Bounds on negativity of superpositions

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For pure bipartite superposed states, the entanglement quantified by negativity is studied. If the entanglement is quantified by concurrence, we show that two pure states with high fidelity to one another have nearly the same entanglement. We deduce an inequality in which the concurrence is known to be a continuous function in infinite dimensions. The main result of this paper is to give the bounds on the negativity of a bipartite state in terms of the entanglement of the states being superposed. These bounds may be used in estimating the entanglement of a given state.

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Quantum entanglement plays an important role both in many aspects of quantum information theory [1] and in describing quantum phase transition in quantum many-body systems [2,3]. As such, characterization of quantum entanglement is a fundamental issue. Consequently, the legitimate measures of entanglement are desirable as a first step. The existing well-known measures of entanglement for two-qubit systems with an elegant formula are the concurrence derived analytically by Wootters [4] and the entanglement of formation [5,6] which is a monotonically increasing function of the concurrence. In general, for a multipartite or higher-dimensional system, it is a formidable task to quantify its entanglement since a complicate convex-roof extension is needed. In recent decades, some important properties of quantum entanglement were found, one of which is the monogamy property described by the Coffman-Kundu-Wootters inequality in terms of concurrence [7]. In our previous work, we have shown that the monogamy inequality can not be generalized to higher-dimensional systems [8] and we established a monogamy inequality in terms of negativity, giving a different residual entanglement [9].

On the other hand, quantum entanglement is a direct consequence of the superposition principle of quantum mechanics. It is an interesting physical phenomenon that the superposition of two separable states may give birth to an entangled state, on the contrary, the superposition of two entangled states may give birth to a separable state. The relationships between the entanglement of a given state and that of the individual terms which by superposition yield the state have been studied, where the entanglement is quantified by the von Neumann entropy [10], the concurrence [12], and witnessed entanglement [13], respectively. Recently, it was generalized to the superposition of more than two components [14]. If the entanglement is quantified by negativity, it would be interesting to establish the analogous relation and obtain the bounds of entanglement for the superposition states. In this paper, we first deduce an inequality to guarantee that the concurrence is a continuous function even in infinite dimensions. Next, we give the bounds of negativity of the superposition states. The discussions and conclusions are presented in the final part.

Before giving the main results of this paper, we first prove Theorem 1, which determines how much the entanglement of a state ρ changes when we vary ρ by a small amount. The authors in [10] have shown that two states of high fidelity to one another may not have the same entanglement, i.e., $|\langle\psi|\phi\rangle|^2\rightarrow 1$ may not generally result in $E(\psi)\rightarrow E(\phi)$, where E is the von Neumann entropy. For a bipartite pure state $|\Phi\rangle_{AB}$, the von Neumann entropy is defined as

$$E(\Phi_{AB}) \equiv S(\mathrm{Tr}_B|\Phi\rangle_{AB}\langle\Phi|) = S(\mathrm{Tr}_A|\Phi\rangle_{AB}\langle\Phi|), \tag{1}$$

where $S(\rho)=-\mathrm{Tr}(\rho\log\rho)$, and the concurrence is defined as

$$\mathcal{C}(\Phi_{AB}) \equiv \sqrt{2(1-\mathrm{Tr}\rho_A^2)} = \sqrt{2\left(1-\sum_i \mu_i^2\right)}, \tag{2}$$

where $\rho_A=\mathrm{Tr}_B|\Phi\rangle_{AB}\langle\Phi|$ with the eigenvalues μ_i. However, if we employ the concurrence to quantify the entanglement, $|\langle\psi|\phi\rangle|^2\rightarrow 1$ must result in $\mathcal{C}(\psi)\rightarrow\mathcal{C}(\phi)$. Let us see their example

$$|\phi\rangle_{AB} = |00\rangle, \tag{3}$$

and

$$|\psi\rangle_{AB} = \sqrt{1-\epsilon}|\phi\rangle_{AB} + \sqrt{\frac{\epsilon}{d}}[|11\rangle + |22\rangle + \cdots + |dd\rangle]. \tag{4}$$

It is obvious that $E(\phi_{AB})=\mathcal{C}(\phi_{AB})=0$, while according to [10] the von Neumann entropy of the state $|\psi\rangle_{AB}$ is

$$E(\psi_{AB}) \approx \epsilon\log_2 d. \tag{5}$$

For sufficiently large d and small fixed ϵ, $E(\psi_{AB})\rightarrow\infty$. The concurrence in Eq. (2) gives the result

$$\mathcal{C}^2(\psi_{AB}) = 2\left(2\epsilon-\epsilon^2-\frac{\epsilon^2}{d}\right). \tag{6}$$

