

Thermodynamical properties of a rotating ideal Bose gas

Sebastian Kling*

Institut für Angewandte Physik, Universität Bonn, Wegelerstrasse 8, 53115 Bonn, Germany

Axel Pelster†

Fachbereich Physik, Campus Duisburg, Universität Duisburg-Essen, Lotharstrasse 1, 47048 Duisburg, Germany

(Received 6 April 2006; revised manuscript received 23 March 2007; published 24 August 2007)

In a recent experiment, a Bose-Einstein condensate was trapped in an anharmonic potential that is well approximated by a harmonic and a quartic part. The condensate was set into such a fast rotation that the centrifugal force in the corotating frame overcompensates the harmonic part in the plane perpendicular to the rotation axis. Thus, the resulting trap potential becomes sombrero shaped. We present an analysis for an ideal Bose gas that is confined in such an anharmonic rotating trap within a semiclassical approximation, where we calculate the critical temperature, the condensate fraction, and the heat capacity. In particular, we examine in detail how these thermodynamical quantities depend on the rotation frequency.

DOI: [10.1103/PhysRevA.76.023609](https://doi.org/10.1103/PhysRevA.76.023609)

PACS number(s): 03.75.Hh, 31.15.Gy, 51.30.+i

I. INTRODUCTION

The rotation of a quantum fluid leads to interesting problems. The effect of an angular momentum can be compared with the motion of a charged particle in a magnetic field. Because of the close relation between superfluidity, superconductivity, and Bose-Einstein condensation (BEC), ultracold dilute Bose gases have also been rotated. Initial experiments [1,2] have shown that, analogously to helium II [3], vortices nucleate at some critical rotation frequency. The main difference between earlier studies about quantum fluids and BEC's is the trap in which the condensate is confined. This inspired Fetter [4] to suggest adding a quartic term to the harmonic trap potential. His idea was to rotate the condensate so fast that the centrifugal force may overcompensate the harmonic trapping potential. In a harmonic trap, the condensate gets lost when the rotation frequency comes close to the harmonic trap frequency [5], but the additional anharmonicity ensures the confining of the condensate. Such a trap was realized in an experiment in Paris by Dalibard and co-workers in 2004 [6]. The experimental setup was the usual magneto-optical trap with frequencies $\omega_x = \omega_y = 2\pi \times 75.5$ Hz and $\omega_z = 2\pi \times 11.0$ Hz. The additional quartic anharmonicity was generated with a Gaussian laser beam propagating in the z direction, which created a potential $U = U_0 \exp(-2r_\perp^2/w^2)$ with the perpendicular radius $r_\perp = \sqrt{x^2 + y^2}$, the laser's waist $w = 25 \mu\text{m}$, and the intensity $U_0 \sim (k_B \times 90)$ nK. Due to the experimentally realized condition $r_\perp < w/2$, this potential is well approximated by $U = U_0 - (2U_0/w^2)r_\perp^2 + (2U_0/w^4)r_\perp^4$. In Ref. [6] an amount of 3×10^5 atoms of ^{87}Rb is set into rotation with another laser beam acting as the stirrer. The laser creates an anisotropic potential in the xy plane, which rotates with frequency Ω . In the corotating frame, the resulting trapping potential can be written as

$$V_{\text{rot}}(\mathbf{x}, \Omega) = \frac{M}{2}(\omega_\perp^2 - \Omega^2)r_\perp^2 + \frac{M}{2}\omega_z^2 z^2 + \frac{k}{4}r_\perp^4, \quad (1)$$

where M is the atomic mass and $\omega_\perp = \omega_x - 4U_0M^{-1}w^{-2} = 2\pi \times 64.8$ Hz. The last term in (1) corresponds to the quartic anharmonicity with $k = 8U_0w^{-4} = 2.6 \times 10^{-11}$ J m $^{-4}$. In the following we treat the Bose gas in the anharmonic trap (1) within the grand-canonical ensemble and determine the critical temperature, the condensate fraction, and the heat capacity of the Bose gas within a semiclassical approximation. In our discussion the rotation frequency Ω appears as a control parameter. The experiment of Dalibard *et al.* [6] allows rotation frequencies Ω up to $1.04\omega_\perp$. However, in the present theoretical discussion, we consider arbitrarily large rotation frequencies. In Fig. 1 we depict how the trapping potential (1) varies with increasing rotation frequency Ω . For small rotation frequencies $\Omega < \omega_\perp$, the potential (1) is convex; for the critical rotation frequency $\Omega = \omega_\perp$ it is purely quartic in the perpendicular plane, and for a fast rotation frequency $\Omega > \omega_\perp$ the trap has the shape of a sombrero. In our discussion, we focus on two particular rotation frequencies, namely, the critical rotation frequency $\Omega = \omega_\perp$ and the limit of an infinitely fast rotation frequency $\Omega \rightarrow \infty$, as the poten-

