Continuous-variable entanglement in a nondegenerate three-level laser with a parametric oscillator

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We consider a nondegenerate three-level cascade laser with a subthreshold nondegenerate parametric oscillator coupled to a vacuum reservoir. Applying the pertinent master equation, we analyze the squeezing and entanglement properties of the two-mode light produced by this quantum optical system inside and outside the cavity. We also determine the normalized second-order correlation function for the two-mode light as well as for individual mode. We find that the light generated by this system is in a two-mode squeezed state and the state of the system is strongly entangled at steady state. Moreover, the presence of the parametric oscillator leads to an increase in the degree of squeezing and entanglement. We also find that the intermode correlation decreases as the injected atomic coherence decreases in the system.

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I. INTRODUCTION

Extensive research has been carried out on the quantum analysis of the light generated by optical parametric oscillators [1–14]. These studies show that optical parametric amplifiers generate light in a squeezed state with a maximum intracavity squeezing of 50% below the vacuum level. It has also been studied that three-level cascade lasers can be considered as a source of squeezed light under certain conditions. In such lasers, the squeezing is due to the atomic coherence which can be introduced either by initially preparing the three-level atoms in a coherent superposition of the top and bottom level [15–20] or coupling these levels by external coherent light [21–23]. When a three-level atom makes a transition from the top to bottom level through the intermediate level, two highly correlated photons are generated. If these photons are identical, the laser is referred to as a degenerate three-level laser, and a nondegenerate three-level laser, which is considered in this paper, otherwise.

In recent years, the topic of continuous-variable entanglement has received a significant amount of attention as it plays an important role in all branches of quantum information and communication protocols [24]. The efficiency of quantum information schemes highly depends on the degree of entanglement. A nondegenerate parametric oscillator operating below, near, and above threshold has been theoretically predicted to be a source of light in an entangled state [12,25]. Recently, the experimental realization of the entanglement in nondegenerate parametric oscillator has been demonstrated by Zhang *et al.* [26]. On the other hand, Xiong *et al.* [27] have recently proposed a scheme for an entanglement amplifier based on a nondegenerate three-level laser when the three-level atoms are injected at the lower level and the top and bottom levels are coupled by a strong coherent light. They have found that a nondegenerate three-level laser can generate light in macroscopic entangled state applying the entanglement criterion for bipartite continuous-variable states [28]. Moreover, Tan *et al.* [29] have extended the work of Xiong *et al.* and studied the generation and evolution of the entangled light in the Wigner representation using the sufficient and necessary inseparability criterion for a two-mode Gaussian state proposed by Duan *et al.* [28] and Simon [30]. Tesfa [31] has considered a similar system when the atomic coherence is induced by superposition of atomic states and studied the entanglement at steady state. More recently, Ooi [32] has studied the steady state entanglement in a two-mode Λ laser.

In this paper, we seek to analyze a nondegenerate three-level cascade laser with a subthreshold nondegenerate parametric oscillator coupled to a vacuum reservoir. In contrary to the previous studies [27,29] where they have considered driven atomic coherence, we consider the injected atomic coherence which can be introduced by initially preparing the atoms in a coherent superposition of the top and bottom levels. In addition to exhibiting a two-mode squeezed light, this combined system produces light in an entangled state. To this end, the aim of this paper is to study the squeezing and entanglement properties of the two-mode light inside and outside the cavity at steady state.

Applying the master equation, we obtain equations of evolution of the expectation values for the cavity mode variables. The resulting equations are then used to calculate the quadrature variance for the two-mode light at steady state. Moreover, using the entanglement measure developed in Ref. [28], we investigate the entanglement of the two modes inside and outside the cavity at steady state. We also determine the second-order correlation function for the individual mode and for the superposition of the two modes. Finally, we calculate the linear correlation coefficient between the two modes.

II. HAMILTONIAN AND MASTER EQUATION

We consider a nondegenerate three-level cascade laser with a subthreshold nondegenerate parametric oscillator coupled to a vacuum reservoir through the port mirror. The three-level cascade atoms initially prepared in a coherent su-

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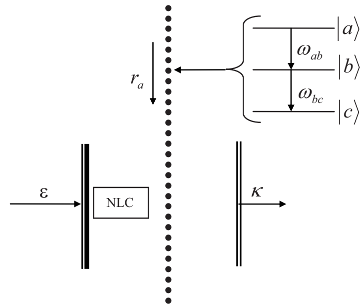

FIG. 1. Schematics of a nondegenerate three-level cascade laser with a subthreshold nondegenerate parametric oscillator.

perposition of the top and bottom levels are injected into the cavity at a constant rate r_a and removed from the cavity after a certain time τ. We represent the top, intermediate, and bottom levels of a three-level atom by $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively. We assume that the transitions between levels $|a\rangle$ and $|b\rangle$ and between levels $|b\rangle$ and $|c\rangle$ to be dipole allowed, with direct transitions between levels $|a\rangle$ and $|c\rangle$ to be dipole forbidden. We consider the case for which the two cavity modes are at resonance with the two transitions $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$ having transition frequencies ω_{ab} and ω_{bc}, respectively (see Fig. 1). On the other hand, a pump mode photon of frequency $\omega = \omega_{ab} + \omega_{bc}$ directly interacts with the nonlinear crystal (NLC) to produce the signal-idler photon pairs having the same frequencies as the two cavity modes. Furthermore, we consider the case for which the pump mode emerging from the NLC does not couple the top and bottom levels. This could be realized by putting on the right-hand side of the NLC a screen which absorbs the pump mode.

