Canonical formalism and quantization of the perturbative sector of higher-derivative theories

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The theories defined by Lagrangians containing a second time derivative are considered. It is shown that if the second derivatives enter only the terms multiplied by coupling constant one can consistently define the perturbative sector via Dirac procedure. The possibility of introducing standard canonical variables is analyzed in detail. The ambiguities in quantization procedure are pointed out.

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I. INTRODUCTION

Higher-derivative theories were introduced quite early in an attempt to regularize the ultraviolet divergencies of quantum field theories [1].

Another context in which higher-derivative and nonlocal theories appear naturally is the description of the low-energy phenomena in terms of effective action which is nonlocal as a result of integrating out high energy degrees of freedom [2]. Moreover, theories with infinite degree derivatives do appear in the framework of string theory [3,4] and as modified theories of gravity [5].

In recent years, the emergence of noncommutative field theories [6] has revived discussion concerning higher-derivative theories. Apart from their string-theoretical origin noncommutative field theories can be viewed as an attempt to describe the dynamics at the scales where the very notion of space-time point lacks its meaning. Such theories, when modeled with the help of commutative space-time endowed with star product, lead at once to nonlocal Lagrangians. If noncommutativity involves time variables the theory becomes nonlocal in time and is plagued with unitarity and causality problems, at least when quantized with the help of naive Feynman rules [7]. There exist alternative quantization schemes which seem to cure the unitarity problems [8]; however, they are claimed to lead to new troubles [9].

In view of this state of the art it seems necessary to reconsider the quantization problem starting from first principles. The first step is to put the theory in Hamiltonian form. The relevant framework is provided by Ostrogradski formalism [10] for higher derivative theories and its sophisticated version [11] for nonlocal ones. The main problem with such procedures is that the resulting Hamiltonians are necessarily unbounded from below due to their behavior at the infinity of phase space. This implies that the quantum theory, if it exists, has no stable ground state.

Still, some hope exists because, in most interesting cases, the nonlocality enters only through the interaction term. Then one can pose the problem of quantizing the perturbative sector of the theory [4,12]. The initial value problem for perturbative solutions involves basic variables and their first time derivatives so the phase space for such solutions resembles the standard one. Moreover, for perturbation theory only the vicinity of the phase space is relevant, and we can hope that the Hamiltonian is bounded here from below leading to a stable perturbative vacuum.

In the present paper, inspired by Refs. [4] and [12], we study in some detail the Hamiltonian formalism and quantization for a simple system described by the Lagrangian containing a second time derivative in the interaction term. In Sec. II we show in full generality that the perturbative sector of our theory can be described with the help of the Dirac method. There are two constraints of the second kind which allow one to eliminate perturbatively Ostrogradski momenta in favor of coordinates q and $\dot{q}$. The Dirac bracket $\{q,\dot{q}\}_D$ can then be perturbatively computed to arbitrary order in coupling constant. It is, however, rather complicated. Therefore in Sec. III we analyze the possibility of simplifying the form of the Dirac bracket and the Hamiltonian. We show that it is indeed possible to define perturbatively, order by order, the variables $x,\dot{x}$ such that: (i) the Dirac bracket takes the standard form $\{x,\dot{x}\}=1$, and (ii) the Hamiltonian is the sum of kinetic and potential energy. We show that there is a large freedom in defining x and $\dot{x}$ obeying (i) and (ii); in fact, at any order of perturbative expansion for x one can add many terms with new, also dimensional, constants. These constants are spurious in the sense that they disappear after coming back to the original dynamical variables. However, this might be not the case in quantum theory as we explain in Sec. VI. Section IV is devoted to the special case of homogeneous (monomial) potentials. The form of the transformation $(q,\dot{q})\rightarrow(x,\dot{x})$ is studied in some detail. In particular, it is shown that if the degree of homogeneity is odd the above transformation can be chosen such that the resulting Hamiltonian is parity invariant. This implies that the initial theory, when restricted to the perturbative sector, posseses some complicated discrete symmetry. The form of symmetry transformation can be determined, order by order; however, we would like to have a simpler and more straightforward explanation of its emergence. In Sec. V, we study the simplest example of homogeneous potential of the third degree, considered already in Refs. [4] and [12]. We find explicitly, up to fourth order, the transformation relating q and x as well as the Hamiltonian to this order, expressed in terms of $x,\dot{x}$ variables. It appears that the resulting parity invariant potential is positive term by term, up to fourth order. On the other hand, the initial Hamiltonian, considered to the same order, is not positively definite. There is no contradiction here because our expansions are at best asymptotic and valid at the vicinity of phase space. Moreover, we do not know whether the property of positivity of parity invariant potential persists in higher orders. If this is the case, the theory is perturbatively stable.

Finally, in Sec. VI we study the quantum theory of the system described in Sec. V. To this end we consider (up to the second order) the transformation converting the Dirac bracket and the Hamiltonian into the standard form. As it has been already stressed such a transformation is not uniquely defined. We consider a one-parameter family of transformations and show that for different values of the parameter the resulting quantum theories are not equivalent. Specifically, the energy eigenvalues differ by an overall constant. Therefore the additional parameter, spurious in the classical case, becomes meaningful when quantum corrections are taken into account. This shows that the quantization is a subtle and ambiguous procedure.