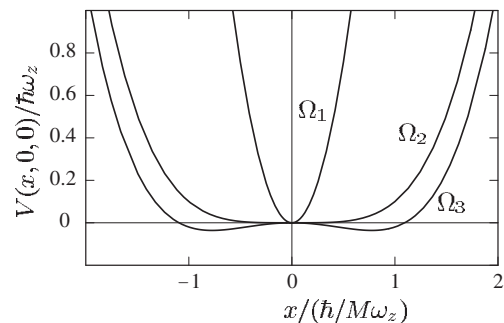


FIG. 1. Trapping potential (1) in the $(x, 0, 0)$ direction for the values of the experiment [6] and varying rotation frequencies $\Omega_1 = 0$, $\Omega_2 = \omega_\perp$, $\Omega_3 = 1.04\omega_\perp$. The last is the largest experimentally realized rotation frequency.

*Electronic address: kling@iap.uni-bonn.de

†Electronic address: axel.pelster@uni-due.de

tial reduces to power laws in both cases. Such potentials were investigated some time ago as they have comparatively simple analytic formulas for the respective thermodynamical properties [7,8]. Note that the case $\Omega \rightarrow \infty$ corresponds to a trap where the bosons are confined to a cylinder of radius $r_{\text{cyl}} = \sqrt{M^2 \omega_z (\Omega^2 - \omega_\perp^2) / (k\hbar)}$. Thus, the above mentioned experimental restriction $r_{\text{cyl}} < w/2$ allows us to determine a maximum rotation frequency Ω_{max} for which the anharmonic potential (1) of the experiment of Dalibard and co-workers is valid. The resulting value $\Omega_{\text{max}} = 1.08\omega_\perp$ shows that considering an infinite rotation frequency $\Omega \rightarrow \infty$ is not suitable for this experiment.

II. IDEAL BOSE GAS IN ROTATING TRAP

We consider N particles of an ideal Bose gas which are distributed over various quantum states ν of the system. These states are characterized through the population $n_{\mathbf{n}}$ of the one-particle state \mathbf{n} of the trap (1), such that the energy levels are $E_\nu = \sum_{\mathbf{n}} n_{\mathbf{n}} E_{\mathbf{n}}$, where $E_{\mathbf{n}}$ denotes the one-particle energy. Correspondingly, the number of particles in state ν is given by $N_\nu = \sum_{\mathbf{n}} n_{\mathbf{n}}$. The resulting grand-canonical ensemble is determined by its partition function

$$\mathcal{Z} = \sum_{\nu} \exp[-\beta(E_\nu - \mu N_\nu)], \quad (2)$$

where $\beta = 1/(k_B T)$ denotes the inverse temperature, k_B is Boltzmann's constant, and μ is the chemical potential. The corresponding grand-canonical free energy $\mathcal{F} = -(1/\beta) \ln \mathcal{Z}$ allows us to calculate all relevant thermodynamical quantities [9,10]. Around the minimum, the trap (1) has a small curvature so that the energy levels are close to each other. With increasing rotation frequency, the curvature decreases until the critical rotation frequency $\Omega = \omega_\perp$ is reached. Not until the rotation frequency overcompensates the harmonic part of the trap does the curvature increase again. Thus, for all experimentally realized rotation frequencies $0 \leq \Omega \leq 1.04\omega_\perp$, our system can be described by the discrete ground-state E_0 , which must be retained quantum-mechanically, plus a continuum of states above E_0 . Within this semiclassical approximation, we can set the ground-state energy E_0 to zero so that the grand-canonical free energy of the ideal Bose gas reads

$$\mathcal{F} = N_0(\mu_c - \mu) - \sum_{j=1}^{\infty} \frac{1}{\beta j} \int \frac{d^3x d^3p}{(2\pi\hbar)^3} \exp\{-\beta j[H(\mathbf{x}, \mathbf{p}) - \mu]\}, \quad (3)$$

where the energy levels are replaced by the classical Hamiltonian

$$H(\mathbf{x}, \mathbf{p}) = \frac{\mathbf{p}^2}{2M} + V_{\text{rot}}(\mathbf{x}, \Omega). \quad (4)$$

The critical chemical potential μ_c , where the condensation emerges, is determined by the condition $H(\mathbf{x}, \mathbf{p}) - \mu > 0$ and is therefore given by $\mu_c = \min_{\mathbf{x}} V_{\text{rot}}(\mathbf{x}, \Omega)$. Due to (1) it reads explicitly

$$\mu_c = \begin{cases} 0, & \Omega \leq \omega_\perp \\ -\frac{M^2}{4k}(\omega_\perp^2 - \Omega^2)^2, & \Omega > \omega_\perp. \end{cases} \quad (5)$$