A subthreshold nondegenerate parametric oscillator, with the pump mode treated classically, can be described by the Hamiltonian

$$\hat{H}_1 = i\varepsilon(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b}), \tag{1}$$

in which ε, considered to be real and constant, is proportional to the amplitude of the coherent light, and $\hat{a}$ and $\hat{b}$ are the annihilation operators for the two cavity modes. The master equation associated with this Hamiltonian has the form

$$\frac{d}{dt}\hat{\rho} = \varepsilon(\hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b}\hat{\rho} + \hat{\rho}\hat{a}\hat{b}). \tag{2}$$

Moreover, the interaction of a three-level atom with the two cavity modes can be described by the Hamiltonian

$$\hat{H}_2 = ig[\hat{a}^\dagger|b\rangle\langle a| + \hat{b}^\dagger|c\rangle\langle b| - \hat{a}|a\rangle\langle b| - \hat{b}|b\rangle\langle c|], \tag{3}$$

where g is the atom-cavity mode coupling constant assumed to be the same for both transitions. In this paper, we take the initial state of a single three-level atom to be

$$|\psi_A(0)\rangle = C_a(0)|a\rangle + C_c(0)|c\rangle \tag{4}$$

and, hence, the density operator for a single atom is

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \tag{5}$$

where $\rho_{aa}^{(0)} = |C_a(0)|^2$, $\rho_{cc}^{(0)} = |C_c(0)|^2$ are, respectively, the probabilities for the atom to be initially in the upper and lower levels, and $\rho_{ac}^{(0)} = C_a(0)C_c^*(0) = \rho_{ca}^{(0)*}$ represents the initial atomic coherence of the atom. Using Eqs. (3) and (5) and taking into account the interaction of the cavity modes with the vacuum reservoir, we obtain the master equation for the laser cavity modes in the good-cavity limit ($\kappa \ll g$) and in the linear and adiabatic approximation schemes. By combining the resulting equation with Eq. (2), the master equation for the cavity modes of the system under consideration turns out to be

$$\begin{aligned}
\frac{d}{dt}\hat{\rho} = {} & \varepsilon(\hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b}\hat{\rho} + \hat{\rho}\hat{a}\hat{b}) \\
& + \frac{A\rho_{aa}^{(0)}}{2}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) \\
& + \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) \\
& + \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}) \\
& + \frac{A\rho_{ac}^{(0)}}{2}(\hat{\rho}\hat{a}^\dagger\hat{b}^\dagger + \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger) \\
& + \frac{A\rho_{ca}^{(0)}}{2}(\hat{\rho}\hat{a}\hat{b} + \hat{a}\hat{b}\hat{\rho} - 2\hat{b}\hat{\rho}\hat{a}),
\end{aligned} \tag{6a}$$

where

$$A = 2g^2 r_a/\gamma^2 \tag{6b}$$

is the linear gain coefficient, κ, assumed to be the same for the two modes, is the cavity damping constant, and γ is the spontaneous atomic decay rate assumed to be the same for all three levels. The terms proportional to $\rho_{aa}^{(0)}$ and $\rho_{cc}^{(0)}$, respectively, describe the gain for mode a and loss for mode b, the terms proportional to the $\rho_{ac}^{(0)}$, describe the coupling of the two modes due to atomic coherence induced by the initial superposition of the top and bottom levels of the three-level atoms and are responsible for the squeezing obtained in the cascade laser system. Furthermore, the terms proportional to κ describe the cavity modes' loss through the port mirror. Applying this master equation we have derived the equations of evolution for the moments of the cavity mode variables in Appendix A.

Now it proves to be more convenient to introduce a new parameter defined by

$$\rho_{aa}^{(0)} = \frac{1-\eta}{2} \tag{7}$$

with $-1 \leq \eta \leq 1$. In view of the fact that $\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1$ and $\rho_{ac}^{(0)2} = \rho_{aa}^{(0)}\rho_{cc}^{(0)}$, one easily finds

$$\rho_{cc}^{(0)} = \frac{1+\eta}{2} \tag{8}$$

and

$$\rho_{ac}^{(0)} = \frac{1}{2}\sqrt{1-\eta^2}, \tag{9}$$

where we have assumed $\rho_{ac}^{(0)}$ to be real for convenience. We want to carry out our analysis at steady state. Thus, with the aid of Eqs. (7)–(9), the steady-state solutions of Eqs. (A2)–(A9) can be expressed as

$$\langle \hat{a}\rangle_{ss} = \langle \hat{b}\rangle_{ss} = 0, \tag{10}$$

$$\langle \hat{a}^2\rangle_{ss} = \langle \hat{b}^2\rangle_{ss} = \langle \hat{a}\hat{b}^\dagger\rangle_{ss} = 0, \tag{11}$$

$$\langle \hat{a}^\dagger\hat{a}\rangle_{ss} = \frac{(\kappa + A\eta)[16\varepsilon^2 - A^2(1-\eta^2)]}{4(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)} + \frac{A(1-\eta)(2\kappa + A\eta)[2\kappa + A(1+\eta)]}{4(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)}, \tag{12}$$

$$\langle \hat{b}^\dagger\hat{b}\rangle_{ss} = \frac{\kappa(4\varepsilon + A\sqrt{1-\eta^2})^2}{4(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)}, \tag{13}$$

$$\langle \hat{a}\hat{b}\rangle_{ss} = \frac{\kappa(4\varepsilon + A\sqrt{1-\eta^2})(2\kappa + A(1+\eta))}{4(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)}. \tag{14}$$

We observe that Eqs. (12) and (13) respectively represent the steady state mean photon number of the cavity modes a and b. These equations are physically meaningful if $(\kappa^2 + \kappa A\eta - 4\varepsilon^2) > 0$, where ss stands for steady state. We then interpret

$$4\varepsilon^2 = \kappa^2 + \kappa A\eta \tag{15}$$

as the threshold condition for the system under consideration. According to Eq. (15), the expression on the right-hand side cannot be negative. This holds true provided that $\eta = (\rho_{cc}^{(0)} - \rho_{aa}^{(0)}) \geq 0$ (since $\kappa \ll A$). Moreover, since we have considered a subthreshold nondegenerate parametric oscillator, the value of ε is constrained by the inequality $\varepsilon < \kappa/2$.

III. QUADRATURE SQUEEZING

In this section, we seek to determine the quadrature variance for the two-mode light inside and outside the cavity.