II. HAMILTONIAN FORMALISM FOR THE PERTURBATIVE SECTOR

Let us consider the following Lagrangian

$$L = \frac{1}{2}\dot{q}^2 - \frac{\omega^2}{2}q^2 - gV(q,\dot{q},\ddot{q}), \quad \frac{\partial^2 V}{\partial \ddot{q}^2} \neq 0. \tag{1}$$

It depends on second derivative $\ddot{q}$; however, $\ddot{q}$ enters L only through V which, in turn, is multiplied by the coupling constant g.

The corresponding Euler-Lagrange equation reads

$$\ddot{q} + \omega^2 q + g\left(\frac{\partial V}{\partial q} - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial V}{\partial \ddot{q}}\right)\right) = 0. \tag{2}$$

This is a fourth-order differential equation. In order to obtain a unique solution one has to impose the initial conditions involving q and its first three derivatives. Correspondingly, the phase space of the system must be four dimensional.

The canonical formalism for our system can be introduced according to the Ostrogradski prescription [10]. To this end we define the canonical variables

$$q_1 = q, \quad q_2 = \dot{q}, \tag{3}$$

$$p_1 = \frac{\delta L}{\delta \dot{q}} \equiv \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right) = \dot{q} + g\left(-\frac{\partial V}{\partial \dot{q}} + \frac{d}{dt}\left(\frac{\partial V}{\partial \ddot{q}}\right)\right) \equiv p_1(q,\dot{q},\ddot{q},\dddot{q}),$$

$$p_2 = \frac{\delta L}{\delta \ddot{q}} \equiv \frac{\partial L}{\partial \ddot{q}} = -g\frac{\partial V}{\partial \ddot{q}} \equiv p_2(q,\dot{q},\ddot{q}),$$

and the Hamiltonian

$$H \equiv p_1 q_2 + p_2 \ddot{q}(q_1,q_2,p_2) - L(q_1,q_2,\ddot{q}(q_1,q_2,p_2)), \tag{4}$$

where $\ddot{q}(q_1,q_2,p_2)$ is the solution to the last Eq. (3).

The main disadvantage of H is that p_1 enters it linearly so it is unbounded from below; the system is unstable. One can try to cure this by imposing constraints confining the system to some submanifold of phase space. A natural choice is to consider only perturbative solutions to Eq. (2). Due to the fact that third and fourth derivatives enter only the terms multiplied by the coupling constant, the perturbative solution is uniquely determined by imposing the initial conditions on q and $\dot{q}$. In particular, higher derivatives can be expressed in terms of q and $\dot{q}$. In fact, one can write [12]

$$\ddot{q} = f(q,\dot{q}),$$

$$\dddot{q} = \left(\dot{q}\frac{\partial}{\partial q} + f(q,\dot{q})\frac{\partial}{\partial \dot{q}}\right)f \equiv Df \tag{5}$$

$$\vdots$$

$$q^{(n)} = D^{n-2}f.$$

The form of $f(q,\dot{q})$ is determined by demanding that it is consistent with Euler-Lagrange equations. Let $F(q,\dot{q},\ddot{q},\ldots,q^{(n)})$ be any function. Define [12]

$$[F](q,\dot{q}) \equiv F(q,\dot{q},Df,\ldots,D^{n-2}f). \tag{6}$$

Some properties of the bracket $[\cdot]$ are discussed in the Appendix.

The consistency condition for f reads

$$\left[\ddot{q} + \omega^2 q + g\left(\frac{\partial V}{\partial q} - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial V}{\partial \ddot{q}}\right)\right)\right] = 0. \tag{7}$$

Assume now that we have found some $f(q,\dot{q})$ obeying Eq. (7). The definitions of $p_{1,2}$ can be now converted into constraints

$$\varphi_1 \equiv p_1 - [p_1(q,\dot{q},\ddot{q},\dddot{q})](q_1,q_2) = 0, \tag{8}$$

$$\varphi_2 \equiv p_2 - [p_2(q,\dot{q},\ddot{q})](q_1,q_2) = 0.$$

Differentiating φ_1,φ_2 with respect to time and using Eqs. (7) and (8) we find that there are no secondary constraints.

