Regularization of the singular inverse square potential in quantum mechanics with a minimal length

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We study the problem of the attractive inverse square potential in quantum mechanics with a generalized uncertainty relation. Using the momentum representation, we show that this potential is regular in this framework. We solve analytically the *s*-wave bound states equation in terms of Heun's functions. We discuss in detail the bound states spectrum for a specific form of the generalized uncertainty relation. The minimal length may be interpreted as characterizing the dimension of the system.

I. INTRODUCTION

It is well known that in quantum gravity and string theory, there is a lower bound to the possible resolution of distances, i.e., a minimal observable length on the scale of the Planck length of 10^{-35} m. This minimal length may be introduced as an additional uncertainty in position measurements, so that the standard Heisenberg uncertainty relation becomes $(\Delta X) \times (\Delta P) \geq \frac{\hbar}{2}[1 + \beta(\Delta P)^2 + \cdots]$, where β is a small positive parameter [1–3]. It is clear that in this new relation, (ΔX) is always larger than $(\Delta X)_{\min} = \hbar\sqrt{\beta}$. It was shown in Refs. [4–7] that the introduction of specific corrections to the usual canonical commutation relations between position and momentum operators imply this new generalized uncertainty relation in a natural way. This formalism, based on a noncommutative Heisenberg algebra, together with the new concepts it implies, has been discussed in one and more dimensions [4]. Quantum field theory (QFT) has also been reformulated within this framework, and it has been shown, in particular, that this minimal length may regularize unwanted divergencies [8,9].

In addition to its importance in QFT, a minimal length may be of great interest in nonrelativistic or relativistic quantum mechanics. Indeed, it has been argued [5,10] that this length may be viewed as an intrinsic scale characterizing the system under study. Consequently, the formalism based on these deformed commutation relations may provide a new model for an effective description of complex systems such as quasiparticles and various collective excitations in solids, or composite particles such as nucleons, nuclei, and molecules [5]. Various topics were studied over the last ten years, in connection with this formalism: the spectrum of the hydrogen atom has been obtained perturbatively in coordinate space by several authors [11–14], whereas its momentum space treatment was done in Ref. [15]. The authors found an upper bound of about 0.1 fm for the minimal length by exploiting the experimental data from precision hydrogen spectroscopy (the Lamb shift). The harmonic oscillator potential has also been solved exactly in arbitrary dimensions [16] and perturbatively [4,5,11]. In Ref. [16], an upper bound

for the minimal length has been calculated by confronting theoretical results to precision measurement of electrons trapped in a strong magnetic field; it is of the same order of magnitude as the result obtained in the hydrogen atom problem. The influence of the minimal length on the Casimir energy between two parallel plates has also been examined [17,18]. The problem of a charged particle of spin one-half moving in a constant magnetic field has been treated within the minimal length formalism, and the thermal properties of the system at high temperatures have been investigated [19]. The minimal length was introduced in the Dirac equation in Ref. [20], where a one-dimensional Dirac oscillator has been solved exactly; in three dimensions, this problem has been solved using supersymmetric quantum mechanics [21]. Finally, the modifications of the gyromagnetic moment of electrons and muons due to the minimal length have been discussed in Ref. [22]. For a review of different approaches of theories with a minimal length scale and the relation between them, we refer the reader to Ref. [23].

In this paper, we study the effect of a minimal length in nonrelativistic quantum mechanics with a potential $V(R)$ of the form $V(R) = -\alpha/R^2$ with $2m\alpha/\hbar^2 > 1/4$ (m is the particle mass). Such a potential is singular when used in conjunction with the usual Schrödinger equation. Specifically, the condition of square integrability of the wave function does not lead to an orthogonal set of eigenfunctions with their corresponding eigenvalues [24,25]. This is due to the fact that the Hamiltonian operator is not self-adjoint [26]; to cure this illness, we must define self-adjoint extensions of the Hamiltonian or equivalently require orthogonality of the wave functions [24]. However, the obtained spectrum is a peculiar one, as the energy eigenvalue may take values from 0 to $-\infty$, so that there is no finite ground state. Landau and Lifshitz associate the occurrence of this infinite bound state to the classical fall to the center of the particle [27]. In addition to this fundamental problem, the expression of the energy spectrum depends on an arbitrary phase parameter, coming from restoring the self-adjointness of the Hamiltonian. For a review of works concerning this potential, we refer the reader to Refs. [30,31].

From a physical point of view, the strongly attractive $1/R^2$ potential is very interesting. Indeed, the problem of atoms interacting with a charged wire, relevant to the fabrication of nanoscale atom optical devices, is known to provide an experimental realization of an attractive $1/R^2$ potential [32,33]. It is a fundamental (long range) part of the potential describing dipole-bound anions in polar molecules [34], and has some applications in black holes physics [35]. Finally, let us note that the Efimov effect in three-body systems [36] arises from the existence of a long range effective interaction $V(R)$ of the form $V(R) \sim c/R^2$ (c some constant), where R is built from the relative distances between the three particles. Further interest in the singular inverse square potential also arose from recent studies showing that it provides a simple example of a renormalization group limit cycle in nonrelativistic quantum mechanics [37–39]. We also mention, for the sake of completeness, other works on the regularization and the renormalization of this potential [40–42].

In this work we study in detail how the introduction of a generalized uncertainty relation regularizes the singular inverse square potential in nonrelativistic quantum mechanics. We show, in particular, that the "elementary length" included in these relations may be interpreted as an effective cutoff regularizing the potential at large momenta. It follows that in this new framework the existence of an elementary length regularizes the $1/R^2$ potential, without introducing any arbitrary cutoff.

Our paper is organized as follows. In Sec. II, we study the attractive $1/R^2$ potential in ordinary quantum mechanics, using the momentum representation. In Sec. III, we derive the corresponding equations in quantum mechanics with a modified uncertainty relation. In Sec. IV, within the formalism of deformed Heisenberg algebra, we solve exactly the Schrödinger equation and extract the energy spectrum. Some concluding remarks are reported in the last section.

II. SINGULAR ATTRACTIVE $1/R^2$ POTENTIAL IN ORDINARY QUANTUM MECHANICS

The singular attractive inverse square potential has been extensively studied in the coordinate representation (see, for instance, [24,25,28–30]). In Ref. [25], the expression of the momentum wave function was given as a Fourier transform of the wave function in configuration space. We use here a simple method for dealing with the attractive $1/R^2$ potential in momentum space, as first applied to the hydrogen atom potential [44].

A. Schrödinger equation in momentum representation

We write the Schrödinger equation for a particle of mass m in the external potential $V(R) = -\alpha/R^2$, $\alpha > 0$ in the form

$$(R^2 P^2 - 2m\alpha)|\psi\rangle = 2mER^2|\psi\rangle, \qquad (1)$$

where $\vec{R}$ and $\vec{P}$ are, respectively, the position and momentum operators. In the momentum representation, the wave function reads [16]

$$\psi(\vec{p}) = Y_{lm}(\theta, \varphi)\psi(p).$$

Without loss of generality, we restrict ourselves to s waves. One then has

$$R^2\psi(p) = -\hbar^2\left(\frac{\partial^2}{\partial p^2} + \frac{2}{p}\frac{\partial}{\partial p}\right)\psi(p),$$

$$P^2\psi(p) = p^2\psi(p).$$

From Eq. (1), we obtain the following differential equation:

$$\frac{d^2\psi}{dp^2} + \frac{2}{p}\left(\frac{3p^2 + k^2}{p^2 + k^2}\right)\frac{d\psi}{dp} + \left(\frac{6 + 2m\alpha/\hbar^2}{p^2 + k^2}\right)\psi = 0, \qquad (2)$$

where $k^2 = -2mE$.