A. Quadrature variance of the two-mode light in the cavity

We define the intracavity quadrature operators for the two-mode light as

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c}, \tag{16}$$

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}), \tag{17}$$

in which

$$\hat{c} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{b}). \tag{18}$$

Using Eqs. (16) and (17) the intracavity quadrature variance of the two-mode light can be expressed as

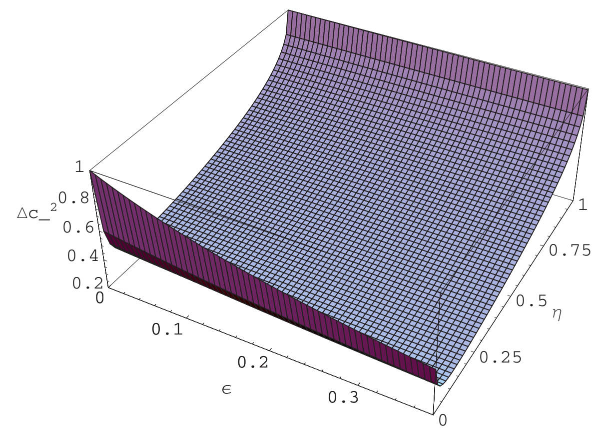

FIG. 2. (Color online) A plot of the intracavity quadrature variance of the two-mode light, Δc_-^2, vs ε and η for $\kappa=0.8$ and for $A=100$.

$$\Delta c_\pm^2 = \langle \hat{c}_\pm^2\rangle - \langle \hat{c}_\pm\rangle^2. \tag{19}$$

In view of Eq. (10) along with Eqs. (16)–(18), we easily see that

$$\langle \hat{c}_\pm\rangle^2 = 0 \tag{20}$$

and hence Eq. (19) reduces to

$$\Delta c_\pm^2 = \langle \hat{c}_\pm^2\rangle. \tag{21}$$

Making use of Eqs. (16)–(18), we easily find

$$\Delta c_\pm^2 = 1 + \langle \hat{a}^\dagger\hat{a}\rangle + \langle \hat{b}^\dagger\hat{b}\rangle + \langle \hat{a}^\dagger\hat{b}\rangle + \langle \hat{a}\hat{b}^\dagger\rangle \pm \tfrac{1}{2}(\langle \hat{a}^2\rangle + \langle \hat{b}^2\rangle + \langle \hat{a}^{\dagger 2}\rangle + \langle \hat{b}^{\dagger 2}\rangle + 2\langle \hat{a}\hat{b}\rangle + 2\langle \hat{a}^\dagger\hat{b}^\dagger\rangle). \tag{22}$$

On account of Eq. (11), the steady-state intracavity quadrature variance of the two-mode light becomes

$$\Delta c_\pm^2 = 1 + \langle \hat{a}^\dagger\hat{a}\rangle_{ss} + \langle \hat{b}^\dagger\hat{b}\rangle_{ss} \pm 2\langle \hat{a}\hat{b}\rangle_{ss}. \tag{23}$$

Thus, with the help of Eqs. (12)–(14), the above equation takes the form

$$\Delta c_\pm^2 = 1 + \frac{(4\varepsilon + A\sqrt{1-\eta^2})[8\kappa\varepsilon + A\eta(4\varepsilon - A\sqrt{1-\eta^2})]}{4(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)} + \frac{A(1-\eta)(2\kappa + A\eta)[2\kappa + A(1+\eta)]}{4(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)} \pm \frac{2\kappa[2\kappa + A(1+\eta)](4\varepsilon + A\sqrt{1-\eta^2})}{4(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)}. \tag{24}$$

In Fig. 2, we plot the intracavity quadrature variance Δc_-^2 for the two-mode light vs ε and η. This figure indicates that the system under consideration exhibits two-mode squeezing and the degree of squeezing increases with the parameter ε which represents the parametric oscillator. Moreover, relatively better squeezing occurs for small values of η, that is, when slightly more atoms are initially in the lower level than in the upper level. On the other hand, Fig. 3 clearly shows that the effect of the presence of the parametric oscillator is to increase the intracavity degree of squeezing for small values of η. The dotted curve in this figure indicates that

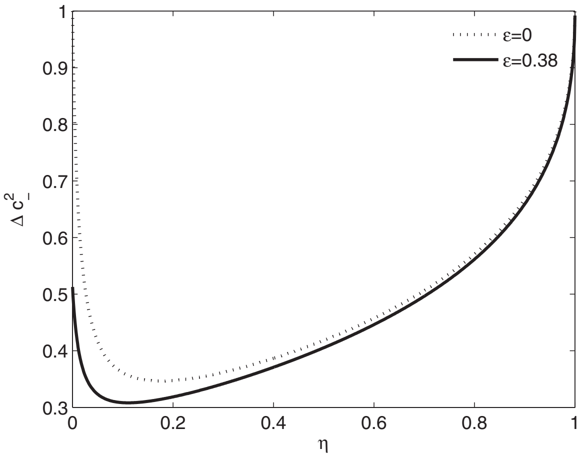

FIG. 3. Plots of the intracavity quadrature variance of the two-mode light, Δc_-^2, vs η for $\kappa=0.8$, $A=100$ and in the presence of the parametric oscillator with $\varepsilon=0.38$ (solid curve) and in the absence of the parametric oscillator ($\varepsilon=0$) (dotted curve).

squeezing vanishes for $\eta=0$ and $\eta=1$ which corresponds to maximum injected atomic coherence, $\rho_{ac}^{(0)}=1/2$, and no injected atomic coherence, $\rho_{ac}^{(0)}=0$, respectively. However, as can be seen from the solid curve, the presence of the parametric oscillator leads to some degree of squeezing for $\eta=0$.

In Fig. 4, we plot the intracavity quadrature variance of the two-mode light vs η for $\varepsilon=0.38$ and for different values of the linear gain coefficient. We easily see from this figure that the degree of squeezing increases with the linear gain coefficient. In addition, as the linear gain coefficient increases, the values of η at which the minimum value of the quadrature variance occurs approaches zero. We thus realize that better squeezing can be achieved by initially preparing the atoms in such a way that slightly more atoms are in the lower level than in the upper level and by increasing the linear gain coefficient.

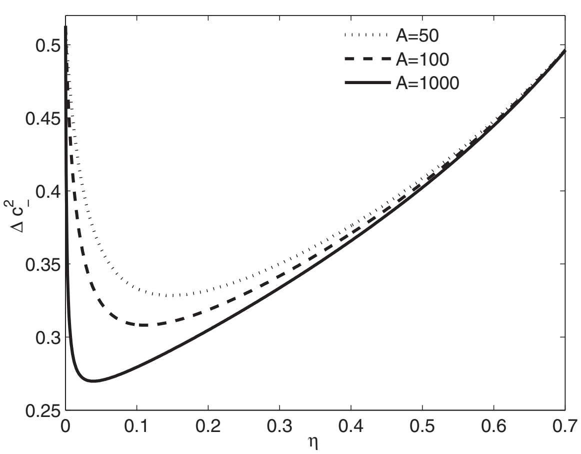

FIG. 4. Plots of the intracavity quadrature variance of the two-mode light Δc_-^2 vs η for $\kappa=0.8$, $\varepsilon=0.38$ and for different values of the linear gain coefficient.