Performing the phase-space integral in (3), we obtain

$$\mathcal{F} = N_0(\mu_c - \mu) - \frac{\zeta_4(e^{\beta\mu}, \Omega)}{\beta^4 \hbar^3 \omega_z (\omega_\perp^2 - \Omega^2)}, \quad (6)$$

where we have introduced the generalized ζ function

$$\zeta_\nu(e^{\beta\mu}, \Omega) = \sum_{j=1}^{\infty} \frac{e^{j\beta\mu}}{j^\nu} \sqrt{j\pi\gamma_T} (\omega_\perp^2 - \Omega^2) \times \exp[j\gamma_T(\omega_\perp^2 - \Omega^2)] \text{erfc}[\sqrt{j\gamma_T}(\omega_\perp^2 - \Omega^2)] \quad (7)$$

with the complementary error function

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty dt e^{-t^2}. \quad (8)$$

We have also shortened the notation by using $\gamma_T = M^2/(4kk_B T)$ as another inverse temperature. We remark that in the limit of a vanishing anharmonicity $k \downarrow 0$ with an undercritical rotation frequency $\Omega < \omega_\perp$ the generalized ζ function (7) reduces to the polylogarithmic function

$$\lim_{\substack{k \downarrow 0 \\ \Omega < \omega_\perp}} \zeta_\nu(z, \Omega) = \zeta_\nu(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^\nu}, \quad (9)$$

which is related to the Riemann ζ function via

$$\zeta_\nu(1) = \zeta(\nu) = \sum_{j=1}^{\infty} \frac{1}{j^\nu}. \quad (10)$$

Furthermore, we note that, in the limit of the critical rotation frequency $\Omega \rightarrow \omega_\perp$, the generalized ζ function (7) reads

$$\lim_{\Omega \rightarrow \omega_\perp} \frac{\zeta_\nu(z, \Omega)}{\omega_\perp^2 - \Omega^2} = \sqrt{\pi\gamma_T} \zeta_{\nu-1/2}(z), \quad (11)$$

and in the limit of an infinite fast rotation it reads

$$\frac{\zeta_\nu(z, \Omega)}{\omega_\perp^2 - \Omega^2} \approx 2\sqrt{\pi\gamma_T} \sum_{j=1}^{\infty} \frac{z^j e^{j\gamma_T(\omega_\perp^2 - \Omega^2)}}{j^{\nu-1/2}}, \quad \Omega \rightarrow \infty. \quad (12)$$

III. CONDENSATE DENSITY

From the grand-canonical free energy (6), we read off that the number of particles $N = -(\partial\mathcal{F}/\partial\mu)_{T, V}$ of an ideal Bose gas is given by the sum of $N = N_0 + N_e$ of particles in the ground state N_0 and particles in excited states N_e :

$$N = N_0 + \frac{\zeta_3(e^{\beta\mu}, \Omega)}{\beta^3 \hbar^3 \omega_z (\omega_\perp^2 - \Omega^2)}. \quad (13)$$

The critical temperature T_c at which the condensation emerges can be found from Eq. (13) by setting $N_0 = 0$ and

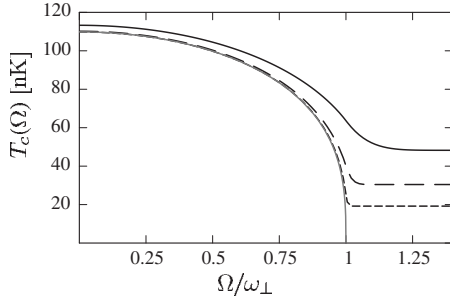


FIG. 2. Critical temperature versus rotation frequency Ω . The highest-lying line (solid) corresponds to the data of the experiment [6] [see data below Eq. (1)]. For the deeper-lying lines we varied the anharmonicity parameter k : $k \rightarrow k/10$ (long dashes), $k \rightarrow k/100$ (short dashes), and the harmonic limit $k \downarrow 0$ (gray solid).

$\mu = \mu_c$. For undercritical rotation frequencies $\Omega < \omega_\perp$ and vanishing anharmonicity $k \downarrow 0$, we apply (9) so that the critical temperature reads

$$T_c = \frac{\hbar \omega_z}{k_B} \left(\frac{(\omega_\perp^2 - \Omega^2)N}{\omega_z^2 \zeta(3)} \right)^{1/3}, \quad k = 0. \quad (14)$$

For a nonvanishing anharmonicity k , the critical temperature cannot be determined explicitly because it appears in Eq. (13) transcendently. However, there are two special cases in which we obtain an analytical expression for the critical temperature. First, for the critical rotation $\Omega = \omega_\perp$, we find with (11) [11]

$$T_c = \frac{\hbar \omega_z}{k_B} \left(\frac{4k\hbar}{\pi M^2 \omega_z^3} \right)^{1/5} \left(\frac{N}{\zeta(5/2)} \right)^{2/5}. \quad (15)$$

