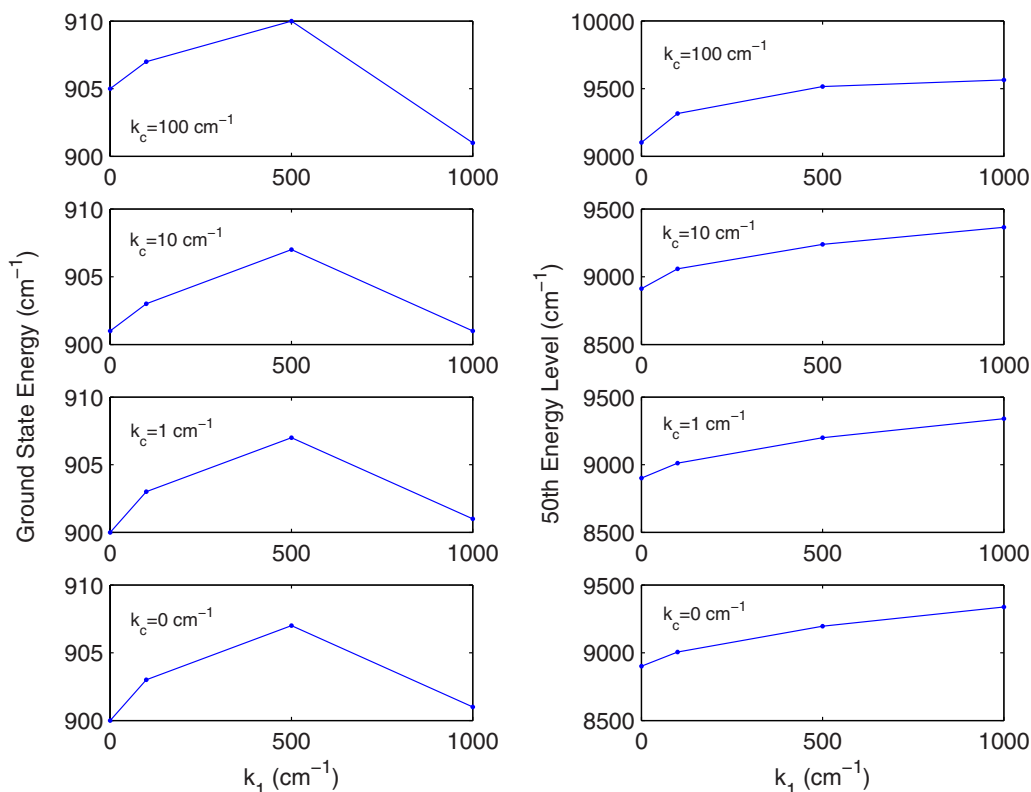| λ	ω_1^2	ω_2^2	E_{00}	E_{00}^W	E_{10}	E_{10}^W	E_{20}	E_{20}^W
−0.10	0.36	1.96	0.9939	0.9939	1.5809	1.5809	2.1613	2.1612
−0.10	0.49	1.69	0.9955	0.9955	1.6870	1.6870	2.3750	2.3750
−0.08	0.81	1.21	0.9980	0.9980	1.8944	1.8944	2.7899	2.7899

FIG. 1. (Color online) Energy eigenvalues of the system of coupled cubic-quartic anharmonic oscillators for the case of $\hbar=m_1=m_2=1$, $\omega_1=1000$ cm^{-1}, $\omega_2=800$ cm^{-1}, $c_{30}=k_1/6$, $c_{40}=k_1/24$, $c_{03}=c_{04}=0$, $c_{12}=c_{21}=k_c/2$, $c_{22}=k_c/4$, $\lambda=1$.

diminished the magnitude of the perturbing anharmonic potential. This implies that one can analyze the rescaled system by means of perturbation theory about the squeezed vacuum product state. By treating the squeezed vacuum state as the zero-order vacuum product state, we examine into the zero-order squeezing effects on the coordinate and momentum uncertainties. After which, we proceed to study into the squeezing effects on these uncertainties for the coupled anharmonic oscillators based on the standard first-order perturbation theory.