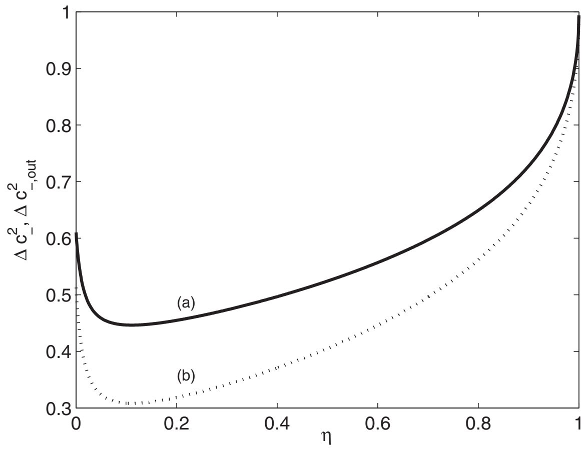

FIG. 5. (a) A plot of the quadrature variance of the two-mode light outside the cavity, $\Delta c_{-,out}^2$, vs η for $\kappa=0.8$, $\varepsilon=0.38$, and $A=100$. (b) A plot of the quadrature variance of the two-mode light inside the cavity, Δc_-^2, vs η for $\kappa=0.8$, $\varepsilon=0.38$, and $A=100$.

B. Quadrature variance of the two-mode light outside the cavity

The variance of the output mode quadrature operators

$$\hat{c}_+^{out} = \hat{c}_{out}^\dagger + \hat{c}_{out}, \tag{25}$$

$$\hat{c}_-^{out} = i(\hat{c}_{out}^\dagger - \hat{c}_{out}), \tag{26}$$

in which

$$\hat{c}_{out} = \frac{1}{\sqrt{2}}(\hat{a}_{out} + \hat{b}_{out}) \tag{27}$$

with

$$\hat{a}_{out} = \sqrt{\kappa}\hat{a}, \quad \hat{b}_{out} = \sqrt{\kappa}\hat{b} \tag{28}$$

can be expressed as

$$\Delta c_{\pm,out}^2 = \langle \hat{c}_{\pm,out}^2 \rangle - \langle \hat{c}_{\pm,out} \rangle^2. \tag{29}$$

Now applying Eqs. (25)–(28) and (20), the steady quadrature variance of the two-mode light outside the cavity can be put in the form

$$\Delta c_{\pm,out}^2 = 1 + \kappa\langle \hat{a}^\dagger \hat{a}\rangle_{ss} + \kappa\langle \hat{b}^\dagger \hat{b}\rangle_{ss} \pm 2\kappa\langle \hat{a}\hat{b}\rangle_{ss}. \tag{30}$$

Upon substituting Eqs. (21) and (23) into Eq. (30), we get

$$\begin{aligned}\Delta c_{\pm,out}^2 = 1 &+ \frac{\kappa A(1-\eta)(2\kappa + A\eta)[2\kappa + A(1+\eta)]}{4(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)} \\ &+ \frac{\kappa(4\varepsilon + A\sqrt{1-\eta^2})[8\kappa\varepsilon + A\eta(4\varepsilon - A\sqrt{1-\eta^2})]}{4(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)} \\ &\pm \frac{2\kappa^2[2\kappa + A(1+\eta)](4\varepsilon + A\sqrt{1-\eta^2})}{4(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)}.\end{aligned} \tag{31}$$

We clearly see from Fig. 5 that the degree of squeezing of the two-mode light outside the cavity is less than that of the intracavity. For instance, for $A=100$, $\kappa=0.8$, and $\varepsilon=0.38$ the intracavity squeezing is found to be 69.2% which occurs at

η=0.108 and for the same parameters the output mode squeezing is calculated to be 55.3% at η=0.110. We then see that the output mode squeezing is less by nearly 14% than that of the intracavity. It is known that the photons in the two cavity modes are generated pairwise and highly correlated and this correlation is the source of squeezing in this system. However, when photons leave the cavity through the port mirror, correlated pairs of photons may not leave at the same time, and hence, there is some finite probability of observing an odd number of photons inside and outside the cavity. It is obvious that the probability of observing an odd number of photons outside the cavity is greater than the intracavity one [21]. As a result of this, the correlation between the photon pairs could be destroyed to some extent depending on the property of the port mirror (or the value of κ). This leads to the decrease in the degree of squeezing of the output mode.

IV. ENTANGLEMENT ANALYSIS OF THE TWO-MODE LIGHT

In this section, we wish to study entanglement properties of the two modes inside and outside the cavity. It is known that a state of a system ρ of two modes a and b is said to be entangled or not separable if it is not possible to express in the form

$$\rho = \sum_i P_i \rho_i^{(a)} \otimes \rho_i^{(b)}, \tag{32}$$

where $\rho_i^{(a)}$ and $\rho_i^{(b)}$ are assumed to be the normalized density operators of modes a and b, respectively, with $P_i \geq 0$ and $\Sigma_i P_i = 1$. A maximally entangled continuous variable state can be expressed as a co-eigenstate of a pair of Einstein-Podolsky-Rosen (EPR)-type operators [33] such as $\hat{x}_a - \hat{x}_b$ and $\hat{p}_a + \hat{p}_b$. Thus the sum of the variances of these operators is reduced to zero for the maximally entangled continuous variable state [28].