Introducing the dimensionless variable y, defined by

$$y = -\frac{p^2}{k^2},$$

the Schrödinger equation (2) takes the following form:

$$y(1-y)\frac{d^2\psi}{dy^2} + \left(\frac{3}{2} - \frac{7}{2}y\right)\frac{d\psi}{dy} - \frac{1}{2}\left(3 + \frac{m\alpha}{\hbar^2}\right)\psi = 0. \qquad (3)$$

This equation is in the form of a hypergeometric equation [45] as follows:

$$y(1-y)\frac{d^2\psi}{dy^2} + [c - (a+b+1)y]\frac{d\psi}{dy} - ab\psi = 0,$$

with the parameters

$$a = \frac{5}{4} + \frac{i}{2}\nu,$$

$$b = \frac{5}{4} - \frac{i}{2}\nu,$$

$$c = \frac{3}{2}, \quad \text{where } \nu = \sqrt{\frac{2m\alpha}{\hbar^2} - 1/4}. \qquad (4)$$

The solution to Eq. (3) finite for $p = 0$ is [45]

$$\psi(p) = AF\left(a, b, c; -\frac{p^2}{k^2}\right), \qquad (5)$$

where A is a normalization constant. This solution was obtained in Ref. [25] by taking the Fourier transform of the configuration space wave function $\psi(r)$, with

$$\psi(r) = Ar^{-1/2}K_{i\nu}(kr),$$

where $K_{i\nu}$ is the modified Bessel function.

Let us now examine the asymptotic behavior of solution (5) in the vicinity of $p = 0$ and $p \to \infty$. For $p = 0$, one has $\psi(p) = $ finite constant, as $F(a, b, c; y) \approx_{y \ll 1} 1$; so, it is quadratically integrable at the origin. In the limit $p \to \infty$, by means of the transformation [45]

$$F(a,b,c;y)$$

$$= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-y)^{-a}F\left(a,1-c+a,1-b+a;\frac{1}{y}\right)$$

$$+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-y)^{-b}F\left(b,1-c+b,1-a+b;\frac{1}{y}\right), \tag{6}$$

the wave function (5) is written as

$$\psi(p) = \frac{\Gamma(3/2)\Gamma(-i\nu)}{\Gamma\left(5/4-\dfrac{i\nu}{2}\right)\Gamma\left(1/4-\dfrac{i\nu}{2}\right)}\left(\frac{p}{k}\right)^{-5/2-i\nu}$$

$$\times F\left(a,1-c+a,1-b+a;-\frac{k^2}{p^2}\right)$$

$$+ \frac{\Gamma(3/2)\Gamma(i\nu)}{\Gamma\left(5/4+\dfrac{i\nu}{2}\right)\Gamma\left(1/4+\dfrac{i\nu}{2}\right)}\left(\frac{p}{k}\right)^{-5/2+i\nu}$$

$$\times F\left(b,1-c+b,1-a+b;-\frac{k^2}{p^2}\right). \tag{7}$$

Then the behavior of $\psi(p)$ at infinity is of the form

$$\psi(p) \underset{p\to\infty}{\sim} p^{-5/2}(Ap^{-i\nu} + Bp^{+i\nu}), \tag{8}$$

where A and B are complex constants.

Solution (7) is a linear combination of two solutions that behave in the same manner at infinity and, both of them, are quadratically integrable. Usually the integrability condition suffices to distinguish between the two independent solutions, but this is not the case here. From Eq. (8), one can see that the wave function depends on an arbitrary phase φ as $\psi(p)\underset{p\to\infty}{\sim}p^{-5/2}\cos(\nu\ln p+\varphi)$, for real $\psi(p)$, and then it has an infinite number of oscillations as $p\to\infty$. As was expected, the oscillating behavior of $\psi(p)$ at infinity is analogous to the oscillating behavior of the configuration space wave function $\psi(r)$ in the neighborhood of the origin (see, for example, Ref. [24]).

B. Integral equation

For later comparison with the solution of the Schrödinger equation with a minimal length, we derive now an integral equation equivalent to Eq. (2). Let us observe that Eq. (1) can be written in the form

$$[L+g(p)]\varphi(p) = 0,$$

where

$$\varphi(p) = (p^2+k^2)\psi(p),$$

$$g(p) = \frac{2m\alpha}{\hbar^2}\frac{p^2}{(p^2+k^2)},$$

and L is the self-adjoint operator

$$L = -\frac{p^2}{\hbar^2}R^2 = \frac{d}{dp}\left(p^2\frac{d}{dp}\right). \tag{9}$$

Then $\varphi(p)$, satisfying the boundary conditions $\varphi(0)$ =constant and $\varphi(\infty)=0$, is given by [48]

$$\varphi(p) = \int_0^\infty G(p,p')g(p')\varphi(p')dp'. \tag{10}$$

The Green's function $G(p,p')$ is then given by

$$G(p,p') = \frac{\theta(p-p')}{p} + \frac{\theta(p'-p)}{p'}, \tag{11}$$

and the integral equation satisfied by the wave function $\psi(p)$ is

$$(p^2+k^2)\psi(p) = \frac{2m\alpha}{\hbar^2}\int_0^\infty p'^2\psi(p')G(p,p')dp'. \tag{12}$$

This equation can also be obtained by calculating the Fourier transform of the potential and inserting it in the s-wave integral Schrödinger equation and then integrating over the angles [43].

Note that putting $\psi(p)\sim p^s$ in Eq. (12), we get

$$p^{s+2} \underset{p\sim\infty}{=} \frac{2m\alpha}{\hbar^2}\left\{\frac{1}{p}\int^{p\sim\infty} p'^{s+2}dp' + \int_{p\sim\infty} p'^{s+1}dp'\right\}.$$

After integration we get the characteristic equation

$$1 = \frac{2m\alpha}{\hbar^2}\left[\frac{1}{s+3} - \frac{1}{s+2}\right],$$

which has two roots $s=-\frac{5}{2}\pm i\nu$, corresponding to the two solutions (8).

This is the momentum space illustration of the singular nature of the potential $-\alpha/R^2$: Eq. (12) has square integrable solutions for any value of $k^2>0$.

C. Energy spectrum

For the sake of completeness, we now show, following [49], how a spectrum can be obtained by requiring the functions $\psi(p)$ to be mutually orthogonal.

1. Orthogonality of the eigenfunctions

Let us consider two eigenfunctions $\psi_1(p)$ and $\psi_2(p)$ corresponding, respectively, to the eigenvalues k_1 and k_2. The scalar product between these two functions reads

$$\langle\psi_1|\psi_2\rangle = A_1A_2^*\int_0^\infty p^2 dp\,F\left(a,b,c;-\frac{p^2}{k_1^2}\right)F\left(a,b,c;-\frac{p^2}{k_2^2}\right). \tag{13}$$

Introducing the change of variable $x=p^2$ and using the formula [47]

$$\int_0^\infty x^{c-1}F(a,b,c;-\sigma x)F(a',b',c;-\omega x)dx = \sigma^{-a}\omega^{a-c}\frac{[\Gamma(c)]^2\Gamma(a+a'-c)\Gamma(a+b'-c)\Gamma(a'+b-c)\Gamma(b+b'-c)}{\Gamma(a)\Gamma(b)\Gamma(a')\Gamma(b')\Gamma(a+a'+b+b'-2c)}$$

$$\times F\left(a+a'-c,a+b-c,a+a'+b+b'-2c;1-\frac{\omega}{\sigma}\right),$$

$$\mathrm{Re}\,c, \quad \mathrm{Re}(a+a'-c), \quad \mathrm{Re}(a+b'-c),$$
$$\mathrm{Re}(a'+b-c), \quad \mathrm{Re}(b+b'-c) > 0,$$

$$|\arg\sigma|, \quad |\arg\omega| < \pi,$$

we obtain

$$\langle\psi_1|\psi_2\rangle = \Omega\left(\frac{k_1}{k_2}\right)^{i\nu}F\left(1+i\nu,1,2;1-\frac{k_1^2}{k_2^2}\right),$$

where

$$\Omega = \frac{1}{2}A_1A_2^*k_1^{5/2}k_2^{1/2}\frac{\left[\Gamma\left(\frac{3}{2}\right)\right]^2[\Gamma(1)]^2\Gamma(1+i\nu)\Gamma(1-i\nu)}{\Gamma(2)\left[\Gamma\left(\frac{5}{4}+i\frac{\nu}{2}\right)\right]^2\left[\Gamma\left(\frac{5}{4}-i\frac{\nu}{2}\right)\right]^2}.$$