Second, the limit of an infinitely fast rotation frequency $\Omega \rightarrow \infty$ leads with (12) to the critical temperature

$$T_c = \frac{\hbar \omega_z}{k_B} \left(\frac{k\hbar}{\pi M^2 \omega_z^3} \right)^{1/5} \left(\frac{N}{\zeta(5/2)} \right)^{2/5}, \quad (16)$$

which is by a factor $(1/4)^{1/5} \approx 0.76$ smaller than the previous one. A numerical evaluation of the critical temperature obtained from (13) is shown in Fig. 2 for the values of the experiment of Dalibard and co-workers [12]. For the nonrotating trap, we see that the anharmonicity affects the critical temperature by only a few percent compared to the harmonic potential. With increasing rotation frequency, the critical temperature in both traps decreases and the difference between the traps is clearly to be seen. At the critical rotation frequency $\Omega = \omega_\perp$, the critical temperature is $T_c = 63.5$ nK, which is about three times smaller than the one estimated for the experiment [6].

From the number of particles (13), we also obtain the condensate fraction in the temperature regime $T < T_c$. Here the chemical potential coincides with the critical one given by Eq. (5). For undercritical rotation frequencies $\Omega < \omega_\perp$ and vanishing anharmonicity $k \downarrow 0$, we use (9) so that the condensate fraction is given by

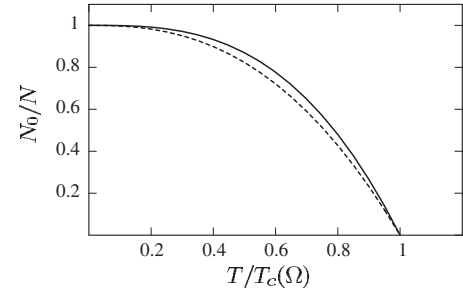


FIG. 3. Condensate fraction versus reduced temperature. The solid line corresponds to the condensate fraction (19) of a Bose gas in the trap (1) for the rotation frequency $\Omega = 0$ and the parameters of the experiment [6]. The dashed line corresponds to the condensate fraction at the critical $\Omega = \omega_\perp$ and at the infinitely fast $\Omega \rightarrow \infty$ rotation frequency given by Eq. (18).

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c} \right)^3, \quad k = 0. \quad (17)$$

Furthermore, applying (11) and (12) in the cases $\Omega = \omega_\perp$ and $\Omega \rightarrow \infty$, respectively, yields with the critical chemical potential (5) in both situations the following condensate fraction:

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c} \right)^{5/2}. \quad (18)$$

Here, T_c is given by (14)–(16), respectively. In general, the condensate fraction is given by

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c} \right)^3 \frac{\zeta_3(e^{\beta \mu_c}, \Omega)}{\zeta_3(e^{\beta \mu_c}, \Omega)}, \quad (19)$$

where $\beta_c = 1/(k_B T_c)$. In the low-temperature limit $T \downarrow 0$, since it coincides with the harmonic limit $k \downarrow 0$ and the infinitely fast rotation $\Omega \rightarrow \infty$, the condensate fraction shows a power-law behavior, which is different in the two regimes of undercritical rotation $\Omega < \omega_\perp$ and overcritical rotation $\Omega > \omega_\perp$. Due to (9) and (12), we obtain

$$\frac{N_0}{N} = \begin{cases} 1 - \frac{k_B^3 \zeta(3)}{N \hbar^3 \omega_z (\omega_\perp^2 - \Omega^2)} T^3, & \Omega < \omega_\perp \\ 1 - \frac{M \sqrt{\pi} k_B^{5/2} \zeta(5/2)}{N \sqrt{k} \hbar^3 \omega_z} T^{5/2}, & \Omega > \omega_\perp. \end{cases} \quad (20)$$

The temperature dependence of the condensate fraction N_0/N following from (19) is shown in Figs. 3 and 4. From this we read off that the temperature dependence of the condensate fraction depends crucially on the rotation frequency and is thus not universal.

IV. HEAT CAPACITY

The heat capacity follows from the grand-canonical free energy $\mathcal{F} = U - TS - \mu N$, where U is the internal energy and S is the entropy, according to

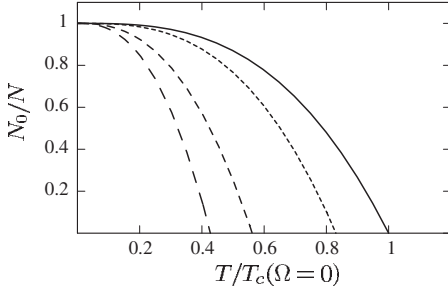


FIG. 4. Condensate fraction versus reduced temperature. The condensate fraction (19) is evaluated for various rotation frequencies Ω . The solid line corresponds to $\Omega=0$ to which the temperature is normalized. The other lines are for $\Omega=\omega_{\perp}/\sqrt{2}$ (short dashes), $\Omega=\omega_{\perp}$ (longer dashes), and $\Omega=\sqrt{3/2}\omega_{\perp}$ (long dashes).

$$C = \left. \frac{\partial U}{\partial T} \right|_{N,V}. \quad (21)$$

Within the grand-canonical ensemble, the heat capacity has to be determined separately in the two regimes $T > T_c$ and $T < T_c$.