A. Entanglement of the two modes in the cavity

In order to verify the entanglement of the two modes, we apply the criterion presented in Ref. [28]. According to this criterion, a quantum state of a system is said to be entangled if the sum of the variances of the two EPR-like operators $\hat{u}$ and $\hat{v}$ of the two modes satisfy the inequality

$$\Delta u^2 + \Delta v^2 < 2, \tag{33}$$

in which

$$\hat{u} = \hat{x}_a - \hat{x}_b, \tag{34}$$

$$\hat{v} = \hat{p}_a + \hat{p}_b, \tag{35}$$

with $\hat{x}_a = (\hat{a}^\dagger + \hat{a})/\sqrt{2}$, $\hat{x}_b = (\hat{b}^\dagger + \hat{b})/\sqrt{2}$, $\hat{p}_a = i(\hat{a}^\dagger - \hat{a})/\sqrt{2}$, and $\hat{p}_b = i(\hat{b}^\dagger - \hat{b})/\sqrt{2}$ being the quadrature operators for modes a and b.

The variance of operators $\hat{u}$ and $\hat{v}$ can be put at steady state in the form

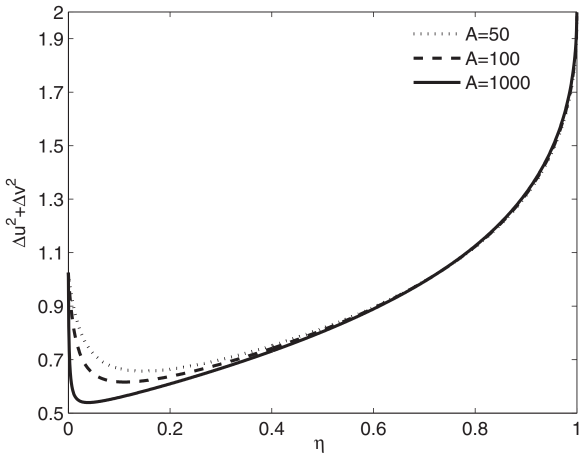

FIG. 6. Plots of $\Delta u^2 + \Delta v^2$ of the two-mode light in the cavity as steady state vs η for κ=0.8, ε=0.38, and for different values of the linear gain coefficient.

$$\Delta u^2 = \Delta v^2 = 1 + \langle \hat{a}^\dagger \hat{a} \rangle_{ss} + \langle \hat{b}^\dagger \hat{b} \rangle_{ss} - 2\langle \hat{a}\hat{b} \rangle_{ss}. \tag{36}$$

Thus, in view of Eqs. (23) and (24), the sum of the variances of the operators $\hat{u}$ and $\hat{v}$ takes the form

$$\begin{aligned}\Delta u^2 + \Delta v^2 = 2\Delta c_-^2 = 2 &+ \frac{A(1-\eta)(2\kappa + A\eta)[2\kappa + A(1+\eta)]}{2(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - \varepsilon^2)} \\ &+ \frac{(4\varepsilon + A\sqrt{1-\eta^2})[4\kappa\varepsilon + A\eta(4\varepsilon - A\sqrt{1-\eta^2})]}{2(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)} \\ &- \frac{\kappa[2\kappa + A(1+\eta)](4\varepsilon + A\sqrt{1-\eta^2})}{(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)}.\end{aligned} \tag{37}$$

We easily see from Fig. 6 that $\Delta u^2 + \Delta v^2$ is less than 2 for all values of η except for η=1, hence the entanglement criterion (33) is satisfied. This indicates that the state of the system is entangled at steady state provided that there is injected atomic coherence. Moreover, the degree of entanglement increases with the linear gain coefficient. On the other hand, comparison of Figs. 4 and 6 shows that there is a strong entanglement of the two modes in the cavity when there is a substantial degree of squeezing. This strong entanglement is observed for relatively small values of η, that is, when slightly more atoms are in the lower level at the initial time. It is also easy to see that the entanglement disappears when the squeezing vanishes. This is due to the fact that the squeezing and entanglement are directly related as given in Eq. (37). Furthermore, Fig. 7 clearly shows that the degree of entanglement increase due to the presence of the parametric oscillator.

B. Entanglement of the two modes outside the cavity

The EPR-like operators for the output modes have the form

$$\hat{u}^{out} = \hat{x}_a^{out} - \hat{x}_b^{out} \tag{38}$$

$$\hat{v}^{out} = \hat{p}_a^{out} + \hat{p}_b^{out} \tag{39}$$

in which $\hat{x}_a^{out} = \sqrt{\kappa}(\hat{a}^\dagger + \hat{a})/\sqrt{2}$, $\hat{x}_b^{out} = \sqrt{\kappa}(\hat{b}^\dagger + \hat{b})/\sqrt{2}$, $\hat{p}_a^{out} = i\sqrt{\kappa}(\hat{a}^\dagger - \hat{a})/\sqrt{2}$, and $\hat{p}_b^{out} = i\sqrt{\kappa}(\hat{b}^\dagger - \hat{b})/\sqrt{2}$ are quadrature operators for the output modes. Hence, the entanglement criterion for the output modes can be written as

$$\Delta u_{out}^2 + \Delta v_{out}^2 < 2. \tag{40}$$

The steady-state variance of the operators $\hat{u}_{out}$ and $\hat{v}_{out}$ can then be expressed as

$$\Delta u_{out}^2 = \Delta v_{out}^2 = 1 + \kappa\langle \hat{a}^\dagger \hat{a}\rangle_{ss} + \kappa\langle \hat{b}^\dagger \hat{b}\rangle_{ss} - 2\kappa\langle \hat{a}\hat{b}\rangle_{ss} \tag{41}$$

so that on account of Eqs. (30) and (31), the total variance of the operators $\hat{u}_{out}$ and $\hat{v}_{out}$ becomes

$$\Delta u_{out}^2 + \Delta v_{out}^2 = 2\Delta c_{-,out}^2 = 2 + \frac{\kappa A(1-\eta)(2\kappa + A\eta)[2\kappa + A(1+\eta)]}{2(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)} + \frac{\kappa(4\varepsilon + A\sqrt{1-\eta^2})[8\kappa\varepsilon + A\eta(4\varepsilon - A\sqrt{1-\eta^2})]}{2(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)} - \frac{\kappa^2[\kappa + A(1+\eta)](4\varepsilon + A\sqrt{1-\eta^2})}{(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)}. \tag{42}$$

Figure 8 shows that the entanglement criterion for the output modes, Eq. (40), is satisfied (except $\eta=1$ as before). In addition, as can be seen from the same figure, the degree of entanglement of the output modes is less than that of the cavity modes. This is because the degree of squeezing of the output modes is less than that of the cavity modes.