A. Zero-order estimation of the coordinate and momentum uncertainties

By using the fact that the squeeze vacuum product state $|\bar{\phi}_1, \bar{\phi}_2\rangle$ is an eigenstate of the TAO, we can simply determine the following expectation values:

TABLE V. Energy eigenvalues of the system of coupled pendulums for the case of $\hbar=1$, $\alpha=50$.

ε	E_{00}	E_{10}, E_{01}	E_{02}, E_{20}	E_{11}
0.0	7.0080	13.9517	20.7651	20.8955
		13.9517	20.7651	
0.5	7.0426	13.9870	20.8395	21.0225
		14.0536	20.8688	
1.0	7.0768	14.0218	20.8867	21.1750
		14.1546	20.9716	

$$\langle x_j \rangle = \langle p_j \rangle = 0, \tag{28}$$

$$\langle a_j^\dagger a_j \rangle = \frac{\bar{t}_j^2}{1-\bar{t}_j^2}, \tag{29}$$

$$\langle a_j^{\dagger 2} \rangle = \frac{\bar{t}_j}{1-\bar{t}_j^2}, \tag{30}$$

$$\langle a_j^2 \rangle = \frac{\bar{t}_j}{1-\bar{t}_j^2}, \tag{31}$$

where $j=1,2$. From Eqs. (28)–(31), we readily obtain the uncertainties of x and p,

$$\sigma_{x_j}^2 = \langle x_j^2 \rangle = \beta_j^2 \left(\frac{1}{2} + \frac{\bar{t}_j}{1-\bar{t}_j} \right) = \frac{\beta_j^2}{2\bar{\Omega}_j}, \tag{32}$$

TABLE VI. Convergence of the energy eigenvalues of the system of three coupled sextic anharmonic oscillators.

Size of matrix	E_0	E_1	E_2
$5^3 \times 5^3$	2.986296	5.339869	5.905519
$9^3 \times 9^3$	2.978875	5.298264	5.867776
$17^3 \times 17^3$	2.978305	5.296000	5.865828
Braun [22]	2.978302	5.295992	5.865822

TABLE VII. Convergence of the energy eigenvalues of the system of four coupled sextic anharmonic oscillators.

Size of matrix	E_0	E_1	E_2
$3^4 \times 3^4$	4.047835	6.523707	7.230376
$5^4 \times 5^4$	3.970225	6.328065	7.057805
$9^4 \times 9^4$	3.960086	6.283305	7.017863
Kaluža [23]	3.959304		

$$\sigma_{p_j}^2 = \langle p_j^2 \rangle = \frac{\hbar^2}{\beta_j^2}\left(\frac{1}{2} - \frac{\bar{t}_j}{1 + \bar{t}_j}\right) = \frac{\hbar^2 \bar{\Omega}_j}{2\beta_j^2}. \quad (33)$$

This gives the zero-order estimation of the coordinate and momentum uncertainties. Equation (32) shows that coordinate squeezing occurs when $-1 < \bar{t}_j < 0$, while Eq. (33) shows that momentum squeezing occurs when $0 < \bar{t}_j < 1$. Since the product of the uncertainties σ_{x_j} and σ_{p_j} is always equal to $\hbar/2$, the squeezed vacuum product state is a coherent state. The same results can also be obtained by operating the position and momentum operators on the squeezed vacuum state with the new bosonic operators $\bar{b}_j^\dagger$ and $\bar{b}_j$ as follows:

$$\langle \bar{\phi}_j | x_j | \bar{\phi} \rangle = \frac{\beta_j}{\sqrt{2\bar{\Omega}_j}} \langle \bar{\phi}_j | \bar{b}_j^\dagger + \bar{b}_j | \bar{\phi}_j \rangle, \quad (34)$$

$$\langle \bar{\phi}_j | p_j | \bar{\phi}_j \rangle = \frac{i\hbar\sqrt{\bar{\Omega}_j}}{\beta_j\sqrt{2}} \langle \bar{\phi}_j | \bar{b}_j^\dagger - \bar{b}_j | \bar{\phi}_j \rangle. \quad (35)$$

B. Position and momentum uncertainties by first-order perturbation theory

Since the squeezed vacuum product state is not the energy eigenstate of the Hamiltonian of the coupled anharmonic oscillators, it cannot adequately describe the ground state of these oscillators. Hence, a first-order perturbation analysis is required to determine their coordinate and momentum uncertainties. In this section, we shall first perform the perturbation analysis on a coupled harmonic oscillators as a check on accuracies, before performing the evaluation on a coupled quartic anharmonic oscillators.