A. Gas phase

At first, we treat the gas phase where $N_0=0$ and determine the entropy via the thermodynamic relation $S = -(\partial F / \partial T)_{V,\mu}$:

$$\frac{S_{>}}{k_B N} = \frac{7\zeta_4(e^{\beta\mu}, \Omega)}{2\zeta_3(e^{\beta\mu}, \Omega)} + \gamma_T(\omega_{\perp}^2 - \Omega^2)^2 \frac{\zeta_3(e^{\beta\mu})}{\zeta_3(e^{\beta\mu}, \Omega)} - [\beta\mu + \gamma_T(\omega_{\perp}^2 - \Omega^2)^2]. \quad (22)$$

The internal energy $U = F + TS + \mu N$ then follows from (6), (13), and (22) and reads

$$\frac{U_{>}}{Nk_B T} = \frac{5\zeta_4(e^{\beta\mu}, \Omega)}{2\zeta_3(e^{\beta\mu}, \Omega)} + \gamma_T(\omega_{\perp}^2 - \Omega^2)^2 \left(\frac{\zeta_3(e^{\beta\mu})}{\zeta_3(e^{\beta\mu}, \Omega)} - 1 \right). \quad (23)$$

Finally, the heat capacity (21) for temperatures above T_c is given by

$$\begin{aligned} \frac{C_{>}}{k_B N} &= \frac{35\zeta_4(e^{\beta\mu}, \Omega)}{4\zeta_3(e^{\beta\mu}, \Omega)} - \frac{25\zeta_3(e^{\beta\mu}, \Omega)}{4\zeta_2(e^{\beta\mu}, \Omega)} + \gamma_T(\omega_{\perp}^2 - \Omega^2)^2 \\ &\times \left(\frac{11\zeta_3(e^{\beta\mu})}{2\zeta_3(e^{\beta\mu}, \Omega)} - 5 \frac{\zeta_2(e^{\beta\mu})}{\zeta_2(e^{\beta\mu}, \Omega)} \right) + \gamma_T^2(\omega_{\perp}^2 - \Omega^2)^4 \\ &\times \frac{\zeta_2(e^{\beta\mu})\zeta_2(e^{\beta\mu}, \Omega) - \zeta_2^2(e^{\beta\mu})}{\zeta_2(e^{\beta\mu}, \Omega)\zeta_3(e^{\beta\mu}, \Omega)}. \end{aligned} \quad (24)$$

To obtain this result, we have determined the derivative $(\partial \beta\mu / \partial T)_{N,V}$ from (13). In the limit $k \downarrow 0$ of a harmonic trap, the heat capacity (24) reduces to the well-known result

$$\frac{C_{>}}{k_B N} = 12 \frac{\zeta_4(e^{\beta\mu}, \Omega)}{\zeta_3(e^{\beta\mu}, \Omega)} - 9 \frac{\zeta_3(e^{\beta\mu}, \Omega)}{\zeta_2(e^{\beta\mu}, \Omega)}, \quad k = 0. \quad (25)$$

Again, both cases $\Omega = \omega_{\perp}$ and $\Omega \rightarrow \infty$ yield the same analytic expression:

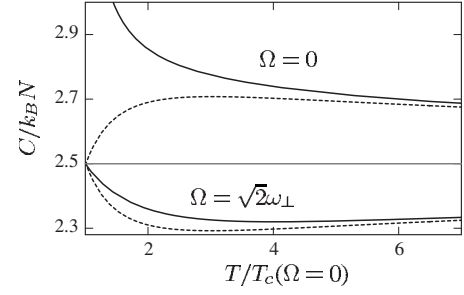


FIG. 5. Approach of the heat capacity (24), solid lines, and its approximation (28), dashed lines, to the Dulong-Petit law, horizontal line.

$$\frac{C_{>}}{k_B N} = \frac{35}{4} \frac{\zeta_{7/2}(e^{\beta\mu + \gamma_T(\omega_{\perp}^2 - \Omega^2)})}{\zeta_{5/2}(e^{\beta\mu + \gamma_T(\omega_{\perp}^2 - \Omega^2)})} - \frac{25}{4} \frac{\zeta_{5/2}(e^{\beta\mu + \gamma_T(\omega_{\perp}^2 - \Omega^2)})}{\zeta_{3/2}(e^{\beta\mu + \gamma_T(\omega_{\perp}^2 - \Omega^2)})}. \quad (26)$$