V. NORMALIZED SECOND-ORDER CORRELATION FUNCTIONS

In this section we analyze the second-order correlation function for the separate mode as well as for the superposition of the two modes. Moreover, we calculate the linear correlation coefficient between the cavity modes. The normalized second-order correlation function for the two-mode light can be expressed as

$$g_{(a,b)}^{(2)}(0) = \frac{\langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b}\rangle}{\langle \hat{a}^\dagger \hat{a}\rangle\langle \hat{b}^\dagger \hat{b}\rangle}. \tag{43}$$

Since Eqs. (A2) and (A3) are linear differential equations, we see that $\hat{a}$ and $\hat{b}$ are Gaussian variables. Moreover, on account of Eq. (10), $\hat{a}$ and $\hat{b}$ are Gaussian variables with vanishing mean. One can then express Eq. (50) in the form [34]

$$g_{(a,b)}^{(2)}(0) = 1 + \frac{\langle \hat{a}\hat{b}\rangle\langle \hat{a}^\dagger \hat{b}^\dagger\rangle + \langle \hat{a}\hat{b}^\dagger\rangle\langle \hat{a}^\dagger \hat{b}\rangle}{\langle \hat{a}^\dagger \hat{a}\rangle\langle \hat{b}^\dagger \hat{b}\rangle}. \tag{44}$$

With the aid of Eqs. (11)–(14), the steady-state second-order correlation function takes the form

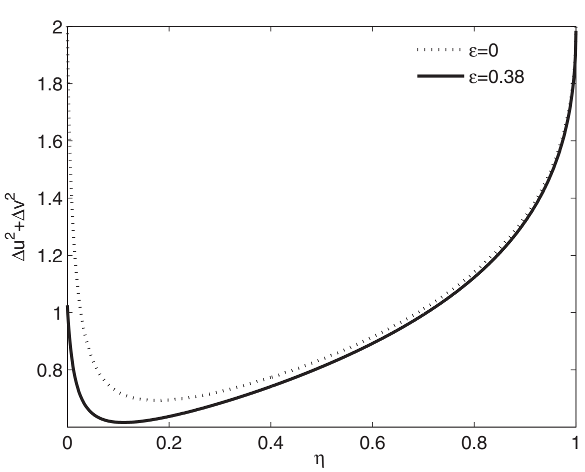

FIG. 7. Plots of $\Delta u^2 + \Delta v^2$ of the two-mode light in the cavity at steady state vs η for $\kappa=0.8$, $A=100$, and in the presence of the parametric oscillator with $\varepsilon=0.38$ (solid curve) and in the absence of the parametric oscillator $\varepsilon=0$ (dotted curve).

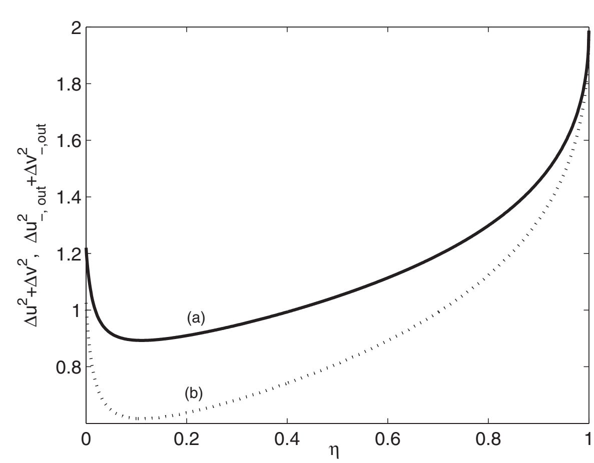

FIG. 8. (a) A plot of $\Delta u_{out}^2 + \Delta v_{out}^2$ of the two-mode light outside the cavity at steady state vs η for $\kappa=0.8$, $A=100$, and $\varepsilon=0.38$. (b) A plot of $\Delta u^2 + \Delta v^2$ of the two-mode light in the cavity at steady state vs η for $\kappa=0.8$, $A=100$, and $\varepsilon=0.38$.

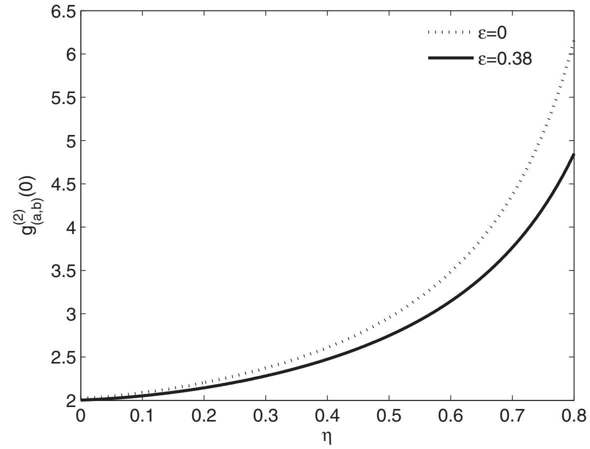

FIG. 9. Plots of the normalized second-order correlation function for the two-mode light, $g^{(2)}_{(a,b)}(0)$, at steady state vs η for $\kappa=0.8$, and $A=10$, in the absence (dotted curve) and in the presence (solid curve) of the parametric oscillator.

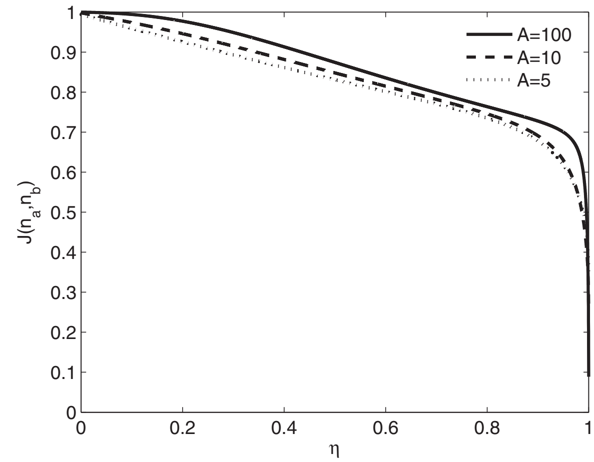

FIG. 10. Plots of the linear correlation coefficient $J(\hat{n}_a,\hat{n}_b)$ at steady state vs η for $\kappa=0.8$, $\varepsilon=0.38$, and for different values of the linear gain coefficient.