Using the formula [45]

$$F(a,b,c;z) = \frac{1}{b-1-(c-a-1)z}[(b-c)F(a,b-1,c;z)$$
$$+ (c-1)(1-z)F(a,b,c-1;z)],$$

and

$$F(a,b,b;z) = (1-z)^{-a}, \quad F(0,b,c;z) = F(a,0,c;z) = 1,$$

we get, finally, the following expression for the scalar product:

$$\langle\psi_1|\psi_2\rangle = \frac{\Omega}{i\nu\left(\frac{k_1^2}{k_2^2}-1\right)}\left[\left(\frac{k_1}{k_2}\right)^{+i\nu}-\left(\frac{k_1}{k_2}\right)^{-i\nu}\right]$$

$$= \frac{2\Omega}{\nu\left(\frac{k_1^2}{k_2^2}-1\right)}\sin\left[\nu\ln\left(\frac{k_1}{k_2}\right)\right]. \tag{14}$$

It is clear that ψ_1 and ψ_2 are orthogonal, if the following condition is satisfied:

$$\nu\ln\left(\frac{k_1}{k_2}\right) = n\pi, \quad n = 0, \pm 1, \ldots. \tag{15}$$

This condition leads to the following discrete spectrum:

$$E_n = E_1\exp\left[-\frac{2n\pi}{\nu}\right], \quad n = 0, \pm 1, \ldots. \tag{16}$$

It is the same result as obtained in coordinate space by Case [24]. Thus a requirement that the state functions for bound states, for $2m\alpha/\hbar^2 > 1/4$, be a mutually orthogonal

set imposes a quantization of energy. It does not uniquely fix the levels, but it fixes the levels relative to one another. If we fix E_1, then the bound levels extend to $-\infty$ and have an accumulation point at zero energy [49].

Now we show that the energy spectrum can be obtained by introducing a momentum space cutoff $\Lambda \gg k$ with the boundary condition $\psi(\Lambda) = 0$. We note that this regularization procedure was used in Refs. [40–42], in coordinate space. This regularization is equivalent to replacing the potential at short distances with an infinitely repulsive barrier.

2. Regularization by an ultraviolet cutoff

Let us go back to the wave function (7), by writing the boundary condition $\psi(\Lambda) =_{\Lambda \gg k} 0$. Bearing in mind that $F(a,b,c;y) \approx_{y \ll 1} 1$, we obtain

$$\left(\frac{\Lambda}{k}\right)^{-5/2-i\nu}\exp[-i\arg(A)] + \left(\frac{\Lambda}{k}\right)^{-5/2+i\nu}\exp[i\arg(A)] = 0, \tag{17}$$

where

$$A \equiv \frac{\Gamma(i\nu)}{\Gamma\left(5/4+\frac{i\nu}{2}\right)\Gamma\left(1/4+\frac{i\nu}{2}\right)} = |A|\exp[i\arg(A)],$$

Eq. (17) can be written as

$$\cos\left[\arg(A) + \nu\ln\left(\frac{\Lambda}{k}\right)\right] = 0, \tag{18}$$

which gives the following bound states:

$$E_n = -\frac{k^2}{2m} = -\frac{\Lambda^2}{2m}\exp\frac{2}{\nu}\left[\arg(A) - \left(n+\frac{1}{2}\right)\pi\right],$$

$$n = 0, +1, +2, \ldots. \tag{19}$$

Consequently, this regularization leads to a quantized energy spectrum, which now possesses a finite ground state for the singular attractive $1/R^2$ potential.

III. QUANTUM MECHANICS WITH A GENERALIZED UNCERTAINTY RELATION

Let us consider the following modified commutation relation between the position and momentum operators:

$$[\hat{X},\hat{P}] = i\hbar(1+\beta\hat{P}^2), \quad \beta > 0. \tag{20}$$

This commutation relation leads to the generalized uncertainty relation [4]

$$(\Delta X)(\Delta P) \geqslant \frac{\hbar}{2}[1 + \beta(\Delta P)^2 + \beta\langle\hat{P}\rangle^2], \qquad (21)$$

which implies a lower bound for ΔX or a minimal length, given by

$$(\Delta X)_{\min} = \hbar\sqrt{\beta}. \qquad (22)$$

The striking feature of Eq. (21) is the UV/IR mixing: when ΔP is large (UV), ΔX is proportional to ΔP and, therefore, is also large (IR). This phenomenon is said to be necessary to understand the cosmological constant problem or the observable implications of short distance physics on inflationary cosmology; it has appeared in several contexts, for example, in noncommutative field theory [46]. Another fundamental consequence of the minimal length is the loss of localization in coordinate space, so that momentum space is more convenient in order to solve any eigenvalue problem.

An explicit form for $\hat{X}$ and $\hat{P}$ satisfying Eq. (20) is given by

$$\hat{X} = i\hbar\left[(1 + \beta p^2)\frac{\partial}{\partial p} + \gamma p\right],$$

$$\hat{P} = p, \qquad (23)$$

where a constant γ does not affect the observable quantities; it determines only the weight function in the definition of the scalar product [16] as follows:

$$\langle\varphi|\psi\rangle = \int_{-\infty}^{+\infty} \frac{dp}{(1 + \beta p^2)^{1-\gamma/\beta}}\varphi^*(p)\psi(p). \qquad (24)$$

A generalization of Eq. (20) to D dimensions is [4,5,16,46]

$$[\hat{X}_i, \hat{P}_j] = i\hbar[(1 + \beta\hat{P}^2)\delta_{ij} + \beta'\hat{P}_i\hat{P}_j], \quad (\beta, \beta') > 0. \qquad (25)$$

If we assume that

$$[\hat{P}_i, \hat{P}_j] = 0, \qquad (26)$$

then the Jacobi identity determines the commutation relations among the coordinates $\hat{X}_i$ as

$$[\hat{X}_i, \hat{X}_j] = i\hbar\frac{2\beta - \beta' + \beta(2\beta + \beta')\hat{P}^2}{1 + \beta\hat{P}^2}(\hat{P}_i\hat{X}_j - \hat{P}_j\hat{X}_i). \qquad (27)$$

The generalized uncertainty relation implied by Eq. (25) is

$$(\Delta X_i)(\Delta P_i)$$
$$\geqslant \frac{\hbar}{2}\left(1 + \beta\sum_{j=1}^{D}[(\Delta P_j)^2 + \langle\hat{P}_j\rangle^2] + \beta'[(\Delta P_i)^2 + \langle\hat{P}_i\rangle^2]\right). \qquad (28)$$

This relation leads to a lower bound of ΔX_i, given by

$$(\Delta X_i)_{\min} = \hbar\sqrt{(D\beta + \beta')}, \quad \forall\ i. \qquad (29)$$

In the momentum representation, the following realization satisfies the above commutation relations:

$$\hat{X}_i = i\hbar\left((1 + \beta p^2)\frac{\partial}{\partial p_i} + \beta'p_ip_j\frac{\partial}{\partial p_j} + \gamma p_i\right),$$

$$\hat{P}_i = p_i. \qquad (30)$$

As in one dimension, the arbitrary constant γ does not affect the observable quantities; its choice determines the weight factor in the definition of the scalar product as follows:

$$\langle\varphi|\psi\rangle = \int \frac{d^D p}{[1 + (\beta + \beta')p^2]^{1-\alpha}}\varphi^*(p)\psi(p),$$

$$\alpha = \frac{\gamma - \beta'\left(\dfrac{D-1}{2}\right)}{\beta + \beta'}. \qquad (31)$$