Now, we investigate the heat capacity (24) in the high-temperature limit $T \rightarrow \infty$. To this end we use a large- T expansion of the generalized ζ function (7):

$$\begin{aligned} \zeta_{\nu}(z, \Omega) &\approx e^{\beta\mu} [\sqrt{\pi} \gamma_T(\omega_{\perp}^2 - \Omega^2) - 2\gamma_T(\omega_{\perp}^2 - \Omega^2)^2 + \dots] \\ &+ \frac{e^{2\beta\mu}}{2^{\nu}} [\sqrt{2} \gamma_T(\omega_{\perp}^2 - \Omega^2) - 4\gamma_T(\omega_{\perp}^2 - \Omega^2)^2 + \dots] \\ &+ \dots \end{aligned} \quad (27)$$

Inserting the expansion (27) into the number of particles (13), we find for the first order of the fugacity $e^{\beta\mu} \approx 2N\sqrt{\hbar^6 \omega_{\perp}^2 k} / \sqrt{\pi M^2 k_B^5 T^5}$, so that the heat capacity (24) behaves as

$$\frac{C_{>}}{k_B N} \approx \frac{5}{2} + \frac{\gamma_T(\omega_{\perp}^2 - \Omega^2)}{4\sqrt{\pi}} - \frac{\gamma_T^3(4 - \pi)(\omega_{\perp}^2 - \Omega^2)^3}{32\pi^{3/2}}. \quad (28)$$

Thus, the heat capacity approaches the Dulong-Petit law in an anharmonic trap $\lim_{T \rightarrow \infty} C_{>}/(k_B N) = 5/2$ which is 1/2 smaller than the corresponding one in the harmonic trap. Furthermore, the first Ω dependent term in (28) changes its behavior, from being larger ($\Omega < \omega_{\perp}$) than the limit to being smaller ($\Omega > \omega_{\perp}$), see Fig. 5.

At the critical point, the harmonic heat capacity (25) reduces for small rotation frequencies $\Omega < \omega_{\perp}$ to

$$\lim_{T \uparrow T_c} \frac{C_{>}}{k_B N} = 12 \frac{\zeta(4)}{\zeta(3)} - 9 \frac{\zeta(3)}{\zeta(2)} \approx 4.23, \quad k = 0. \quad (29)$$

In both limits $\Omega = \omega_{\perp}$ and $\Omega \rightarrow \infty$, the heat capacity (26) at the critical point is given by

$$\lim_{T \uparrow T_c} \frac{C_{>}}{k_B N} = \frac{35}{4} \frac{\zeta(7/2)}{\zeta(5/2)} - \frac{25}{4} \frac{\zeta(5/2)}{\zeta(3/2)} \approx 4.14. \quad (30)$$

B. Condensate phase

Now we turn to the condensate phase $T < T_c$ where the chemical potential is given by (5). For the entropy, we obtain

$$\frac{S_{<}}{k_B N} = \left(\frac{T}{T_c}\right)^3 \left[\frac{7\zeta_4(e^{\beta\mu_c}, \Omega)}{2\zeta_3(e^{\beta\mu_c}, \Omega)} - \beta\mu_c \frac{\zeta_3(e^{\beta\mu_c}, \Omega)}{\zeta_3(e^{\beta\mu_c}, \Omega)} \right. \\ \left. + \gamma_T(\omega_\perp^2 - \Omega^2)^2 \left(\frac{\zeta_3(e^{\beta\mu_c}) - \zeta_3(e^{\beta\mu_c}, \Omega)}{\zeta_3(e^{\beta\mu_c}, \Omega)} \right) \right]. \quad (31)$$

The internal energy below the critical temperature reads

$$\frac{U_{<}}{Nk_B T} = \left(\frac{T}{T_c}\right)^3 \left[\frac{5\zeta_4(e^{\beta\mu_c}, \Omega)}{2\zeta_3(e^{\beta\mu_c}, \Omega)} + \gamma_T(\omega_\perp^2 - \Omega^2)^2 \right. \\ \left. \times \left(\frac{\zeta_3(e^{\beta\mu_c}) - \Theta(\omega_\perp - \Omega)\zeta_3(e^{\beta\mu_c}, \Omega)}{\zeta_3(e^{\beta\mu_c}, \Omega)} \right) \right] + \mu_c N, \quad (32)$$

where Θ denotes the Heaviside function. Thus, we find for the heat capacity below the critical temperature with the help of Eq. (5)