The constraints φ_1,φ_2 are second class ones:

$$\{\varphi_1,\varphi_2\} = -g\frac{\partial}{\partial q_1}\left[\frac{\partial V}{\partial \ddot{q}}\right] - g\frac{\partial}{\partial q_2}\left[-\frac{\partial V}{\partial \dot{q}} + \frac{d}{dt}\left(\frac{\partial V}{\partial \ddot{q}}\right)\right]. \tag{9}$$

Due to the form of constraints φ_i, the momenta p_i can be expressed in terms of q_1 and q_2 which parametrize the reduced phase space. The Dirac bracket reads

$$\{A,B\}_D = \{A,B\} + \{A,\varphi_1\}\{\varphi_1,\varphi_2\}^{-1}\{\varphi_2,B\} - \{A,\varphi_2\}\{\varphi_1,\varphi_2\}^{-1}\{\varphi_1,B\}. \tag{10}$$

In particular

$$\{q_1,q_2\}_D = -\{\varphi_1,\varphi_2\}^{-1}. \tag{11}$$

The same result is obtained by considering the symplectic form

$$\Omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 \tag{12}$$

reduced to our submanifold. Indeed, we find

$$\Omega_{red} = \left(\frac{\partial[p_1]}{\partial q_2} - \frac{\partial[p_2]}{\partial q_1}\right)dq_2 \wedge dq_1 \tag{13}$$

so that

$$\{q_1,q_2\}_{red} = \left(\frac{\partial[p_1]}{\partial q_2} - \frac{\partial[p_2]}{\partial q_1}\right)^{-1} \tag{14}$$

which, by Eq. (9), coincides with Eq. (11). One can also check the validity of the Hamiltonian equations. It is convenient to come back to initial notation $q_1=q$, $q_2=\dot{q}$. Simple computation gives

$$\frac{\partial[H]}{\partial q} = -f(q,\dot{q})\{q,\dot{q}\}_D^{-1}, \tag{15}$$

$$\frac{\partial[H]}{\partial \dot{q}} = \dot{q}\{q,\dot{q}\}_D^{-1},$$

where Eq. (7) has been used. Now, the first Hamiltonian equation

$$\dot{q} = \{q,[H]\}_D \tag{16}$$

gives the identity $\dot{q}=\dot{q}$ while the second one

$$\ddot{q} = \{\dot{q},[H]\}_D \tag{17}$$

leads to the constraint equation

$$\ddot{q} = f(q,\dot{q}). \tag{18}$$

III. SIMPLIFYING DYNAMICS

The form of reduced dynamics presented above is rather complicated; in particular, due to the nontrivial form of the basic Poisson (Dirac) bracket (11) the quantization poses a nontrivial ordering problem. In order to avoid this problem one can adopt the following strategy [4,12]: instead of direct quantization one first makes the Darboux transformation which simplifies Ω_{red}, $\Omega_{red}=d\dot{x}\wedge dx$. Such a transformation is not unique; in fact, it is defined up to a canonical transformation. The question arises whether this freedom can be used to simplify also the Hamiltonian or even to put it in standard form: kinetic plus potential energy.

In order to analyze this problem we start with the lowest order approximation. Let us first note that for the Lagrangian (1) the zeroth-order approximation to $f(q,\dot{q})$ reads

$$f_0(q,\dot{q}) = -\omega^2 q. \tag{19}$$

The corresponding approximation to the time-derivative operator D will be denoted by D_0,

$$D_0 \equiv \dot{q}\frac{\partial}{\partial q} - \omega^2 q\frac{\partial}{\partial \dot{q}}. \tag{20}$$

Finally, $[\cdot]_0$ denotes $[\cdot]$ given by Eq. (6) with D replaced by D_0.

Our aim is to define the transformation $(q,\dot{q})\to(x,\dot{x})$ simplifying both the Dirac bracket and Hamiltonian. To the first order in g one can write

$$q = x + gm(x,\dot{x}) \tag{21}$$

or

$$x = q - gm(q,\dot{q}). \tag{22}$$

To this order we have also

$$\dot{x} = \dot{q} - gD_0 m(q,\dot{q}) \tag{23}$$

or

$$\dot{q} = \dot{x} + gD_0 m(x,\dot{x}), \tag{24}$$

where D_0 on the right-hand side of Eq. (24) is given by Eq. (20) with q replaced by x.

We start by writing the reduced symplectic form to the first order in g:

$$\Omega_{red(1)} = \left(1 + g\left(\frac{\partial}{\partial \dot{q}}\left(-\left[\frac{\partial V}{\partial \dot{q}}\right]_0 + \left[\frac{d}{dt}\left(\frac{\partial V}{\partial \ddot{q}}\right)\right]_0\right) + \frac{\partial}{\partial q}\left[\frac{\partial V}{\partial \ddot{q}}\right]_0\right)\right) d\dot{q}\wedge dq. \tag{25}$$

We are looking for $m(x,\dot{x})$ such that the transformations (21) and (24) lead to $\Omega_{red(1)}=d\dot{x}\wedge dx$. As a result of simple computation we obtain the following equation for $m(x,\dot{x})$:

$$\frac{\partial(D_0 m)}{\partial \dot{x}} + \frac{\partial m}{\partial x} + \frac{\partial}{\partial \dot{x}}\left(-\left[\frac{\partial V}{\partial \dot{x}}\right]_0 + \left[\frac{d}{dt}\left(\frac{\partial V}{\partial \ddot{x}}\right)\right]_0\right) + \frac{\partial}{\partial x}\left(\left[\frac{\partial V}{\partial \ddot{x}}\right]_0\right) = 0, \tag{26}$$

which we rewrite as

$$\frac{\partial}{\partial \dot{x}}\left(D_0\left(m + \left[\frac{\partial V}{\partial \ddot{x}}\right]_0\right) - \left[\frac{\partial V}{\partial \dot{x}}\right]_0\right) + \frac{\partial}{\partial x}\left(m + \left[\frac{\partial V}{\partial \ddot{x}}\right]_0\right) = 0. \tag{27}$$