1. Coupled harmonic oscillators

The Hamiltonian for a system of coupled harmonic oscillators can be written as follows:

$$H = \frac{p_1^2}{2m_1} + \frac{1}{2}m_1\omega_1^2 x_1^2 + \frac{p_2^2}{2m_2} + \frac{1}{2}m_2\omega_2^2 x_2^2 + \lambda(c_{20}x_1^2 + c_{11}x_1x_2 + c_{02}x_2^2). \quad (36)$$

By transforming to the harmonic oscillator states characterized by the bosonic operators $\bar{b}_j^\dagger$ and $\bar{b}_j$, the Hamiltonian can be formulated into an unperturbed H_0 and a small perturbation H_1 quantified by the parameter λ', as was done in Sec. II for the general case,

$$H = H_0 + \lambda' H_1, \quad (37)$$

where

$$H_0 = \sum_{j=1}^{2} \hbar\omega_j \bar{\Omega}_j \left(\frac{1}{2} + \bar{b}_j^\dagger \bar{b}_j\right), \quad (38)$$

$$H_1 = (\bar{b}_2^\dagger \bar{b}_1^\dagger + \bar{b}_2^\dagger \bar{b}_1 + \bar{b}_2 \bar{b}_1^\dagger + \bar{b}_2 \bar{b}_1), \quad (39)$$

and $\lambda' = \lambda c_{11}\beta_1\beta_2/(2\sqrt{\bar{\Omega}_1 \bar{\Omega}_2})$.

Next, we expand the vacuum state $|0\ 0\rangle$ of the coupled harmonic oscillators about the optimal squeezed vacuum state $|\overline{00}^{(0)}\rangle$ perturbatively as follows:

$$|00\rangle = |\overline{00}^{(0)}\rangle + \lambda'|\overline{00}^{(1)}\rangle + \lambda'^2|\overline{00}^{(2)}\rangle + \cdots. \quad (40)$$

This allows us to calculate the position and momentum uncertainties for the first oscillator perturbatively,

$$\sigma_x = \sqrt{\langle 00|x_1^2|00\rangle - \langle 00|x_1|00\rangle^2}, \quad (41)$$

$$\sigma_p = \sqrt{\langle 00|p_1^2|00\rangle - \langle 00|p_1|00\rangle^2}, \quad (42)$$

up to first order. Our calculated results for σ_x and σ_p are shown in Table VIII. Compared to those calculated by MacDermott and Redmount [11] (σ^R), our first-order approximations are not as accurate for larger λ, but are sufficiently good for us to investigate into the squeezing of the state. To obtain more accurate results, one has to perform higher-order perturbative calculations. From Table VIII, we observe that our results lead to the same conclusion as MacDermott: the coupling squeezes the uncertainty in x, while it raises the uncertainty in p. The overall effect is an increase in the product of the uncertainties.

2. Coupled quartic anharmonic oscillators

Let us consider again the pair of coupled quartic anharmonic oscillators with Hamiltonian given by Eq. (22),

TABLE VIII. Position and momentum uncertainties of the system of coupled harmonic oscillators for the case of $\hbar = m_1 = m_2 = \omega_1 = \omega_2 = 1$, $c_{20} = 0.5$, $c_{02} = 0.5$, $c_{11} = -1$.