IV. SINGULAR ATTRACTIVE $1/R^2$ POTENTIAL IN QUANTUM MECHANICS WITH A GENERALIZED UNCERTAINTY RELATION

A. Schrödinger equation

We proceed, as in Sec. II, by writing the Schrödinger equation, for a particle of mass m in the external potential $V(R) = -\alpha/R^2$, $\alpha > 0$, in the form

$$(R^2P^2 - 2m\alpha)|\psi\rangle = 2mER^2|\psi\rangle. \qquad (32)$$

Restricting ourselves to the $l=0$ wave function and using Eq. (30) with $\gamma = 0$, we obtain the following expression for $R^2 \equiv \Sigma_{i=1}^3 X_i X_i$:

$$R^2 = (i\hbar)^2\left\{[1 + (\beta + \beta')p^2]^2\frac{d^2}{dp^2}\right.$$
$$\left. + [1 + (\beta + \beta')p^2]\left[2(2\beta + \beta')p + \frac{2}{p}\right]\frac{d}{dp}\right\}. \qquad (33)$$

From Eqs. (32) and (33) the Schrödinger equation for the $-\alpha/R^2$ potential in the presence of a minimal length takes the form

$$\frac{d^2\psi(p)}{dp^2} + \frac{2}{p}\left\{4\left[\frac{p^2 - mE}{p^2 - 2mE}\right] - \frac{1 + \beta'p^2}{1 + (\beta + \beta')p^2}\right\}\frac{d\psi(p)}{dp}$$
$$+ \left\{\frac{6 + (10\beta + 6\beta')p^2}{[1 + (\beta + \beta')p^2]} + \frac{2m\alpha/\hbar^2}{[1 + (\beta + \beta')p^2]^2}\right\}\frac{\psi(p)}{(p^2 - 2mE)}$$
$$= 0. \qquad (34)$$

In the case $\beta = \beta' = 0$, this equation reduces to Eq. (2) of ordinary quantum mechanics.

We can again transform Eq. (34) to an integral equation. We write Eq. (32) in the form

$$R^2\varphi(p) = 2m\alpha\psi(p), \qquad (35)$$

where

$$\varphi(p) = (p^2 - 2mE)\psi(p).$$

Then R^2 can be written as

$$R^2 = -\hbar^2 p^{-2}[1 + (\beta + \beta')p^2]^{2-[\beta/(\beta+\beta')]}\tilde{L},$$

where $\tilde{L}$ is the following self-adjoint operator:

$$\tilde{L} = \frac{d}{dp}\left(K(p)\frac{d}{dp}\right), \qquad (36)$$

with

$$K(p) = p^2[1 + (\beta + \beta')p^2]^{\beta/(\beta+\beta')}.$$

Equation (35) is then transformed to the following nonhomogeneous Sturm-Liouville equation:

$$[\tilde{L} + g(p)]\varphi(p) = 0, \qquad (37)$$

where

$$g(p) = \frac{2m\alpha}{\hbar^2}\frac{p^2}{p^2 - 2mE}[1 + (\beta + \beta')p^2]^{[\beta/(\beta+\beta')]-2}. \quad (38)$$

Then $\varphi(p)$ is given by the integral [48]

$$\varphi(p) = \int_0^\infty G(p,p')g(p')\varphi(p')dp'$$
$$+ \left[\varphi(0)K(0)\frac{dG(p,p')}{dp'}\right]_{p'=0}$$
$$- \left[\varphi(\infty)K(\infty)\frac{dG(p,p')}{dp'}\right]_{p'=\infty}. \qquad (39)$$

$G(p,p')$ is the corresponding Green's function.

In order to have a homogeneous integral equation in the form of an eigenvalue problem

$$\varphi(p) = \int_0^\infty G(p,p')g(p')\varphi(p')dp', \qquad (40)$$

$\varphi(p)$ must vanish at infinity. The wave function $\psi(p)$ is then required to satisfy the boundary condition

$$p^2\psi(p)\underset{p\to\infty}{=} 0. \qquad (41)$$

The explicit form of $G(p,p')$, using the boundary conditions (41), and $\psi(0)=$constant, is found to be

$G(p,p')$
$$= \begin{cases} \dfrac{1}{p}F\left(-\dfrac{1}{2},\dfrac{\beta}{\beta+\beta'},\dfrac{1}{2};-[\beta+\beta']p^2\right) - C, & p > p', \\[3mm] \dfrac{1}{p'}F\left(-\dfrac{1}{2},\dfrac{\beta}{\beta+\beta'},\dfrac{1}{2};-[\beta+\beta']p'^2\right) - C, & p < p', \end{cases}$$

where C is the constant

$$C = \frac{(\beta+\beta')^{1/2}\Gamma\left(\dfrac{1}{2}\right)\Gamma\left(\dfrac{1}{2}+\dfrac{\beta}{\beta+\beta'}\right)}{\Gamma(1)\Gamma\left(\dfrac{\beta}{\beta+\beta'}\right)}. \qquad (42)$$

Finally, the integral equation satisfied by the wave function $\psi(p)$ is

$$(p^2 - 2mE)\psi(p) = \frac{2m\alpha}{\hbar^2}\int_0^\infty p'^2[1 + (\beta+\beta')p'^2]^{[\beta/(\beta+\beta')]-2}$$
$$\times G(p,p')\psi(p')dp'. \qquad (43)$$

In the limit $\beta=\beta'=0$, Eq. (43) reduces to Eq. (12) of ordinary quantum mechanics.

Let us return now to the differential equation (34); by introducing the dimensionless variable z, defined as

$$z = \frac{(\beta+\beta')p^2 - 1}{(\beta+\beta')p^2 + 1}, \qquad (44)$$

which varies from -1 to $+1$, and using the following notations:

$$\omega_1 = \beta + \beta', \quad \omega_2 = \beta + 2\beta', \quad \omega_3 = 2\beta + 3\beta',$$

$$\omega_4 = \frac{\beta}{\beta+\beta'}, \quad \omega = -m(\beta+\beta')E, \quad \kappa = \frac{m\alpha}{2\hbar^2}, \quad (45)$$

we obtain the differential equation

$$(1 - z^2)\frac{d^2\psi}{dz^2} + \left[8\frac{(1+\omega)+(1-\omega)z}{(1+2\omega)+(1-2\omega)z} - \frac{1}{\omega_1}(\omega_2 z + \omega_3)\right]\frac{d\psi}{dz}$$
$$+ \left[\frac{\kappa z^2 + 2(\omega_4-\kappa)z + (6+2\omega_4+\kappa)}{(-1+2\omega)z^2 - 4\omega z + (1+2\omega)}\right]\psi = 0. \qquad (46)$$

To rewrite this equation in the form of a known differential equation, we make the following transformation:

$$\psi(z) = (1-z)^\lambda(1+z)^{\lambda'}f(z), \qquad (47)$$

where λ and λ' are arbitrary constants. Then, the equation for $f(z)$ is

$$\frac{d^2f}{dz^2} + \left\{\frac{-2\lambda}{(1-z)} + \frac{2\lambda'}{(1+z)} + \frac{8[(1+\omega)+(1-\omega)z]}{(1-z^2)[(1+2\omega)+(1-2\omega)z]}\right.$$
$$\left. - \frac{(\omega_2 z + \omega_3)}{\omega_1(1-z^2)}\right\}\frac{df}{dz} + \left\{\frac{\lambda(\lambda-1)}{(1-z)^2} + \frac{\lambda'(\lambda'-1)}{(1+z)^2}\right.$$
$$- \frac{8\lambda[(1+\omega)+(1-\omega)z]}{(1-z^2)(1-z)[(1+2\omega)+(1-2\omega)z]}$$
$$+ \frac{8\lambda'[(1+\omega)+(1-\omega)z]}{(1-z^2)(1+z)[(1+2\omega)+(1-2\omega)z]}$$
$$+ \lambda\frac{(\omega_2 z + \omega_3)}{\omega_1(1-z)^2(1+z)} - \lambda'\frac{(\omega_2 z + \omega_3)}{\omega_1(1-z)(1+z)^2}$$
$$\left. + \frac{\kappa z^2 + 2(\omega_4-\kappa)z + (6+2\omega_4+\kappa)}{(1-z^2)[(-1+2\omega)z^2 - 4\omega z + (1+2\omega)]} - \frac{2\lambda\lambda'}{(1-z^2)}\right\}f$$
$$= 0. \qquad (48)$$

This equation constitutes our starting point for studying the attractive $1/R^2$ potential in quantum mechanics with a minimal length. We shall be interested in the singularity structure of this equation and the effect of the finite length. For this purpose, let us begin with the case $E=0$.