$$g^{(2)}_{(a,b)}(0) = 1 + \frac{\kappa[2\kappa + A(1+\eta)]^2}{\chi}, \tag{45a}$$

where

$$\chi = (\kappa + A\eta)[16\varepsilon^2 - A^2(1-\eta^2)] + A(1-\eta)(2\kappa + A\eta)[2\kappa + A(1+\eta)]. \tag{45b}$$

We plot, in Fig. 9, the second-order correlation function for the two-mode light in the presence and absence of the parametric oscillator. We easily see from this figure that $g^{(2)}_{(a,b)}(0)$ increases with η in both cases. We also see from the same figure that the effect of the parametric oscillator is to decrease the second-order correlation function. As this function deals with the correlation between the photon numbers, as already seen in the plots, it would not be a direct measure of the squeezing as well as entanglement in this system. However, as we will see below, it could be an indirect inference for the existence of squeezing and entanglement as it exhibits quantum correlation that violates certain classical inequalities.

It is then essential to calculate the second-order correlation function for the individual mode to have an insight for the previous result. To this end, the second order correlation function for mode a is given by

$$g^{(2)}_{(a,a)}(0) = \frac{\langle :\hat{n}_a\hat{n}_a: \rangle}{\langle \hat{n}_a \rangle^2}, \tag{46}$$

where :: represent normal ordering and $\hat{n}_a=\hat{a}^\dagger\hat{a}$ is the photon number operator for mode a. Since $\hat{a}$ is a Gaussian variable with vanishing mean, one can easily verify with the help of Eq. (11) that

$$g^{(2)}_{(a,a)}(0) = 2. \tag{47}$$

Similarly, the second-order correlation function for mode b is found to be

$$g^{(2)}_{(b,b)}(0) = 2. \tag{48}$$

We note that expressions (47) and (48) represent the second-order correlation function for light in a chaotic state. Thus, the cavity modes are separately in a chaotic or thermal state. Now comparing $[g^{(2)}_{(a,b)}(0)]^2$ with $g^{(2)}_{(a,a)}(0)g^{(2)}_{(b,b)}(0)$ we easily see that $[g^{(2)}_{(a,b)}(0)]^2 > g^{(2)}_{(a,a)}(0)g^{(2)}_{(b,b)}(0)$. This is a violation of the Cauchy-Schwarz inequality [34]. Therefore, we infer from this result that the two-mode light exhibits quantum-mechanical correlation which is the source for the squeezing as well as entanglement in our system.

In order to quantify the correlation between the two modes, we introduce the linear correlation coefficient defined as [35]

$$J(\hat{n}_a,\hat{n}_b) = \frac{\mathrm{cov}(\hat{n}_a,\hat{n}_b)}{\sqrt{\Delta n_a^2}\sqrt{\Delta n_b^2}}, \tag{49}$$

where Δn_a^2 and Δn_b^2 are the variances of the photon number for modes a and b, respectively. The covariance of the photon numbers is defined by

$$\mathrm{cov}(\hat{n}_a,\hat{n}_b) = \langle \hat{n}_a\hat{n}_b \rangle - \langle \hat{n}_a \rangle\langle \hat{n}_b \rangle. \tag{50}$$

It can be verified, using the fact that $\hat{a}$ and $\hat{b}$ are Gaussian variables, in the steady state that

$$\mathrm{cov}(\hat{n}_a,\hat{n}_b) = \langle \hat{a}\hat{b} \rangle_{ss}\langle \hat{a}^\dagger\hat{b}^\dagger \rangle_{ss}. \tag{51}$$

Further, since the cavity modes are separately in a chaotic state the variances of the photon numbers obey the relation for a chaotic state, $\Delta n_a^2=\langle \hat{n}_a\rangle+\langle \hat{n}_a\rangle^2$ and $\Delta n_b^2=\langle \hat{n}_b\rangle+\langle \hat{n}_b\rangle^2$. On account of this fact and Eq. (51), the linear correlation coefficient takes the form

$$J(\hat{n}_a,\hat{n}_b) = \frac{\langle \hat{a}\hat{b} \rangle_{ss}\langle \hat{a}^\dagger\hat{b}^\dagger \rangle_{ss}}{\sqrt{\langle \hat{n}_a\rangle_{ss} + \langle \hat{n}_a\rangle_{ss}^2}\sqrt{\langle \hat{n}_b\rangle_{ss} + \langle \hat{n}_b\rangle_{ss}^2}}. \tag{52}$$

In Fig. 10, we plot the linear correlation coefficient vs the parameter η. We clearly see from this figure that the intermode correlation is maximum at $\eta=0$ and minimum at $\eta=0$ which correspond to the maximum injected atomic co-

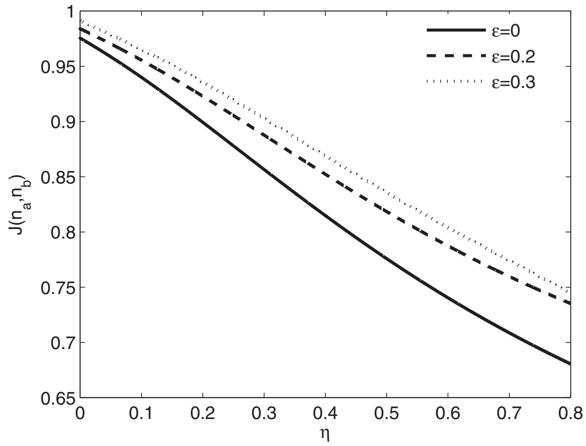

FIG. 11. Plots of the linear correlation coefficient $J(\hat{n}_a,\hat{n}_b)$ at steady state vs η for $\kappa=0.8$, $A=10$, and for different values of ε.

herence, $\rho_{ac}^{(0)}=1/2$, and no injected atomic coherence, $\rho_{ac}^{(0)}=0$. It is also easy to see that the degree of intermode correlation decreases with η similar to the injected atomic coherence which varies according to Eq. (9). Moreover, the degree of intermode correlation increases with the linear gain coefficient which is consistent with the squeezing and entanglement. We notice that there is a critical value of the atomic coherence (or degree of intermode correlation) that exhibits maximum squeezing and entanglement in this system. In general, the degree of squeezing and entanglement increases with the atomic coherence (or degree of intermode correlation). Furthermore, to see the effect of the parametric oscillator on the degree of intermode correlation, we plot in Fig. 11, the linear correlation coefficient vs η for different values of ε (which represents the parametric oscillator in our system). We notice from this figure that the presence of the parametric oscillator enhances the intermode correlations. This is due to the fact that the nondegenerate parametric oscillator produces highly correlated pairs of photons.