Equation (27) implies that

$$m + \left[\frac{\partial V}{\partial \ddot{x}}\right]_0 = \frac{\partial \Phi(x,\dot{x})}{\partial \dot{x}},$$

$$D_0\left(m + \left[\frac{\partial V}{\partial \ddot{x}}\right]_0\right) - \left[\frac{\partial V}{\partial \dot{x}}\right]_0 = -\frac{\partial \Phi(x,\dot{x})}{\partial x} \tag{28}$$

for some function Φ. By virtue of Eq. (28), Φ obeys

$$\frac{\partial(D_0\Phi)}{\partial \dot{x}} = \left[\frac{\partial V}{\partial \dot{x}}\right]_0 = \frac{\partial[V]_0}{\partial \dot{x}} \tag{29}$$

or

$$D_0\Phi = [V]_0 - \tilde{V}(x), \tag{30}$$

$\tilde{V}(x)$ being an arbitrary (up to now) function of x alone.

In order to answer the question whether we can always find, to the first order in g, the transformation which puts $\Omega_{red(1)}$ in Darboux form let us note that we are looking for a transformation which, up to a given order, is defined globally in the phase space [optimally, $m(x,\dot{x})$ is some polynomial provided V is].

Let us introduce the polar coordinates

$$x = r\cos\Theta, \tag{31}$$

$$\dot{x} = \omega r\sin\Theta.$$

In terms of new coordinates Eq. (30) reads

$$\frac{\partial \Phi}{\partial \Theta} = -\omega([V]_0(r\cos\Theta, \omega r \sin\Theta) - \tilde{V}(r\cos\Theta)) \quad (32)$$

The right-hand side is some periodic function of Θ. Therefore, one has

$$\frac{\partial \Phi}{\partial \Theta} = \sum_{n>0} [a_n(r)e^{in\Theta} + \overline{a_n(r)}e^{-in\Theta}] + a_0(r) \quad (33)$$

and Φ is globally defined (periodic) provided $a_0(r)=0$. Consider the first term on the right-hand side (rhs) of Eq. (32). It is easy to see that the Θ-independent term must be a function of r^2. Consider the particular contribution of the form $\alpha_k r^{2k}$; it can be canceled by the term $\alpha_k \frac{2^{2k}}{\binom{2k}{k}} x^{2k}$ entering $\tilde{V}(x)$. We conclude that $\tilde{V}(x)$ can be chosen in such a way that no Θ-independent term appears on the rhs of Eq. (32). With such a choice $m(x,\dot{x})$, defined first by Eq. (28), defines the transformation leading to standard symplectic form. Let us note that there is a considerable freedom in the choice of $\tilde{V}(x)$.

In order to find the meaning of $\tilde{V}(x)$ let us note that Eqs. (28) and (30) imply the following identity:

$$(D_0^2 + \omega^2)m + \left[\frac{\partial V}{\partial x}\right]_0 - \left[\frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)\right]_0 + \left[\frac{d^2}{dt^2}\left(\frac{\partial V}{\partial \ddot{x}}\right)\right]_0 = \frac{\partial \tilde{V}}{\partial x}. \quad (34)$$

Now, by computing $\ddot{x}$ from Eq. (23), keeping terms up to the first order and using Eq. (33) we arrive at the equation of motion for x:

$$\ddot{x} = -\omega^2 x - g\frac{\partial \tilde{V}(x)}{\partial x}. \quad (35)$$

Therefore due to $\{x,\dot{x}\}=1$, the Hamiltonian computed to the first order in g has the form

$$H = \left(\frac{1}{2}\dot{x}^2 + \frac{\omega^2 x^2}{2}\right) + g\tilde{V}(x). \quad (36)$$

Let us generalize our analysis to arbitrary order in g. To this end we write

$$x = q - \sum_{n=1}^{\infty} g^n m_n(q,\dot{q}) \equiv q - M(q,\dot{q}), \quad (37)$$

$$\dot{x} = \dot{q} - \sum_{n=1}^{\infty} g^n Dm_n(q,\dot{q}) \equiv \dot{q} - DM(q,\dot{q}),$$

Let us note that the second formula does not represent an explicit expansion in powers of coupling constant g. This is due to the fact that D itself contains $f(q,\dot{q})$ which is also given as a power series in g.