B. Zero energy solution

The simplicity of the zero energy Schrödinger equation allows us to investigate whether the "deformed" version of the $-\alpha/R^2$ potential in momentum space from Eq. (48) remains singular.

Let us rewrite Eq. (48), in the case $\omega=0$ in the following form:

$$(1-z^2)\frac{d^2f}{dz^2} + \{(-2\lambda + 2\lambda' + 5 + \omega_4)$$

$$- (2\lambda + 2\lambda' + 2 - \omega_4)z\}\frac{df}{dz}$$

$$+ \frac{1}{(1-z^2)}\{(1+z)[\lambda(\lambda-1)(1+z) - 2\lambda\lambda'(1-z)$$

$$- \lambda(5+\omega_4) + \lambda(2-\omega_4)z + 2\omega_4]$$

$$+ (1-z)[\lambda'(\lambda'-1)(1-z) + \lambda'(5+\omega_4)$$

$$- \lambda'(2-\omega_4)z + \kappa(1-z)] + 6\}f = 0. \quad (49)$$

We choose λ and λ' by requiring that the coefficient of $f(z)$ in Eq. (49) vanishes for $z = \pm 1$; this leads to the two equations for λ and λ' as follows:

$$\lambda^2 - \left(\frac{5}{2} + \omega_4\right)\lambda + \frac{3}{2} + \omega_4 = 0,$$

$$\lambda'^2 + \frac{5}{2}\lambda' + \kappa + \frac{3}{2} = 0. \quad (50)$$

The values of λ and λ' satisfying this system are

$$\lambda = 1, \quad \left(\frac{3}{2} + \omega_4\right),$$

$$\lambda' = \left(-\frac{5}{4} - i\frac{\nu}{2}\right), \quad \left(-\frac{5}{4} + i\frac{\nu}{2}\right),$$

where $\nu = \sqrt{4\kappa - 1/4}$. We note that there are four possible choices concerning (λ, λ') leading to the same solution of the Schrödinger equation. We select the set $\left(1, -\frac{5}{4} - i\frac{\nu}{2}\right)$; so the transformation (47) becomes

$$\psi(z) = (1-z)(1+z)^{[-(5/4)-i(\nu/2)]}f(z). \quad (51)$$

By substituting λ and λ' with their values in Eq. (49), we obtain

$$(1-z^2)\frac{d^2f}{dz^2} + \left\{\left(\frac{1}{2} + \omega_4 - i\nu\right) - \left(\frac{3}{2} - \omega_4 - i\nu\right)z\right\}\frac{df}{dz}$$

$$+ \left\{\left(\frac{1}{8} - \frac{\omega_4}{4}\right) + i\nu\left(\frac{1}{4} - \frac{\omega_4}{2}\right)\right\}f = 0. \quad (52)$$

This equation is a second-order differential equation with three (regular) singular points $z = 1, -1, \infty$. Consequently, it may be written in a canonical form of a hypergeometric equation, merely by transforming the singular points to $z = 0, 1, \infty$. We can do this by means of the simple following change of variable:

$$\xi = \frac{z+1}{2}. \quad (53)$$

Thus, Eq. (52) becomes

$$\xi(1-\xi)f''(\xi) + [c - (a+b+1)\xi]f'(\xi) - abf(\xi) = 0, \quad (54)$$

with the parameters

$$a = \frac{1}{4} - \frac{\omega_4}{2} - \frac{\mu}{2} - i\frac{\nu}{2},$$

$$b = \frac{1}{4} - \frac{\omega_4}{2} + \frac{\mu}{2} - i\frac{\nu}{2},$$

$$c = 1 - i\nu, \quad \nu = \sqrt{4\kappa - 1/4},$$

$$\mu = \sqrt{(\omega_4 - 1)^2 - 4\kappa}, \quad \kappa = m\alpha/2\hbar^2. \quad (55)$$

Equation (54) is a hypergeometric equation which has, in the neighborhood of $\xi=0$, the following two solutions [45]:

$$f_1(\xi) = F(a,b,c;\xi), \quad (56)$$

$$f_2(\xi) = \xi^{1-c}F(a-c+1,b-c+1,2-c;\xi), \quad (57)$$

where $F(a,b,c;\xi) \equiv {}_2F_1(a,b,c;\xi)$ is the hypergeometric function.

Finally, from Eq. (51), we obtain two solutions $\psi_1(\xi)$ and $\psi_2(\xi)$, each solution being the complex conjugate of the other. Thus, the general solution is

$$\psi(\xi) = (1-\xi)\xi^{-5/4}[A\xi^{-i(\nu/2)}F(a,b,c;\xi) + B\xi^{i(\nu/2)}F(a-c+1,b-c+1,2-c;\xi)]. \quad (58)$$

In the particular case where $\omega_4 = 1/2$ ($\beta = \beta'$), we have $\mu = i\nu$ and $b = 0$. As $F(a,0,c;\xi) = 1$, the wave function $\psi(\xi)$ simplifies to

$$\psi_{\beta=\beta'}(\xi) = (1-\xi)\xi^{-5/4}[A\xi^{-i(\nu/2)} + B\xi^{+i(\nu/2)}]. \quad (59)$$

In the limit $\beta, \beta' \ll 1$, one has $\xi \sim \omega_1 p^2 \ll 1$, so that $1 - \xi \sim 1$ and $F(a,b,c;\xi) \sim 1$. Consequently, Eq. (58) becomes

$$\psi(p) \underset{\omega_1 \ll 1}{\sim} p^{-5/2}(Ap^{-i\nu} + Bp^{+i\nu}). \quad (60)$$

This is exactly the zero energy solution of ordinary quantum mechanics, which has the same form as the solution in the limit $p \to \infty$ [see Eq. (8)].

Solutions (58) have the same behavior near $\xi=0$. This is not so, however, for $p \to \infty$ ($\xi \to 1$). Using [45]

$$f_1(\xi) = F(a,b,a+b+1-c;1-\xi), \quad (61)$$

$$f_2(\xi) = (1-\xi)^{c-a-b}F(c-b,c-a,c-a-b+1;1-\xi), \quad (62)$$

we find in the limit $\xi \to 1$, $f_1(\xi) \sim 1$ and $f_2(\xi) \sim (1-\xi)^{c-a-b}$. On the other hand, $(1-\xi) \sim p^{-2}$, so by replacing $f_1(\xi)$ and $f_2(\xi)$ in Eq. (51), we obtain the following behavior of the two solutions:

$$\psi_1(p) \underset{p\to\infty}{\sim} p^{-2}, \tag{63}$$

$$\psi_2(p) \underset{p\to\infty}{\sim} p^{-3-2\omega_4}. \tag{64}$$

These two solutions can be found by considering the Schrödinger equation (34) in the limit $p \to \infty$ and seeking a solution in the form p^s.

This behavior is completely different from that of ordinary quantum mechanics: both solutions are independent of the coupling constant; moreover, the solution with asymptotic behavior (63) does not depend on the deformation parameters and falls off more slowly than ψ_2. This implies that ψ_1 does not satisfy the boundary condition (41), imposed by the integral equation, and so must be rejected. We conclude that the physical wave function is ψ_2 with behavior at infinity given by

$$p^2\psi(p) \underset{p\to\infty}{\sim} p^{-1-2\omega_4}. \tag{65}$$

The main conclusion, which we draw from this section, is that the singular attractive $1/R^2$ potential is regularized by this minimal length, so that the boundary condition (65) will suffice to extract the energy spectrum, as will be shown in Sec. IV D.