When ϵ is small, $\mathcal{C}^2(\psi_{AB})\rightarrow 0$. In this situation, by contrast to $E(\psi_{AB})$ in Eq. (5), the contribution of d to $\mathcal{C}^2(\psi_{AB})$ in Eq. (6) can be ignored. Note that when ϵ is small the two states have high fidelity $|\langle\psi|\phi\rangle|^2=1-\epsilon\rightarrow 1$. Comparing Eq. (5) to Eq. (6), we can still draw a conclusion that if the entanglement is quantified by the concurrence, two states of high fidelity with one another still have nearly the same entanglement.

Indeed, the difference of the von Neumann entropy between two pure states of fixed dimension can be bounded using Fannes' inequality [11], while the von Neumann entropy is not a continuous function and no such bound applies in infinite dimensions. However, as we will show here, a similar bound still works if the entanglement is quantified by the concurrence and the concurrence is a continuous function even in infinite dimensions. In order to explain our above viewpoint we present the following Theorem which is similar to the original Fannes' inequality except that the entanglement is quantified by the concurrence.

Theorem 1. Suppose ρ_{AB} and σ_{AB} are density matrices of two bipartite pure states in arbitrary dimensions. For the trace distance $T(\rho_A,\sigma_A)\equiv \mathrm{Tr}|\rho_A-\sigma_B|$ between $\rho_A=\mathrm{Tr}_B\rho_{AB}$ and $\sigma_A=\mathrm{Tr}_B\sigma_{AB}$ we have

$$|\mathcal{C}^2(\rho_{AB})-\mathcal{C}^2(\sigma_{AB})|\leqslant 4T(\rho_A,\sigma_A). \tag{7}$$

Proof. Let $r_1\geqslant r_2\geqslant\cdots\geqslant r_d$ be the eigenvalues of ρ_A, in decreasing order, and $s_1\geqslant s_2\geqslant\cdots\geqslant s_d$ be the eigenvalues of σ_A, also in decreasing order. According to [11], it follows that

$$\sum_i |r_i-s_i|\leqslant T(\rho_A,\sigma_A). \tag{8}$$

From the observation of the definition of the concurrence in Eq. (2), we can rewrite the left-hand side of Eq. (7) as

$$\begin{aligned}|\mathcal{C}^2(\rho_{AB})-\mathcal{C}^2(\sigma_{AB})| &= 2\left|\sum_i (r_i^2-s_i^2)\right| \leqslant 2\sum_i |r_i^2-s_i^2|\\ &= 2\sum_i |r_i+s_i||r_i-s_i|\leqslant 4\sum_i |r_i-s_i|.\end{aligned} \tag{9}$$

The second formula is obtained from the observation that $|a+b+\cdots+k|\leqslant|a|+|b|+\cdots+|k|$ for any complex quantities $a,b,\cdots,k$. In the derivation of the last formula we have taken into account the fact that $|r_i+s_i|\leqslant 2$ since each eigenvalue of r_i and s_i is not greater than one. Combining Eqs. (8) and (9) can give Eq. (7). Thus the proof is completed.

From Theorem 1 it can be seen that the difference of the concurrences of two pure states is a function of fidelity and can be bounded by Eq. (7). What's more, by contrast to the von Neumann entropy [10], the concurrence is a continuous function and such a bound still works in infinite dimensions. Note that the question of whether a similar bound in Eq. (7) holds for the negativity is still open. In the following we are devoted to deducing the bounds on the negativity of any bipartite pure state as a superposition of two terms $|\Gamma\rangle_{AB}=\alpha|\Psi\rangle+\beta|\Phi\rangle$.

Before embarking on this study, we first recall some basic definitions of the negativity [15]. As for detecting an entangled state in higher-dimensional Hilbert space, Peres-Horodecki criterion based on partial transpose [16,17] is a convenient method. Given a density matrix ρ of a bipartite state in A and B, the partial transpose with respect to the A subsystem is described by $(\rho^{T_A})_{ij,kl}=(\rho)_{kj,il}$ and the negativity is defined as

$$\mathcal{N}=\frac{1}{2}(\|\rho^{T_A}\|-1). \tag{10}$$

The trace norm $\|R\|$ is given by $\|R\|=\mathrm{Tr}\sqrt{RR^\dagger}$. Note that $\mathcal{N}>0$ is the necessary and sufficient condition for inseparable bipartite pure states.