$$\frac{C_{<}}{k_B N} = \left(\frac{T}{T_c}\right)^3 \left[\frac{35\zeta_4(e^{\beta\mu_c}, \Omega)}{4\zeta_3(e^{\beta\mu_c}, \Omega)} + \gamma_T(\omega_\perp^2 - \Omega^2)^2 \right. \\ \left. \times \left(\frac{11\zeta_3(e^{\beta\mu_c})}{2\zeta_3(e^{\beta\mu_c}, \Omega)} - 5\Theta(\omega_\perp - \Omega) \frac{\zeta_3(e^{\beta\mu_c}, \Omega)}{\zeta_3(e^{\beta\mu_c}, \Omega)} \right) \right. \\ \left. + \gamma_T^2(\omega_\perp^2 - \Omega^2)^4 \left(\Theta(\Omega - \omega_\perp) \frac{\zeta_2(e^{\beta\mu_c})}{\zeta_3(e^{\beta\mu_c}, \Omega)} \right. \right. \\ \left. \left. - \Theta(\omega_\perp - \Omega) \frac{\zeta_2(e^{\beta\mu_c}) - \zeta_2(e^{\beta\mu_c}, \Omega)}{\zeta_3(e^{\beta\mu_c}, \Omega)} \right) \right]. \quad (33)$$

In the limit $k \downarrow 0$, Eq. (33) simplifies to

$$\frac{C_{<}}{k_B N} = 12 \frac{\zeta(4)}{\zeta(3)} \left(\frac{T}{T_c}\right)^3, \quad k = 0. \quad (34)$$

Thus, at the critical point it has the value

$$\lim_{T \uparrow T_c} \frac{C_{<}}{k_B N} = 12 \frac{\zeta(4)}{\zeta(3)} \approx 10.80, \quad k = 0. \quad (35)$$

In both limits $\Omega = \omega_\perp$ and $\Omega \rightarrow \infty$, the heat capacity is given by

$$\frac{C_{<}}{k_B N} = \frac{35}{4} \frac{\zeta(7/2)}{\zeta(5/2)} \left(\frac{T}{T_c}\right)^{5/2}, \quad (36)$$

so that the heat capacity (33) at the critical point reduces to

$$\lim_{T \uparrow T_c} \frac{C_{<}}{k_B N} = \frac{35}{4} \frac{\zeta(7/2)}{\zeta(5/2)} \approx 7.35. \quad (37)$$

In the low-temperature limit $T \downarrow 0$, the heat capacity (33) tends to zero in accordance with the third law of thermodynamics. We note that the low-temperature limit of the heat capacity (38) has the same power-law behavior as the corresponding one of the condensate fraction (20):

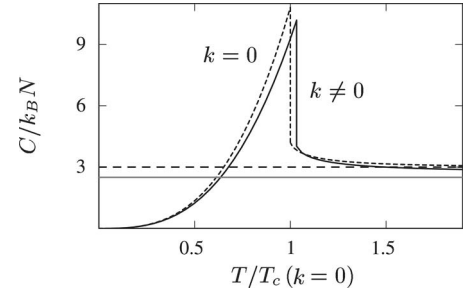


FIG. 6. Heat capacity versus temperature, reduced to the critical temperature (14) of the trap (1) with $k=0$. The dashed line corresponds to the harmonic trap heat capacity (25) and (34) for the trap (1) of Dalibard and co-workers with $k=0$. The black solid line is the heat capacity (24) and (33) for the anharmonic trap of [6] with no rotation, $\Omega=0$. The horizontal lines correspond to the Dulong-Petit law: harmonic trap (dashed) and anharmonic trap (gray).

$$\frac{C_{<}}{k_B N} \approx \begin{cases} \frac{35k_B^3 \zeta(4)}{4N\hbar^3 \omega_z (\omega_\perp^2 - \Omega^2)} T^3, & \Omega < \omega_\perp \\ \frac{35M \sqrt{\pi} k_B^{5/2} \zeta(7/2)}{4N \sqrt{k} \hbar^3 \omega_z} T^{5/2}, & \Omega \geq \omega_\perp. \end{cases} \quad (38)$$

Figure 6 shows the temperature dependence of the heat capacity for the values of the Dalibard *et al.* experiment without rotation [6]. We see that the effect of the anharmonicity is rather small. According to Ehrenfest's classification, the discontinuity at the critical temperature characterizes the phase transition as of second order. In Fig. 7 we show how the heat capacity depends on the rotation frequency Ω . Here, the rotation has a big influence on the heat capacity.