C. Full solution

By the same technique as in the case $E=0$, Eq. (48) can be rewritten in a form of a known differential equation by choosing conveniently the parameters λ and λ' of transformation (47). Taking $\lambda=1$ and $\lambda'=0$, Eq. (47) reads

$$\psi(z) = (1-z)f(z), \tag{66}$$

and Eq. (48) becomes after some calculations

$$\frac{d^2f(z)}{dz^2} + \left[\frac{2}{(z-1)} + \frac{8[(1+\omega)+(1-\omega)z]}{(2\omega-1)(z^2-1)(z-z_0)} + \frac{(\omega_2 z + \omega_3)}{\omega_1(z^2-1)} \right]\frac{df(z)}{dz}$$

$$+ \left[\frac{\left(2-\omega_4+\dfrac{\kappa}{1-2\omega}\right)z - \left(1+\omega_4+\dfrac{\kappa}{1-2\omega}\right)}{(z+1)(z-1)(z-z_0)} \right]f(z) = 0, \tag{67}$$

with the notations defined by Eq. (45), and

$$z_0 = \frac{2\omega+1}{2\omega-1}. $$

Equation (67) is a linear homogeneous second-order differential equation with four singularities $z=-1,1,z_0,\infty$, all regular. So, Eq. (67) belongs to the class of Fuchsian equations, and can be transformed into the canonical form of Heun's equation, having regular singularities at $z=0,1,\xi_0,\infty$ [50,51]. The simple change of variable

$$\xi = \frac{z+1}{2}$$

leads to the following canonical form of Heun's equation:

$$\frac{d^2f(\xi)}{d\xi^2} + \left(\frac{c}{\xi} + \frac{e}{\xi-1} + \frac{d}{\xi-\xi_0}\right)\frac{df(\xi)}{d\xi} + \left(\frac{ab\xi+q}{\xi(\xi-1)(\xi-\xi_0)}\right)f(\xi)$$
$$= 0, \tag{68}$$

with the parameters

$$a = \frac{1}{2}(3-\omega_4-\tilde{\nu}), \quad \tilde{\nu} = \left[(\omega_4-1)^2 - \frac{4\kappa}{1-2\omega}\right]^{1/2},$$

$$b = \frac{1}{2}(3-\omega_4+\tilde{\nu}), \quad \xi_0 = \frac{2\omega}{2\omega-1},$$

$$c = \frac{3}{2}, \quad d = 2, \quad e = \frac{1}{2}-\omega_4,$$

$$q = -\left(\frac{3}{2} + \frac{\kappa}{1-2\omega}\right), \tag{69}$$

which are linked by the Fuchsian condition

$$a+b+1 = c+d+e. \tag{70}$$

In the neighborhood of $\xi=0$, the two linearly independent solutions of Eq. (68) are [51]

$$f_1(\xi) = H(\xi_0,q,a,b,c,d;\xi), \tag{71}$$

$$f_2(\xi) = \xi^{1-c}H(\xi_0,q',1+a-c,1+b-c,2-c,d;\xi), \tag{72}$$

where

$$q' = q - (1-c)[d + \xi_0(1+a+b-c-d)].$$

$H(\xi_0,q,a,b,c,d;\xi)$ is the Heun function defined by the series

$$H(\xi_0,q,a,b,c,d;\xi) = 1 - \frac{q}{c\xi_0}\xi + \sum_{n=2}^{\infty} C_n\xi^n, \tag{73}$$

where the coefficients C_n are determined by the difference equation

$$(n+2)(n+1+c)\xi_0 C_{n+2}$$
$$= \{(n+1)^2(\xi_0+1)$$
$$\quad + (n+1)[c+d-1+(a+b-d)\xi_0] - q\}C_{n+1}$$
$$\quad - (n+a)(n+b)C_n, \tag{74}$$

with the initial conditions

$$C_0 = 1, \quad C_1 = \frac{-q}{c\xi_0}, \quad \text{and} \quad C_n = 0, \quad \text{if } n < 0.$$

Now, we can write the full solution of the deformed Schrödinger equation (46). Thus, by using Eq. (66) the solution $\psi(\xi)$, which is regular (finite) in the neighborhood of $\xi=0$, is given by

$$\psi(\xi) = A(1 - \xi)H(\xi_0, q, a, b, c, d; \xi), \qquad (75)$$

where A is a normalization constant.

We show in the Appendix that in the limit $\beta, \beta' \ll 1$, we recover the result of ordinary quantum mechanics, given by Eq. (5); in the limit $E \to 0$, the zero energy solution (58) is obtained, and finally, as was expected, Eq. (75) has the same behavior as Eq. (58) in the limit $p \to \infty$.

D. Eigenvalue problem

We now study in more detail the solution to Eq. (68), to show how the introduction of a minimal length regularizes the singular attractive $1/R^2$ potential. For this purpose, we begin by the special case $\beta = \beta'$.

1. Special case $\beta = \beta'$

In this case, the Heun equation (68) is reduced to a hypergeometric equation. Indeed, we have $\omega_4 = \frac{1}{2}$, and hence

$$e = 0,$$

$$ab = -q = \frac{3}{2} + \frac{\kappa}{1 - 2\omega},$$

and the Fuchsian condition (70) becomes

$$a + b + 1 = c + d.$$

Using the change of variable

$$x = \frac{\xi}{\xi_0},$$

Eq. (68) takes the form of a hypergeometric differential equation [45]

$$x(1 - x)f''(x) + [c - (a + b + 1)x]f'(x) - abf(x) = 0, \qquad (76)$$

with the parameters

$$a = \frac{5}{4} - \frac{\tilde{\nu}}{2},$$

$$b = \frac{5}{4} + \frac{\tilde{\nu}}{2},$$

$$c = \frac{3}{2}, \quad \tilde{\nu} = \left[\frac{1}{4} - \frac{4\kappa}{1 - 2\omega}\right]^{1/2}. \qquad (77)$$

The solution to the Schrödinger equation, which is finite in the vicinity of $\xi = 0$, is

$$\psi_{\beta = \beta'}(\xi) = A(1 - \xi)F(a, b, c; \xi/\xi_0). \qquad (78)$$

2. Energy spectrum

To compute the energy spectrum, we merely require that the wave function (78) satisfies the boundary condition (41). Since

$$1 - \xi = \frac{1}{1 + 2\beta p^2} \underset{p \to \infty}{\sim} p^{-2},$$

$$\frac{\xi}{\xi_0} = \frac{2\omega - 1}{2\omega} \frac{\omega_1 p^2}{1 + \omega_1 p^2} \underset{p \to \infty}{\sim} \frac{2\omega - 1}{2\omega},$$

the wave function (78) behaves like

$$\psi_{\beta = \beta'} \underset{p \to \infty}{\sim} p^{-2} F\left(a, b, c; \frac{2\omega - 1}{2\omega}\right).$$

From the boundary condition $p^2 \psi \to_{p \to \infty} 0$, we then obtain the following condition:

$$F\left(a, b, c; \frac{2\omega - 1}{2\omega}\right) = 0. \qquad (79)$$

This equation constitutes the quantization condition; the eigenvalues ω are the zeros of the hypergeometric function.

Let us now consider the limit $\omega \equiv -2m\beta E \ll 1$, i.e.,

$$\left|\frac{2\omega - 1}{2\omega}\right| \sim \frac{1}{2\omega} \gg 1.$$

By means of the transformation (6), and by taking into account that $F(a, b, c; -2\omega) \approx_{\omega \ll 1} 1$, Eq. (79) can be written in the following form:

$$\omega^{5/4}\left\{\exp\left[i\left(\arg[A] - \frac{\nu}{2}\ln[2\omega]\right)\right] + \exp\left[-i\left(\arg[A] - \frac{\nu}{2}\ln[2\omega]\right)\right]\right\} = 0, \qquad (80)$$

where we have used the notations

$$A = \frac{\Gamma(i\nu)}{\Gamma\left(\frac{5}{4} + i\frac{\nu}{2}\right)\Gamma\left(\frac{1}{4} + i\frac{\nu}{2}\right)} = |A|\exp[i\arg(A)],$$

$$\nu = \sqrt{4\kappa - 1/4}.$$

From Eq. (80), we have

$$\cos\left[\arg(A) - \frac{\nu}{2}\ln(2\omega)\right] = 0, \qquad (81)$$

which gives the following expression of the energy spectrum:

$$E_n = \frac{-1}{4m\beta}\exp\left\{\frac{2}{\nu}\left[\arg(A) - \left(n + \frac{1}{2}\right)\pi\right]\right\}, \qquad (82)$$

as one has

$$|E_n| \leq \frac{1}{4m\beta}, \quad n = 0, 1, 2, \ldots .$$

We recall that the deformation parameter β is related to the minimal length via Eq. (29), hence $(\Delta r)_{min} = 2\hbar\sqrt{\beta}$.