λ	σ_x ($\sqrt{\hbar/2}$)	σ_x^R ($\sqrt{\hbar/2}$)	σ_p ($\sqrt{\hbar/2}$)	σ_p^R ($\sqrt{\hbar/2}$)	$\sigma_x\sigma_p$ ($\hbar/2$)	$\sigma_x^R\sigma_p^R$ ($\hbar/2$)
0.05	0.98809	0.98830	1.01249	1.01213	1.00043	1.00029
0.10	0.97721	0.97798	1.02491	1.02385	1.00155	1.00130
0.50	0.91297	0.92388	1.11815	1.09868	1.02083	1.01505
1.00	0.86038	0.88807	1.21676	1.16877	1.04688	1.03795

TABLE IX. Comparison of the position and momentum uncertainties of the system of coupled harmonic oscillators (CHO), the system of coupled anharmonic oscillators (CAHO), and the single anharmonic oscillator (SAHO).

Oscillator	σ_x $(\sqrt{\hbar/2})$	σ_p $(\sqrt{\hbar/2})$	$\sigma_x\sigma_p$ $(\hbar/2)$
CHO	0.8604	1.2168	1.0469
CAHO	0.6815	1.4763	1.0060
SAHO	0.7089	1.4178	1.0051

$$H = \frac{p_1^2}{2} + \frac{1}{2}x_1^2 + \frac{p_2^2}{2} + \frac{1}{2}x_2^2 + \lambda(c_{40}x_1^4 + c_{22}x_1^2x_2^2 + c_{04}x_2^4).$$
(43)

As before, we formulate the Hamiltonian into the unperturbed harmonic oscillator part H_0 and a small perturbation H_1 quantified by λ',

$$H = H_0 + \lambda' H_1,$$
(44)

where

$$H_0 = E_0 + \sum_{j=1}^{2} \bar{\Omega}_j \bar{b}_j^\dagger \bar{b}_j,$$
(45)

$$E_0 = \sum_{j=1}^{2} \frac{3\bar{\Omega}_j^2 + 1}{8\bar{\Omega}_j},$$
(46)

$$H_1 = \frac{c_{40}}{\bar{\Omega}_1^2}:(\bar{b}_1^\dagger + \bar{b}_1)^4: + \frac{c_{04}}{\bar{\Omega}_2^2}:(\bar{b}_2^\dagger + \bar{b}_2)^4: + \frac{c_{22}}{\bar{\Omega}_1\bar{\Omega}_2}(\bar{b}_1^{\dagger 2}\bar{b}_2^{\dagger 2} + \bar{b}_1^{\dagger 2}\bar{b}_2^2$$
$$+ \bar{b}_2^2\bar{b}_2^{\dagger 2} + \bar{b}_1^2\bar{b}_2^2 + 2\bar{b}_1^{\dagger 2}\bar{b}_2^\dagger\bar{b}_2 + 2\bar{b}_1^2\bar{b}_2^\dagger\bar{b}_2 + 2\bar{b}_2^{\dagger 2}\bar{b}_1^\dagger\bar{b}_1$$
$$+ 2\bar{b}_2^2\bar{b}_1^\dagger\bar{b}_1 + 4\bar{b}_1^\dagger\bar{b}_1\bar{b}_2^\dagger\bar{b}_2),$$
(47)

and $\lambda' = \lambda/4$. Note that the symbol : : denotes normal ordering.

We have again expanded the ground state perturbatively about the squeezed vacuum state. Although a perturbation series in integral powers of λ for the quartic anharmonic oscillators is known to be a divergent series, our approach has rescaled the value of λ and reduced the magnitude of the perturbing anharmonic potential such that a first-order perturbative analysis becomes amenable [24,25]. The resulting truncated series with a variational approach turns out to be convergent, as was exemplified by the converging results of the energy eigenvalues of the coupled quartic oscillators reported in Sec. III A. We have calculated the position and momentum uncertainties of the first oscillator for $\lambda = 1$ based on the standard first-order perturbative analysis. Our result is displayed in Table IX. We have also investigated the individual effects of anharmonicity and coupling on the squeezing properties of the vacuum product state of the coupled quartic oscillators through setting the corresponding parameters to zero. These results are shown in Figs. 2 and 3, respectively, where we have plotted the normalized $\bar{\sigma}_x = \sigma_x/\sigma_x^c$ and $\bar{\sigma}_p = \sigma_p/\sigma_p^c$, as well as their product. Note that σ_x
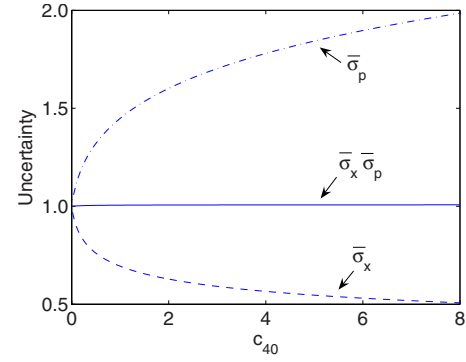