Now, assuming that Ω_{red} takes the standard form when expressed in terms of x and $\dot{x}$, we can write

$$\Omega_{red} = d\dot{x} \wedge dx = \left(1 - \frac{\partial DM}{\partial \dot{q}} - \frac{\partial M}{\partial q} + \frac{\partial DM}{\partial \dot{q}}\frac{\partial M}{\partial q} - \frac{\partial DM}{\partial q}\frac{\partial M}{\partial \dot{q}}\right) d\dot{q} \wedge dq. \quad (38)$$

By virtue of Eqs. (3), (13), and (38) we find that M obeys

$$\frac{\partial DM}{\partial \dot{q}} + \frac{\partial M}{\partial q} - \left(\frac{\partial DM}{\partial \dot{q}}\frac{\partial M}{\partial q} - \frac{\partial DM}{\partial q}\frac{\partial M}{\partial \dot{q}}\right) = g\left(\frac{\partial}{\partial \dot{q}}\left[\frac{\partial V}{\partial q} - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}}\right)\right] - \frac{\partial}{\partial q}\left[\frac{\partial V}{\partial \dot{q}}\right]\right). \quad (39)$$

We want to solve Eq. (39) perturbatively in g. Assume it holds up to nth order and consider the $n+1$ order. Note that the expression in the parentheses is to be computed to nth order only. Moreover, noting that M and DM are both at least $0(g)$ we conclude that the equation for the $n+1$ order contribution to M reads

$$\frac{\partial(D_0 m_{n+1})}{\partial \dot{q}} + \frac{\partial m_{n+1}}{\partial q} = \text{sum of known terms} \equiv \frac{\partial^2 R_{n+1}}{\partial \dot{q}^2}, \quad (40)$$

where the known rhs we have rewritten for further convenience as a second derivative with respect to $\dot{q}$ (which is always possible).

Equation (40) can be written in the form

$$\frac{\partial}{\partial \dot{q}}\left(D_0 m_{n+1} - \frac{\partial R_{n+1}}{\partial \dot{q}}\right) + \frac{\partial m_{n+1}}{\partial q} = 0. \quad (41)$$

Again we conclude that

$$m_{n+1} = \frac{\partial \Phi_{n+1}}{\partial \dot{q}}, \quad (42)$$

$$D_0 m_{n+1} - \frac{\partial R_{n+1}}{\partial \dot{q}} = -\frac{\partial \Phi_{n+1}}{\partial q},$$

for some $\Phi_{n+1}(q,\dot{q})$. Equation (42) leads to the consistency condition for Φ_{n+1}.

$$D_0\frac{\partial \Phi_{n+1}}{\partial \dot{q}} + \frac{\partial \Phi_{n+1}}{\partial q} = \frac{\partial R_{n+1}}{\partial \dot{q}} \quad (43)$$

or

$$D_0 \Phi_{n+1}(q,\dot{q}) = R_{n+1}(q,\dot{q}) + S_{n+1}(q). \quad (44)$$

One can repeat the arguments used in the case of first-order approximation. Namely, Φ_{n+1} is globally well- defined provided S_{n+1} is chosen in such a way that no Θ-independent term [cf. Eqs. (31)] appear on the rhs. This is always possible so we conclude that one can construct the standard canonical variables defined globally to arbitrary order in g.

Let us further note that the transformation $(q,\dot{q}) \to (x,\dot{x})$ of the phase space is defined in such a way that the second canonical variable continues to be the time derivative of the first one (for a given perturbative dynamics). Therefore the first Hamilton equation is an identity which, due to $\{x,\dot{x}\}$

$=1$, leads to the standard form of the Hamiltonian,

$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 + \tilde{V}(x;g). \tag{45}$$

This can be also checked explicitly. We have shown that, order by order, one can reduce to the standard form the perturbative sector of the dynamics defined by the Lagrangian (1).

IV. HOMOGENEOUS POTENTIALS

Let us now consider the special case of homogeneous monomial potentials,

$$V(q,\dot{q},\ddot{q}) = q^k \dot{q}^l \ddot{q}^m, \quad m \geq 2; \tag{46}$$

let us denote $a=k+l+m$. For dimensional reasons one can write

$$f(q,\dot{q}) = \sum_{n=0}^{\infty} g^n f_n(q,\dot{q}), \tag{47}$$

where $f_n(q,\dot{q})$ are homogeneous polynomials of degree $n(a-2)+1$.

Also, one can write the perturbative expansions for other relevant quantities. First, we have

$$\Omega_{red} = \left(1 + \sum_{n=1}^{\infty} g^n \omega_n(q,\dot{q})\right) d\dot{q} \wedge dq, \tag{48}$$