VI. CONCLUSION

We have studied the entanglement and squeezing properties of the two-mode light generated by a nondegenerate three-level laser with a subthreshold nondegenerate parametric oscillator. We have obtained the master equation in the good-cavity limit and in the linear and adiabatic approximation schemes. Applying the master equation, we have derived equations of evolution of the moments of the cavity mode variables. Making use of these equations we have calculated the quadrature variance for the two-mode light inside and outside the cavity at steady state. We have also analyzed at steady state the entanglement of the two modes inside and outside the cavity. The normalized second-order correlation function has been calculated for the individual mode as well as for the superposition of the two modes. Finally, we have calculated the linear correlation coefficient between the two modes.

We have found that the two-mode light exhibits a two-mode squeezing at steady state when the three-level atoms are initially prepared in such a way that more atoms are in the lower level than in the upper level. A relatively better squeezing has been observed for large values of the linear gain coefficient and in the vicinity of $\eta=0$, that is, when very slightly more atoms are in the lower level at the initial time. We have also found that the effect of the parametric oscillator is to increase the degree of squeezing over and above the squeezing obtained from the nondegenerate three-level laser. Furthermore, it is found that the state of the system is strongly entangled at steady state. We have shown that the degree of entanglement in the two-mode light is directly related to the two-mode squeezing. Whenever there is squeezing in the two-mode light, there exists entanglement in the system. Since the parametric oscillator introduces additional squeezing to the system, the degree of entanglement in the system has been enhanced. We have noticed that the normalized second-order correlation of two-mode light increase as the degree of squeezing increase, in general. However, the linear correlation coefficient, which quantifies the intermode correlation present in the system, increases with the degree of squeezing. Moreover, the presence of the parametric oscillator enhances the intermode correlation over the laser system.

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APPENDIX A: EQUATIONS OF EVOLUTION OF CAVITY MODE EXPECTATION VALUES

In this Appendix, we derive equations of evolution for the moments of the cavity mode variables applying the master equation obtained in Sec. II. We note that equation of evolution for expectation value of an operator $\hat{O}$ can be expressed in terms of the master equation as

$$\frac{d}{dt}\langle\hat{O}\rangle=\mathrm{Tr}\left(\frac{d\hat{\rho}}{dt}\hat{O}\right). \tag{A1}$$

Applying the master equation, Eq. (6a) and using this relation, we readily obtain the following equations:

$$\frac{d}{dt}\langle\hat{a}\rangle=-\frac{1}{2}(\kappa-A\rho_{aa}^{(0)})\langle\hat{a}\rangle+\frac{1}{2}(2\varepsilon-A\rho_{ac}^{(0)})\langle\hat{b}^\dagger\rangle, \tag{A2}$$

$$\frac{d}{dt}\langle\hat{b}\rangle=-\frac{1}{2}(\kappa+A\rho_{cc}^{(0)})\langle\hat{b}\rangle+\frac{1}{2}(2\varepsilon+A\rho_{ac}^{(0)})\langle\hat{a}^\dagger\rangle, \tag{A3}$$

$$\frac{d}{dt}\langle\hat{a}^2\rangle=-(\kappa-A\rho_{aa}^{(0)})\langle\hat{a}^2\rangle+(2\varepsilon-A\rho_{ac}^{(0)})\langle\hat{a}\hat{b}^\dagger\rangle, \tag{A4}$$

$$\frac{d}{dt}\langle\hat{b}^2\rangle=-(\kappa+A\rho_{cc}^{(0)})\langle\hat{b}^2\rangle+(2\varepsilon+A\rho_{ac}^{(0)})\langle\hat{a}^\dagger\hat{b}\rangle, \tag{A5}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = -(\kappa - A\rho_{aa}^{(0)})\langle\hat{a}^\dagger\hat{a}\rangle + A\rho_{aa}^{(0)} + \frac{1}{2}(2\varepsilon - A\rho_{ac}^{(0)})(\langle\hat{a}\hat{b}\rangle + \langle a^\dagger\hat{b}^\dagger\rangle), \tag{A6}$$

$$\frac{d}{dt}\langle\hat{b}^\dagger\hat{b}\rangle = -(\kappa + A\rho_{cc}^{(0)})\langle\hat{b}^\dagger\hat{b}\rangle + \frac{1}{2}(2\varepsilon + A\rho_{ac}^{(0)})(\langle\hat{a}\hat{b}\rangle + \langle\hat{a}^\dagger\hat{b}^\dagger\rangle), \tag{A7}$$

$$\frac{d}{dt}\langle\hat{a}\hat{b}^\dagger\rangle = -\frac{1}{2}[2\kappa + A(\rho_{cc}^{(0)} - \rho_{aa}^{(0)})]\langle\hat{a}\hat{b}^\dagger\rangle + \frac{1}{2}(2\varepsilon + A\rho_{ac}^{(0)})\langle\hat{a}^2\rangle + \frac{1}{2}(2\varepsilon - A\rho_{ac}^{(0)})\langle\hat{b}^{\dagger 2}\rangle, \tag{A8}$$

$$\frac{d}{dt}\langle\hat{a}\hat{b}\rangle = -\frac{1}{2}[2\kappa + A(\rho_{cc}^{(0)} - \rho_{aa}^{(0)})]\langle\hat{a}\hat{b}\rangle + \frac{1}{2}(A\rho_{ac}^{(0)} + 2\varepsilon) + \frac{1}{2}(2\varepsilon + A\rho_{ac}^{(0)})\langle\hat{a}^\dagger\hat{a}\rangle + \frac{1}{2}(2\varepsilon - A\rho_{ac}^{(0)})\langle b^\dagger\hat{b}\rangle. \tag{A9}$$

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