FIG. 2. (Color online) Effects of anharmonicity on the squeezing properties of the vacuum product state of the coupled quartic oscillators when $\lambda = 1$. The degree of anharmonicity is quantified by the dimensionless parameter c_{40}. The effect of coupling has been switched off by setting $c_{22} = 0$.

and σ_p are the position and momentum uncertainties of the vacuum product state of the coupled quartic oscillators, while σ_x^c and σ_p^c are the position and momentum uncertainties of the coherent state.

Compared to the uncertainties of a single anharmonic oscillator, the coupling in the coupled anharmonic oscillators has the effect of decreasing the position uncertainty and increasing the momentum uncertainty of the individual oscillator. In addition, it also has the effect of increasing the uncertainty product without violating the Heisenberg uncertainty principle. By comparing with results obtained from the system of coupled harmonic oscillators, the effect of nonlinearity in the anharmonic potential of the coupled anharmonic oscillator is to enhance the squeezing and antisqueezing of the position and momentum uncertainties, respectively, without any significant increase in the uncertainty product.

It is important to note that although the observations given above show that the effect of coupling and anharmonicity leads to a squeezing in x and an antisqueezing in p, this is not always the case. A perturbative calculation for the coupled quantum pendulum given in Sec. III shows that it can indeed happen the other way round, with an antisqueez-
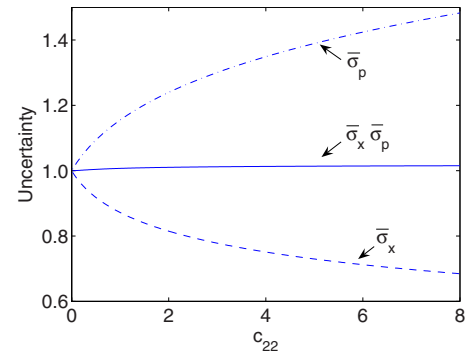

FIG. 3. (Color online) Effects of coupling on the squeezing properties of the vacuum product state of the coupled quartic oscillators when $\lambda = 1$. The degree of coupling is quantified by the dimensionless parameter c_{22}. The effect of anharmonicity has been switched off by setting $c_{40} = c_{04} = 0$.

ing in x ($\sigma_x = 1.009\,04\sqrt{\hbar/2}$) and a squeezing in p ($\sigma_p = 0.991\,04\sqrt{\hbar/2}$).

VI. CONCLUSION

By applying the Bogoliubov transform on the bosonic operators of each oscillator of a system of coupled anharmonic oscillators, we derive a basis set for the description of the quantum states of systems of coupled oscillators. The vacuum state of this basis set is a squeezed coherent product state and is parametrized by a set of independent parameters. We select the optimal squeezed vacuum product state by minimizing the energy of the coupled anharmonic oscillators. This enables us to construct an optimal basis set to evaluate the energy eigenvalues of the systems of N coupled anharmonic oscillators. We have applied this formalism to various systems of two, three and four coupled quantum anharmonic oscillators. We have found that the energy eigenvalues computed by our approach agree closely with those obtained from the literature. In addition, we have also investigated the squeezing of the coupled oscillators vacuum state by performing a first-order perturbation analysis about the optimal squeezed vacuum product state. Our results show that the vacuum state of the coupled nonlinear oscillators is not a coherent state. Furthermore, we have found that the effect of coupling and anharmonicity is to enhance the squeezing in the state without any significant change to the uncertainty product.

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