where $\omega_n(q,\dot{q})$ are homogeneous polynomials of degree $n(a-2)$. On the other hand, we have seen in the last section that there is large freedom in the choice of the functions $m_n(q,\dot{q})$. Indeed, they are determined by the choice of $S_n(q)$ [cf. Eq. (44)]. There is only one condition restricting the admitted form of $S_n(q)$: the sum on the rhs should not contain the Θ-independent term. This is a rather weak condition which allows one to add many terms (say, any homogeneous polynomial of odd degree) containing new (also dimensional) parameters. However, one can show that it is always possible to choose the "minimal" $S_n's$ in the sense that the only constants entering them are g and ω. Assuming this is the case up to the order n we conclude that $\frac{\partial^2 R_{n+1}}{\partial \dot{q}^2}$ is a homogeneous polynomial of degree $(n+1)(a-2)$ depending only on one constant ω. Therefore R_{n+1} can be also chosen as a homogeneous polynomial of degree $(n+1)(a-2)+2$ containing only one dimensional constant ω. As a result, the Θ-independent term in R_{n+1} must be of the form $r^{(n+1)(a-2)+2}$ times a dimensionless constant. Then we can choose $S_{n+1}(q)$ as proportional to $q^{(n+1)(a-2)+2}$, and Φ_{n+1} obeying Eq. (44) can be taken as a homogeneous polynomial of the same degree depending only on ω. So, by Eq. (42) m_{n+1} is homogeneous of degree $n(a-2)+1$. This concludes the inductive proof.

With the minimal choice of the transformation (37) one can easily write out the general form of the potential $\tilde{V}(x;g)$ entering the Hamiltonian (45); it reads

$$\tilde{V}(x;g) = \sum_{n=1}^{\infty} v_n g^n \omega^{(l+2m-2)n+2} x^{(a-2)n+2}. \tag{49}$$

Let us now consider the particular case of odd a. Notice that R_n is of degree $n(a-2)+2$ which is odd for n odd. Therefore R_n is then a homogeneous polynomial of odd degree so it does not contain a Θ-independent term. So S_n can be chosen as $\alpha_n q^{n(a-2)+2}$ with α_n arbitrary (in particular, one can take $\alpha_n=0$). It is not difficult to see that α_n can be chosen perturbatively order by order so that the odd terms in the expansion (49) vanish. Indeed, let

$$S_n = \alpha q^{n(a-2)+2}. \tag{50}$$

Once S_n is selected, one can define, via Eqs. (37), (42), and (44), the variables $x_\alpha, \dot{x}_\alpha$ to nth order. It is easy to see that the relation between $x_0, \dot{x}_0$ (corresponding to the choice $\alpha=0$) and $x_\alpha, \dot{x}_\alpha$, to the same order, reads

$$x_0 = x_\alpha + g^n \Delta m_\alpha(x_\alpha, \dot{x}_\alpha), \tag{51}$$

$$\dot{x}_0 = \dot{x}_\alpha + g^n D_0 \Delta m_\alpha(x_\alpha, \dot{x}_\alpha),$$

with

$$\Delta m_\alpha = \frac{\partial \Delta \Phi_n}{\partial \dot{x}}, \quad D_0 \Delta m_\alpha = \frac{-\partial \Delta \Phi_n}{\partial x},$$

$$D_0 \Delta \Phi_n = \alpha x^{n(a-2)+2}. \tag{52}$$

Therefore adding the term (50) amounts to the following change of the Hamiltonian:

$$\begin{aligned} H &= \frac{1}{2}\dot{x}_0^2 + \frac{1}{2}\omega^2 x_0^2 + \tilde{V}_n(x_0,g) \\ &\simeq \frac{1}{2}\dot{x}_\alpha^2 + \frac{1}{2}\omega^2 x_\alpha^2 + \tilde{V}_n(x_\alpha,g) + g^n \dot{x}_\alpha D_0 \Delta m_\alpha + g^n \omega^2 x_\alpha \Delta m_\alpha \\ &= \frac{1}{2}\dot{x}_\alpha^2 + \frac{1}{2}\omega^2 x_\alpha^2 \\ &\quad + \tilde{V}_n(x_\alpha,g) - g^n \left(\dot{x}_\alpha \frac{\partial}{\partial x_\alpha} - \omega^2 x_\alpha \frac{\partial}{\partial \dot{x}_\alpha}\right) \Delta \Phi_n \\ &= \left(\frac{1}{2}\dot{x}_\alpha^2 + \frac{1}{2}\omega^2 x_\alpha^2 + \tilde{V}_n(x_\alpha,g)\right) - \alpha g^n x_\alpha^{n(a-2)+2}. \end{aligned} \tag{53}$$

Adjusting properly α one can cancel, order by order, all odd terms in $\tilde{V}(x,g)$.

Concluding, we find that for odd monomial $V(q,\dot{q},\ddot{q})$ one can reduce, order by order, the perturbative potential $\tilde{V}(x;g)$ to the form

$$\tilde{V}(x;g) = \sum_{k=1}^{\infty} v_{2k} g^{2k} \omega^{2(l+2m-2)k+2} x^{2(a-2)k+2}. \tag{54}$$

Note that in this case the perturbative sector exhibits some discrete nonlinear symmetry. In fact, the resulting standard Hamiltonian is parity invariant: $x \to -x$, $\dot{x} \to -\dot{x}$ is a symmetry. Then, expressed back in original variables, the parity

transformation produces nonlinear symmetry defined order by order in coupling constant g.