There are two key ingredients to obtain the bounds of the negativity for bipartite superposition pure states. One is that the negativity can be expressed by means of Schmidt coefficients of a pure state. Suppose that a pure $m\otimes n(m\leqslant n)$ quantum state has the standard Schmidt form $|\psi\rangle_{AB}=\Sigma_i\sqrt{\mu_i}|a_ib_i\rangle$, where $\sqrt{\mu_i}$ $(i=1,\cdots,m)$ are the Schmidt coefficients, a_i and b_i are the orthogonal basis in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. For the pure bipartite state we can derive $\|\rho^{T_A}\|=(\Sigma_i\sqrt{\mu_i})^2$ [19], and therefore Eq. (10) can be reexpressed as

$$\mathcal{N}=\frac{1}{2}\left[\left(\sum_i \sqrt{\mu_i}\right)^2-1\right]. \tag{11}$$

In order for later use we can transform Eq. (11) into

$$\left(\sum_i \sqrt{\mu_i}\right)^2=2\mathcal{N}+1. \tag{12}$$

Another one is the Theorem [18] which states that for any two Hermitian matrices H and K defined in $\mathcal{C}^{n\times n}$,

$$\mu_i(H)+\mu_1(K)\leqslant\mu_i(H+K)\leqslant\mu_i(H)+\mu_n(K), \tag{13}$$

holds, where $\mu_i(\cdot)$ are the eigenvalues in increasing order. If $\mu_1(K)\geqslant 0$, from Eq. (13) it is easy to check that

$$\sqrt{\mu_i(H)}\leqslant\sqrt{\mu_i(H+K)}\leqslant\sqrt{\mu_i(H)}+\sqrt{\mu_n(K)}, \tag{14}$$

holds also. Then Eqs. (13) and (14) will be used repeatedly in what follows.

For the negativity of the arbitrary superposition state let us first see the simplest case in which two bipartite states we are superposing, Φ_1 and Ψ_1, are biorthogonal [10], i.e., $\Phi_1\Psi_1^\dagger=\Psi_1\Phi_1^\dagger=0$ [12]. Since the matrix representation of a reduced density matrix will be used, we explain the corresponding notations in the following. For the pure state $|\Phi\rangle_{AB}$ defined in $m\otimes n$ dimensions, generally it can be considered as a vector: $|\Phi\rangle_{AB}=[a_{00},a_{01},\cdots,a_{0m},a_{10},a_{11},\cdots,a_{mn}]^T$ with the superscript T denoting transpose operation. With the matrix notation, the reduced density matrix reads

$$\rho_A=\Phi\Phi^\dagger, \tag{15}$$

whose eigenvalues are μ_i appearing in Eq. (11).

Theorem 2. Suppose that two biorthogonal pure states Φ_1 and Ψ_1 are defined in $m\otimes n(n\leqslant m)$ dimensions. The negativity of their superposed states $\Gamma_1=\alpha\Phi_1+\beta\Psi_1$ with $|\alpha^2|+|\beta|^2=1$ satisfies

$$\begin{aligned}&\frac{2|\alpha|^2\mathcal{N}(\Phi_1)+2|\beta|^2\mathcal{N}(\Psi_1)-1}{4}\\ &\leqslant\mathcal{N}(\alpha\Phi_1+\beta\Psi_1)\\ &\leqslant\frac{2|\alpha|^2\tilde{\mathcal{N}}(\Phi_1)+2|\beta|^2\tilde{\mathcal{N}}(\Psi_1)-1}{4},\end{aligned} \tag{16}$$

where

$$\tilde{\mathcal{N}}(\Phi_1) = \mathcal{N}(\Phi_1) + \frac{n|\beta|\sqrt{\mu_n(\Psi_1)[2\mathcal{N}(\Phi_1)+1]}}{|\alpha|} + \frac{n^2|\beta|^2\mu_n(\Psi_1)}{2|\alpha|^2},$$

and

$$\tilde{\mathcal{N}}(\Psi_1) = \mathcal{N}(\Psi_1) + \frac{n|\alpha|\sqrt{\mu_n(\Phi_1)[2\mathcal{N}(\Psi_1)+1]}}{|\beta|} + \frac{n^2|\alpha|^2\mu_n(\Phi_1)}{2|\beta|^2}.$$