V. CONCLUSIONS

We have determined the critical temperature T_c at which the condensation of a rotating ideal Bose gas occurs in the anharmonic trap (1). We have found that condensation is possible even in the overcritical rotation regime $\Omega > \omega_\perp$,

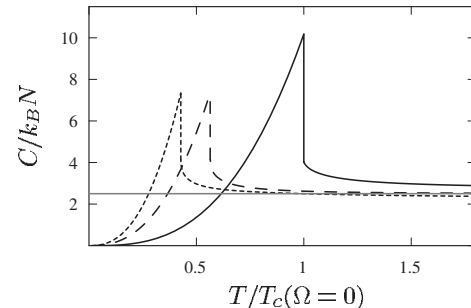


FIG. 7. Heat capacity versus temperature, reduced to the critical temperature $T_c(\Omega=0)$ of the anharmonic trap (1) for varying rotation frequencies. The lines correspond to $\Omega=0$ (solid), $\Omega=\omega_\perp$ (long dashes), and $\Omega=2\omega_\perp$ (short dashes). The gray solid line corresponds to the Dulong-Petit law, the first term of (28). We note that for $\Omega=2\omega_\perp$ and $T > T_c$ the heat capacity approaches the Dulong-Petit limit from below (see Fig. 5).

which is in contrast with the harmonic trap, where the condensate gets lost when the rotation frequency Ω gets close to the trap frequency ω_{\perp} . However, the value $T_c \approx 64$ nK is about three times smaller than the one estimated in the experiment of Dalibard and co-workers [6]. This huge discrepancy could not be explained with the circumstance that our semiclassical analysis of the rotating ideal Bose gas neglects three important aspects, namely, finite-size corrections, interactions between the particles, and the effect of vortices in the condensate. All three should have the effect of lowering the critical temperature. In a harmonic trap, it was found numerically that the finiteness of the system slightly lowers the critical temperature [13], which was also shown analytically [14,15]. Furthermore, an additional weak repulsive two-particle contact interaction leads to a negative shift of just a few percent [16,17]. Therefore, we conclude that our semiclassical findings ask for a revised experimental measurement of the critical temperature for a rotating Bose gas.

For the condensate fraction, we can state that the low-temperature behavior depends on the rotation frequency Ω . It shows a nonuniform temperature dependence that is in between the two power laws $T^{5/2}$ and T^3 .

The heat capacity of the rotating ideal Bose gas is discontinuous at the critical temperature. In agreement with the third law of thermodynamics, it tends to zero in the low-temperature limit $T \downarrow 0$, and approaches the Dulong-Petit law (28) in the high-temperature limit $T \rightarrow \infty$ from above ($\Omega \leq \omega_{\perp}$) or from below ($\Omega > \omega_{\perp}$).

ACKNOWLEDGMENTS

We thank Jean Dalibard and Sabine Stock for critical reading of the manuscript as well as Robert Graham for hospitality at the University Duisburg-Essen. Furthermore, support from the DFG Priority Program SPP 1116 “Interaction in Ultracold Gases of Atoms and Molecules” is acknowledged.

-
- [1] F. Chevy, K. W. Madison, and J. Dalibard, *Phys. Rev. Lett.* **85**, 2223 (2000).
 - [2] C. Raman, J. R. Abo-Shaer, J. M. Vogels, K. Xu, and W. Ketterle, *Phys. Rev. Lett.* **87**, 210402 (2001).
 - [3] R. J. Donnelly, *Quantized Vortices in Helium II* (Cambridge University Press, Cambridge, U.K., 1991).
 - [4] A. L. Fetter, *Phys. Rev. A* **64**, 063608 (2001).
 - [5] P. Rosenbusch, D. S. Petrov, S. Sinha, F. Chevy, V. Bretin, Y. Castin, G. Shlyapnikov, and J. Dalibard, *Phys. Rev. Lett.* **88**, 250403 (2002).
 - [6] V. Bretin, S. Stock, Y. Seurin, and J. Dalibard, *Phys. Rev. Lett.* **92**, 050403 (2004).
 - [7] V. Bagnato, D. E. Pritchard, and D. Kleppner, *Phys. Rev. A* **35**, 4354 (1987).
 - [8] Ph. W. Courteille, V. S. Bagnato, and V. I. Yukalov, *Laser Phys.* **11**, 659 (2001).
 - [9] L. P. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Oxford Science, Oxford, 2003).
 - [10] C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases* (Cambridge University Press, Cambridge, U.K., 2002).
 - [11] S. Stock, B. Battelier, V. Bretin, Z. Hadzibabic, and J. Dalibard, *Laser Phys. Lett.* **2**, 275 (2005).
 - [12] S. Stock, Ph.D. thesis, Université Paris 6, France, 2006 (unpublished).
 - [13] W. Ketterle and N. J. van Druten, *Phys. Rev. A* **54**, 656 (1996).
 - [14] S. Grossmann and M. Holthaus, *Phys. Lett. A* **208**, 188 (1995).
 - [15] T. Haugset, H. Haugerud, and J. O. Andersen, *Phys. Rev. A* **55**, 2922 (1997).
 - [16] S. Giorgini, L. P. Pitaevskii, and S. Stringari, *Phys. Rev. A* **54**, R4633 (1996).
 - [17] F. Gerbier, J. H. Thywissen, S. Richard, M. Hugbart, P. Bouyer, and A. Aspect, *Phys. Rev. Lett.* **92**, 030405 (2004).