The energy spectrum (82) is identical to the one obtained by a cutoff regularization [see Eq. (19)]. The parameter $\beta + \beta' = 2\beta$ is simply the inverse square of the ultraviolet cutoff Λ.
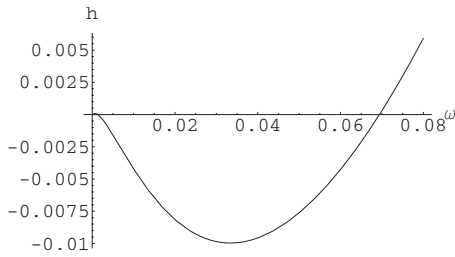
FIG. 1. $h \equiv F(a,b,c;\frac{2\omega-1}{2\omega})$ as a function of ω, for $\kappa=3/4$. All quantities a,b,c,ω,κ are dimensionless.

Equation (82) is accompanied by the condition $|E_n| \ll 1/4m\beta$, which excludes systematically the undesirable values of the number n, so there is now a ground state with finite energy. In the case of a weakly attractive potential ($4\kappa<1/4$), Eq. (80) has no solution.

These results are confirmed by the examination of the exact eigenvalue equation (79). We have plotted the hypergeometric function in Eq. (79) as a function of $\omega=-2m\beta E$ for fixed $\kappa=m\alpha/2\hbar^2$. The energy eigenvalues are the zeros of the function; Figs. 1 and 2 show that the energy of the ground state (ω_1) is finite; for $\kappa=3/4$, $\omega_1 \approx 0.07$ and for $\kappa=2$, $\omega_1 \approx 0.37$. As in ordinary quantum mechanics, there are many, almost identical, excited states with $\omega \simeq 0$ (accumulation point). The energy levels increase as we increase the coupling constant. In Fig. 2, we can see the energy of the first excited state. Figure 3 shows that there are no bound states for $\kappa=1/20$; we find that a critical coupling constant κ^*, below which there are no bound states, has the same value as in ordinary quantum mechanics, i.e., $\kappa^*=1/16$.

An interesting feature of the expression of the energy (82) is that it is inversely proportional to the deformation parameter β; thus if β is a very small parameter the energy of the ground state is very large. Consequently, in the case of the inverse square potential, the minimal length could be viewed as an intrinsic dimension of a system, as argued by Kempf (see, for instance, [5]). However, if this minimal length is obtained from calculations connected with the harmonic oscillator and the hydrogen atom, as in [12,16], namely, ~ 0.1 fm, the energy of the ground state would be so large, and thus it would not be in the energy scale where nonrelativistic quantum mechanics is valid.

3. Generalization to the case $\beta \neq \beta'$
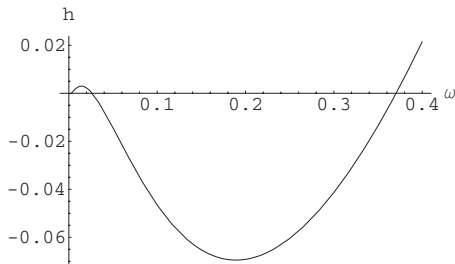
Let us return to the general solution (75)

FIG. 2. $h \equiv F(a,b,c;\frac{2\omega-1}{2\omega})$ as a function of ω, for $\kappa=2$. All quantities a,b,c,ω,κ are dimensionless.
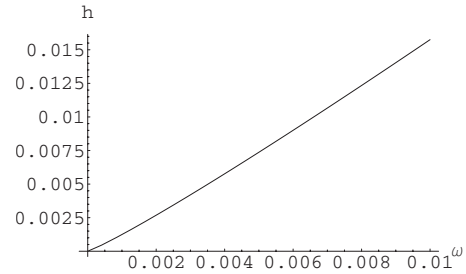

FIG. 3. $h \equiv F(a,b,c;\frac{2\omega-1}{2\omega})$ as a function of ω, for $\kappa=1/20$. All quantities a,b,c,ω,κ are dimensionless.

$$\psi(\xi) = A(1-\xi)H(\xi_0,q,a,b,c,d;\xi).$$

It can be written in the form [51]

$$\psi(\xi) = A(1-\xi)H\left(\frac{1}{\xi_0},\frac{q}{\xi_0},a,b,c,e;\frac{\xi}{\xi_0}\right). \quad (83)$$

As in the case $\beta=\beta'$ we impose the boundary condition (41), and obtain the following quantization condition:

$$H\left(\frac{2\omega-1}{2\omega},\frac{2\omega-1}{2\omega}q,a,b,c,e;\frac{2\omega-1}{2\omega}\right)=0. \quad (84)$$

In the case where $\omega \ll 1$, we set

$$\sigma = \frac{2\omega-1}{2\omega} \approx \frac{-1}{2\omega} \to \infty$$

by means of the following transformation [51]:

$$H(\sigma,\sigma q,a,b,c,e;\xi) \underset{\sigma\to\infty}{=} F(\delta+\sqrt{\delta^2+q},\delta-\sqrt{\delta^2+q},c;\xi),$$

$$c \neq 0,-1,-2,\ldots, \quad (85)$$

where

$$\delta = \frac{a+b-e}{2}.$$

Equation (84) reads

$$F\left(\frac{5}{4}-i\frac{\nu}{2},\frac{5}{4}+i\frac{\nu}{2},\frac{3}{2};\frac{-1}{2\omega}\right)_{\omega=-(\beta+\beta')mE\ll1} = 0. \quad (86)$$

Obviously, we get the same expression of the energy spectrum as in the case $\beta=\beta'$. It is sufficient to replace in Eq. (82) 2β by $\beta+\beta'$.

V. SUMMARY AND CONCLUSION

We have solved exactly the problem of the singular inverse square potential in the framework of quantum mechanics with a generalized uncertainty relation implying the existence of a minimal length. In the momentum representation, the wave function is a Heun function, which reduces to a hypergeometric function for $E=0$ and for $\beta=\beta'$. The potential is regularized in a natural way by this minimal length, so that the energy spectrum is bounded from below. The results of ordinary quantum mechanics with a regularizing

cutoff (Λ) are recovered in the limit $\beta, \beta' \ll 1$; the parameter $\beta + \beta'$ plays the role of the inverse square of Λ.

In conclusion, this study shows that the idea of the introduction of a minimal length, first proposed in high energy physics, could also apply to nonrelativistic quantum mechanics. In the new formalism based on the deformed Heisenberg algebra, the treatment of the singular $1/R^2$ potential is similar to that of regular potentials: we do not need to introduce any arbitrary parameters because β and β' are physical parameters of the formalism, and describe the short distance behavior of the interaction. The formalism includes a natural "cutoff" and modifies the potential at short distances, so that the energy spectrum is computed without imposing any extra condition. The latter result leads us to conclude with Kempf [4,5] that this elementary length should rather be viewed as an intrinsic dimension of a system, at least for the problem considered here.