Proof. From Eq. (15) the reduced density matrix of the state Γ_1 can read

$$\Gamma_1\Gamma_1^\dagger = |\alpha|^2\Phi_1\Phi_1^\dagger + |\beta|^2\Psi_1\Psi_1^\dagger + \alpha\beta^*\Phi_1\Psi_1^\dagger + \alpha^*\beta\Psi_1\Phi_1^\dagger. \tag{17}$$

The biorthogonal condition with $\Phi_1\Psi_1^\dagger=0$ and $\Psi_1\Phi_1^\dagger=0$ makes Eq. (17) reduce to

$$\Gamma_1\Gamma_1^\dagger = |\alpha|^2\Phi_1\Phi_1^\dagger + |\beta|^2\Psi_1\Psi_1^\dagger. \tag{18}$$

Substituting Eq. (18) into the left inequality of Eq. (13) we have

$$|\alpha|^2\mu_i(\Phi_1\Phi_1^\dagger) + |\beta|^2\mu_1(\Psi_1\Psi_1^\dagger) \leqslant \mu_i(\Gamma_1\Gamma_1^\dagger). \tag{19}$$

Since $\Psi_1\Psi_1^\dagger$ is positive semidefinite, $\mu_1(\Psi_1\Psi_1^\dagger)\geqslant 0$. Thus Eq. (19) becomes

$$|\alpha|^2\mu_i(\Phi_1\Phi_1^\dagger) \leqslant \mu_i(\Gamma_1\Gamma_1^\dagger). \tag{20}$$

Taking the square root of both sides in Eq. (20) and the sum of $\sqrt{\mu_i(\cdot)}$ over all index i, we have

$$|\alpha|\sum_i \sqrt{\mu_i(\Phi_1\Phi_1^\dagger)} \leqslant \sum_i \sqrt{\mu_i(\Gamma_1\Gamma_1^\dagger)}. \tag{21}$$

In a similar way, substituting Eq. (18) into the right inequality of Eq. (14) and taking the sum of $\sqrt{\mu_i(\cdot)}$ over all index i, we have

$$\sum_i \sqrt{\mu_i(\Gamma_1\Gamma_1^\dagger)} \leqslant |\alpha|\sum_i \sqrt{\mu_i(\Phi_1\Phi_1^\dagger)} + n|\beta|\sqrt{\mu_n(\Psi_1\Psi_1^\dagger)}. \tag{22}$$

Substituting Eqs. (21) and (22) into Eq. (12), respectively, we can obtain

$$|\alpha|^2\mathcal{N}(\Phi_1) + \frac{|\alpha|^2-1}{2} \leqslant \mathcal{N}(\alpha\Phi_1+\beta\Psi_1) \leqslant |\alpha|^2\tilde{\mathcal{N}}(\Phi_1) + \frac{|\alpha|^2-1}{2}. \tag{23}$$

If we replace the matrix $|\alpha|^2\Phi_1\Phi_1^\dagger$ with $|\beta|^2\Psi_1\Psi_1^\dagger$ in Eqs. (20) and (21), i.e., equivalently exchange the matrixes H and K in Eq. (14), finally we can also obtain

$$|\beta|^2\mathcal{N}(\Psi_1) + \frac{|\beta|^2-1}{2} \leqslant \mathcal{N}(\alpha\Phi_1+\beta\Psi_1) \leqslant |\beta|^2\tilde{\mathcal{N}}(\Psi_1) + \frac{|\beta|^2-1}{2}. \tag{24}$$

Then combining Eqs. (23) and (24) gives Eq. (16). Thus the proof is completed.

Note that the lower bound in Eq. (16) can provide a nonzero value only when $2|\alpha|^2\mathcal{N}(\Phi_1)+2|\beta|^2\mathcal{N}(\Psi_1)>1$. Next we provide an example to illustrate the validity of our bound. Consider the state

$$|\phi\rangle_{AB} = \alpha|\varphi\rangle_{AB} + \beta|\psi\rangle_{AB}, \tag{25}$$

with

$$|\varphi\rangle_{AB} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle, \tag{26}$$

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}|22\rangle + \frac{1}{\sqrt{2}}|33\rangle, \tag{27}$$

where $\alpha=\beta=1/\sqrt{2}$. It is easy to check that $|\varphi\rangle_{AB}$ and $|\psi\rangle_{AB}$ are biorthogonal, $\mathcal{N}(|\phi\rangle_{AB})=3/2$, $\mathcal{N}(|\varphi\rangle_{AB})=\mathcal{N}(|\psi\rangle_{AB})=1/2$, and $\mu_4(|\varphi\rangle_{AB})=\mu_4(|\psi\rangle_{AB})=1/2$. Accordingly from Eq. (16) we obtain the lower and upper bounds

$$0 < \mathcal{N}(|\phi\rangle_{AB}) = \frac{3}{2} < 4, \tag{28}$$

which work well. It is clear that varying the coefficients of α and β would give tighter bounds.