V. SIMPLE EXAMPLE

Let us consider a simple model studied already in Refs. [4,12]:

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 - gq\ddot{q}^2. \tag{55}$$

It belongs to the class of models studied in the last section. Equation (55) leads to the following equation of motion:

$$\ddot{q} + \omega^2 q + g(3\ddot{q}^2 + 4\dot{q}\dddot{q} + 2qq^{(IV)}) = 0. \tag{56}$$

The canonical variables read

$$q_1 = q, \quad q_2 = \dot{q},$$

$$P_1 = \dot{q} + 2g(\dot{q}\ddot{q} + q\dddot{q}), \tag{57}$$

$$P_2 = -2gq\ddot{q}.$$

It is also straightforward to write out the Hamiltonian

$$H = P_1 q_2 - \frac{P_2^2}{4gq_1} - \frac{1}{2}q_2^2 + \frac{1}{2}\omega^2 q_1^2. \tag{58}$$

In order to perform the reduction to the perturbative sector we impose the constraint

$$\ddot{q} = f(q,\dot{q}). \tag{59}$$

Then, by virtue of Eq. (56), $f(q,\dot{q})$ obeys

$$f + \omega^2 q + g\left(3f^2 + 4\dot{q}\left(\dot{q}\frac{\partial f}{\partial q} + f\frac{\partial f}{\partial \dot{q}}\right)\right) + 2q\left(\dot{q}^2\frac{\partial^2 f}{\partial q^2} + \dot{q}\frac{\partial f}{\partial q}\frac{\partial f}{\partial \dot{q}} + f\frac{\partial f}{\partial q} + 2\dot{q}f\frac{\partial^2 f}{\partial q\,\partial\,\dot{q}} + f\left(\frac{\partial f}{\partial \dot{q}}\right)^2 + f^2\frac{\partial^2 f}{\partial \dot{q}^2}\right) = 0. \tag{60}$$

This equation, although quite complicated, can be solved perturbatively order by order in g. For example, to the third order in g one finds

$$f = -\omega^2 q - g(5\omega^4 q^2 - 4\omega^2\dot{q}^2) + g^2(-76\omega^6 q^3 + 140\omega^4 q\dot{q}^2) - g^3(1959\omega^8 q^4 - 6800\omega^6 q^2\dot{q}^2 + 736\omega^4\dot{q}^4). \tag{61}$$

The constraints (8) take the form

$$P_1 - \dot{q} - 2g\left(\dot{q}\dot{f} + q\dot{q}\frac{\partial f}{\partial q} + qf\frac{\partial f}{\partial \dot{q}}\right) \approx 0,$$

$$P_2 + 2gqf \simeq 0, \tag{62}$$

while Ω_{red} is given by

$$\Omega_{red} = \left(1 + 4gf + 4gq\frac{\partial f}{\partial q} + 2g\dot{q}\frac{\partial f}{\partial \dot{q}} + 2gq\dot{q}\frac{\partial^2 f}{\partial q\partial\dot{q}} + 2gq\left(\frac{\partial f}{\partial \dot{q}}\right)^2 + 2gqf\frac{\partial^2 f}{\partial \dot{q}^2}\right)d\dot{q}\wedge dq. \tag{63}$$

Finally, the reduced Hamiltonian reads

$$[H] = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2 q^2 + g\left(-qf^2 + 2\dot{q}^2 f + 2q\dot{q}^2\frac{\partial f}{\partial q} + 2q\dot{q}f\frac{\partial f}{\partial \dot{q}}\right). \tag{64}$$

Now, one can try to find perturbatively the "normal" coordinates $x,\dot{x}$. Following the method outlined in previous sections we found that, to the fourth order,

$$q = x + g(\omega^2 x^2 - 2\dot{x}^2) + g^2\left(\frac{50}{3}\omega^4 x^3 - 18\omega^2 x\dot{x}^2\right) + g^3\left(\frac{760}{3}\omega^6 x^4 - 716\omega^4 x^2\dot{x}^2 - 84\omega^2\dot{x}^4\right) + g^4\left(\frac{111\,422}{15}\omega^8 x^5 - 25\,928\omega^6 x^3\dot{x}^2 + 3030\omega^4 x\dot{x}^4\right) + O(g^5) \tag{65}$$

and

$$[H] = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 + \frac{25}{6}g^2\omega^6 x^4 + \frac{30\,136}{45}g^4\omega^{10}x^6 + O(g^6). \tag{66}$$

We see that our perturbative Hamiltonian, when put in normal form, becomes positively defined, at least up to fourth order in g. We do not know whether this property persists in higher orders. Let us note that our reduced Hamiltonian (64) is not positive. For example, to the first order in g one finds from Eqs. (61) and (64)

$$[H] = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2 q^2 - g\omega^2(\omega^2 q^3 + 4q\dot{q}^2), \tag{67}$$

which is negative for large $q,\dot{q}$.

On the other hand, to the same order $[H]$, when expressed in terms of new coordinates, is simply the energy of the harmonic oscillator. We conclude that, at best, we can expect that our series defining new coordinates are asymptotic [note that $[H]$, as given by Eq. (67), becomes negative for $q,\dot{q}$ of order $\frac{1}{g}$].