APPENDIX A: LIMIT $\beta, \beta' \ll 1$

We write the wave function in the form given by Eq. (83),

$$\psi(\xi) = A(1 - \xi) H\left(\frac{1}{\xi_0}, \frac{q}{\xi_0}, a, b, c, e; \frac{\xi}{\xi_0}\right). \quad \text{(A1)}$$

In the limit $\beta, \beta' \ll 1$, we have

$$\xi = \frac{\omega_1 p^2}{1 + \omega_1 p^2} \approx \omega_1 p^2, \quad \text{where } \omega_1 = (\beta + \beta'),$$

$$\xi_0 = \frac{2\omega}{2\omega - 1} \approx -2\omega, \quad \text{where } \omega = -m\omega_1 E,$$

$$\frac{\xi}{\xi_0} \approx \frac{p^2}{2mE} \quad \text{and} \quad 1 - \xi \approx 1,$$

hence

$$\psi(y) \underset{\beta, \beta' \ll 1}{\approx} H(\sigma, \sigma \tilde{q}, \tilde{a}, \tilde{b}, c, e; y), \quad \text{(A2)}$$

where we have used the notations $\sigma = \frac{1}{2m\omega_1 E}$, $y = \frac{p^2}{2mE}$ and $\tilde{a}, \tilde{b}, \tilde{q}$ are the limits of the parameters a, b, q when $\beta, \beta' \ll 1$.

By means of the transformation (85), Heun's function is transformed to a hypergeometric function, given by

$$H(\sigma, \sigma \tilde{q}, \tilde{a}, \tilde{b}, c, e; y) \underset{\sigma \to \infty}{=} F(\delta + \sqrt{\delta^2 + \tilde{q}}, \delta - \sqrt{\delta^2 + \tilde{q}}, c; y),$$

$$c \neq 0, -1, -2, \ldots, \quad \text{(A3)}$$

where

$$\delta = \frac{\tilde{a} + \tilde{b} - e}{2}.$$

After a direct calculation we get

$$\psi(p) \underset{\beta, \beta' \ll 1}{\approx} F\left(\frac{5}{4} + i\frac{\nu}{2}, \frac{5}{4} - i\frac{\nu}{2}, \frac{3}{2}; \frac{p^2}{2mE}\right).$$

It is exactly the wave function in momentum representation for the attractive $-\alpha/R^2$ potential in ordinary quantum mechanics [see Eq. (5)].

APPENDIX B: LIMIT $p \to \infty$

To examine the behavior of $\psi(\xi)$, when $p \to \infty$ ($\xi \to 1$), we use the well-known relation [51,52]

$$
\begin{aligned}
H(\xi_0, q, a, b, c, d; \xi) &= C_1 H(1 - \xi_0, -q - ab, a, b, e, d; 1 - \xi) \\
&\quad + C_2 (1 - \xi)^{1-e} H(1 - \xi_0, q_2, c + d - a, \\
&\quad c + d - b, 2 - e, d; 1 - \xi), \quad \text{(B1)}
\end{aligned}
$$

where

$$C_1 = H(\xi_0, q, a, b, c, d; 1),$$

$$C_2 = H(\xi_0, q - \xi_0 c[1 - e], c + d - a, c + d - b, c, d; 1),$$

$$q_2 = -q - ab - (1 - e)[d + c(1 - \xi_0)].$$

By adopting the Heun normalization $H(\xi_0, q, a, b, c, d; 0) = 1$, the wave function (75), in the limit $p \to \infty$, behaves as follows:

$$\psi(\xi) \underset{\xi \to 1}{\approx} C_1(1 - \xi) + C_2(1 - \xi)^{2-e},$$

and since

$$1 - \xi \underset{p \to \infty}{\approx} p^{-2}, \quad 2 - e = \frac{3}{2} + \omega_4,$$

then, the asymptotic behavior of $\psi(p)$ in this region is

$$\psi(p) \underset{p \to \infty}{\approx} C_1 p^{-2} + C_2 p^{-(3+2\omega_4)}. \quad \text{(B2)}$$

This behavior is identical to that of the zero energy solution [see Eqs. (63) and (64)] because Schrödinger equations do not depend on the energy in the limit $p \to \infty$.

APPENDIX C: LIMIT $E \to 0$

We show, here, that the zero energy solution (58) can be obtained from the full solution (75) in the limit $E \to 0$. For this purpose, let us return to the transformation (B1). By taking into account that $\xi_0 \to 0$ and $\omega \to 0$ when $E \to 0$, the wave function (75) can be written as

$$\psi(\xi) \underset{E\to 0}{=} A(1-\xi)[C_1 H(1,q_1,a_1,b_1,c_1,d_1;1-\xi)$$
$$+ C_2(1-\xi)^{(1/2)+\omega_4} H(1,q_2,a_2,b_2,c_2,d_2;1-\xi)],$$
$$(C1)$$

with the parameters

$$q_1 = -\frac{1}{2} + \omega_4, \quad q_2 = -\frac{9}{4} + \frac{5\omega_4}{2},$$

$$a_1 = \frac{1}{2}(3 - \omega_4 - \widetilde{\nu}_0), \quad a_2 = 2 + \frac{\omega_4}{2} + \frac{\widetilde{\nu}_0}{2},$$

$$b_1 = \frac{1}{2}(3 - \omega_4 + \widetilde{\nu}_0), \quad b_2 = 2 + \frac{\omega_4}{2} - \frac{\widetilde{\nu}_0}{2},$$

$$c_1 = \frac{1}{2} - \omega_4, \quad c_2 = \frac{3}{2} + \omega_4,$$

$$d_1 = 2, \quad d_2 = 2,$$

where

$$\widetilde{\nu}_0 \equiv \widetilde{\nu}(\omega = 0) = \sqrt{(\omega_4 - 1)^2 - 4\kappa}.$$

We use once again another transformation of the Heun functions [51],

$$H(1,q,a,b,c,d;\xi)$$
$$= (1-\xi)^{[(c-a-b)/2]+\tau} F\left(\frac{c+a-b}{2} + \tau, \frac{c-a+b}{2} + \tau, c; \xi\right),$$

where

$$\tau = \pm \sqrt{\left(\frac{c-a-b}{2}\right)^2 - ab - q}, \quad \text{if } c \neq 0, -1, -2, \dots .$$

For the two Heun's functions in Eq. (C1), a direct calculation gives the following result:

$$\tau_1 = \tau_2 = \pm \sqrt{\frac{1}{16} - \kappa} = \pm i\frac{\nu}{2}.$$

By choosing, for convenience, the sign $(-)$, the wave function (C1) reads as follows:

$$\psi(\xi) \underset{E\to 0}{=} A(1-\xi)\xi^{-(5/4)-i(\nu/2)}[C_1 F(\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_1;1-\xi)$$
$$+ C_2(1-\xi)^{(1/2)+\omega_4} F(\widetilde{a}_2,\widetilde{b}_2,\widetilde{c}_2;1-\xi)], \quad (C2)$$

with the following parameters:

$$\widetilde{a}_1 = \frac{1}{4} - \frac{\omega_4}{2} - \frac{\mu}{2} - i\frac{\nu}{2}, \quad \widetilde{a}_2 = \frac{3}{4} + \frac{\omega_4}{2} + \frac{\mu}{2} - i\frac{\nu}{2},$$

$$\widetilde{b}_1 = \frac{1}{4} - \frac{\omega_4}{2} + \frac{\mu}{2} - i\frac{\nu}{2}, \quad \widetilde{b}_2 = \frac{3}{4} + \frac{\omega_4}{2} - \frac{\mu}{2} - i\frac{\nu}{2},$$

$$\widetilde{c}_1 = \frac{1}{2} - \omega_4, \quad \widetilde{c}_2 = \frac{3}{2} + \omega_4,$$

where

$$\mu = \sqrt{(\omega_4 - 1)^2 - 4\kappa}, \quad \kappa = m\alpha/2\hbar^2.$$

It is easily seen that the expression between brackets in Eq. (C2) is exactly a linear combination of the two solutions (61) and (62), in the vicinity of $\xi = 1$, of the hypergeometric equation (54), with the parameters a,b,c, given by [45]

$$a = \widetilde{a}_1,$$

$$b = \widetilde{b}_1,$$

$$c = 1 - i\nu.$$

Obviously, in the neighborhood of $\xi = 0$, we have the solution (58).

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