VI. QUANTUM THEORY

Our ultimate goal is to quantize the higher derivative dynamical system. The main disadvantage of the Hamiltonian formalism introduced by Ostrogradski is that some momenta enter the Hamiltonian linearly. Therefore it is unbounded from below. Contrary to the case where the Hamiltonian is unbounded in small regions of phase space, this kind of unboundedness cannot be cured with the help of the uncertainty principle. As a result, no stable ground state can exist.

However, one can ask whether it is possible to quantize consistently the higher-derivative theory in the perturbative sector. The first trouble is related here with the complicated form of reduced symplectic structure. It is by far not sure whether one can find the proper ordering procedure which allows one to convert complicated Poisson brackets into commutators obeying Jacobi identity.

The simplest way to define the perturbative quantum theory seems to be the following. First, we construct on the

classical level the transformation in reduced phase space leading to the standard form of the Poisson bracket and the Hamiltonian. Then the quantization can be performed in a straightforward way. Moreover, if the classical Hamiltonian appears to be bounded from below, the quantum theory possesses perturbatively the stable ground state. Once the theory is quantized in "standard" coordinates one defines the quantum counterparts of initial variables by inverting (perturbatively) the classical map and choosing a definite ordering (for example, the Weyl one).

The main problem here is that such a procedure is by far not unique. In fact, we have seen in the previous section that there is a large freedom in defining the classical transformation to standard coordinates. One can hardly believe that the quantum theories resulting from different choices of such transformations are equivalent. Moreover, in the process of defining the perturbative transformation from q to x variables one can introduce new (also dimensional) constants. On the classical level they are spurious and disappear after coming back to the original dynamical variables. This may not be the case after quantization has been performed and the additional parameters may appear to be relevant.

In order to illustrate this phenomenon let us go back to our simple model. Consider the transformation

$$x = q + g(\beta\omega^2 q^2 + (2\beta+4)\dot{q}^2) + g^2\left(\left(-2\beta-\frac{50}{3}\right)\omega^4 q^3 + (-32-2\beta^2-24\beta)\omega^2 q\dot{q}^2\right),$$

$$\dot{x} = \dot{q} - 2g(\beta+4)\omega^2 q\dot{q} + g^2((4\beta^2+22\beta-26)\omega^4 \dot{q}q^2 - 2(\beta^2+4\beta)\omega^2\dot{q}^3) \tag{68}$$

depending on one real parameter β. In terms of new variables the Hamiltonian takes the form

$$[H] = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 - g(\beta+1)\omega^4 x^3 + 5g^2\left(\frac{1}{2}\beta^2+\beta+\frac{4}{3}\right)\omega^6 x^4 + O(g^3). \tag{69}$$

For $\beta=-1$ we obtain the parity invariant form.

Let us now compute the energies to the second order in g. Standard perturbation theory gives

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right) + \frac{25}{8}g^2\hbar^2\omega^4[n^2+(n+1)^2] + \frac{1}{2}g^2\hbar^2\omega^4(\beta+1)^2. \tag{70}$$

We see that the energy eigenvalues depend on β, although it is only an overall shift. It is interesting to note that the energies take minimal values in the parity-invariant case.

The ambiguity considered above is rather mild. We could add other terms, much more complicated and containing new dimensional constants. Let us remind that the only condition imposed, order by order, on new S_n [cf. Eq. (44)] is that the rhs contain no Θ-independent terms. Keeping this in mind one can easily understand that the resulting form of the standard Hamiltonian can vary considerably depending on the particular transformation chosen. This may have a strong impact on the form of energy spectrum. The resulting quantum theories become nonequivalent. This effect can be ultimately ascribed to the ordering problem.

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APPENDIX

The main property of the symbol $[\cdot]$ introduced in Sec. II is expressed by the equation [12]

$$\left[\frac{d[F]}{dt}\right] = D[F] = \left[\frac{dF}{dt}\right]. \tag{A1}$$

To see this let us note that [12]

$$\frac{d[F]}{dt} = \left[\frac{dF}{dt}\right] + \frac{\partial[F]}{\partial\dot{q}}(\ddot{q}-f).$$

Due to $[\ddot{q}-f]=0$ we find

$$\left[\frac{d[F]}{dt}\right] = \left[\frac{dF}{dt}\right].$$

Also $D[F]=[D[F]]$ and $D[F]=[D[F]]=\left[\frac{d[F]}{dt}\right]=\left[\frac{dF}{dt}\right]$.

Iterating Eq. (A1) one obtains

$$\left[\frac{d^2F}{dt^2}\right] = \left[\frac{d}{dt}\left[\frac{dF}{dt}\right]\right] = \left[\frac{d}{dt}\left[\frac{d[F]}{dt}\right]\right] = \left[\frac{d^2[F]}{dt^2}\right]$$

and

$$D^2[F] = D[D[F]] = \left[\frac{d}{dt}(D[F])\right] = \left[\frac{d^2[F]}{dt^2}\right].$$

Therefore

$$\left[\frac{d^n[F]}{dt^n}\right] = D^n[F] = \left[\frac{d^nF}{dt^n}\right].$$

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