Finally we present the main Theorem of this paper, in which the two states being superposed can be biorthogonal, orthogonal, or nonorthogonal.

Theorem 3. Suppose that two arbitrary normalized pure states Φ_2 with rank r_1 and Ψ_2 with rank r_2, which are defined in any dimensions. The negativity of their superposed states $\Gamma_2=\alpha\Phi_2+\beta\Psi_2$ with rank r_3 and $|\alpha^2|+|\beta|^2=1$ satisfies

$$2\|\alpha|\Phi_2\rangle+\beta|\Psi_2\rangle\|^2\mathcal{N}(\alpha\Phi_2+\beta\Psi_2) \leqslant 2|\alpha|^2\tilde{\mathcal{N}}(\Phi_2) + 2|\beta|^2\tilde{\mathcal{N}}(\Psi_2) - \|\alpha|\Phi_2\rangle+\beta|\Psi_2\rangle\|^2 + 1, \tag{29}$$

where

$$\tilde{\mathcal{N}}(\Phi_2) = \mathcal{N}(\Phi_2) + \frac{r|\beta|\sqrt{\mu_n(\Psi_2)[2\mathcal{N}(\Phi_2)+1]}}{|\alpha|} + \frac{r^2|\beta|^2\mu_n(\Psi_2)}{2|\alpha|^2},$$

$$\tilde{\mathcal{N}}(\Psi_2) = \mathcal{N}(\Psi_2) + \frac{r|\alpha|\sqrt{\mu_n(\Phi_2)[2\mathcal{N}(\Psi_2)+1]}}{|\beta|} + \frac{r^2|\alpha|^2\mu_n(\Phi_2)}{2|\beta|^2},$$

where $r=\max\{r_1,r_2,r_3\}$.

Proof. Consider the matrix

$$M=|\alpha|^2\Phi_2\Phi_2^\dagger+|\beta|^2\Psi_2\Psi_2^\dagger, \tag{30}$$

which can be rewritten as

$$M=\frac{\|\Gamma_2\|^2}{2}\hat{\Gamma}_2(\hat{\Gamma}_2)^\dagger+\frac{\|\Gamma_2^-\|^2}{2}\hat{\Gamma}_2^-(\hat{\Gamma}_2^-)^\dagger,$$

where

$$\Gamma_2^-=\alpha\Phi_2-\beta\Psi_2,\quad \hat{\Gamma}_2=\Gamma_2/\|\Gamma_2\|,\quad \text{and}\quad \hat{\Gamma}_2^-=\Gamma_2^-/\|\Gamma_2^-\|. \tag{31}$$

Thus Eqs. (13) show that

$$\begin{aligned}&|\alpha|^2\mu_i(\Phi_2\Phi_2^\dagger)+|\beta|^2\mu_1(\Psi_2\Psi_2^\dagger)\\&\quad\leqslant\mu_i(M)\leqslant|\alpha|^2\mu_i(\Phi_2\Phi_2^\dagger)+|\beta|^2\mu_n(\Psi_2\Psi_2^\dagger),\end{aligned} \tag{32}$$

and

$$\begin{aligned}&\frac{\|\Gamma_2\|^2}{2}\mu_i(\hat{\Gamma}_2\hat{\Gamma}_2^\dagger)+\frac{\|\Gamma_2^-\|^2}{2}\mu_1[\hat{\Gamma}_2^-(\hat{\Gamma}_2^-)^\dagger]\\&\quad\leqslant\mu_i(M)\leqslant\frac{\|\Gamma_2\|^2}{2}\mu_i(\hat{\Gamma}_2\hat{\Gamma}_2^\dagger)+\frac{\|\Gamma_2^-\|^2}{2}\mu_n[\hat{\Gamma}_2^-(\hat{\Gamma}_2^-)^\dagger].\end{aligned} \tag{33}$$

Since $\mu_1(\Psi_2\Psi_2^\dagger)\geqslant 0$ and $\mu_1(\hat{\Gamma}_2^-(\hat{\Gamma}_2^-)^\dagger)\geqslant 0$, observing the left inequality of Eq. (33) and the right inequality in Eq. (32) we have

$$\frac{\|\Gamma_2\|}{\sqrt{2}}\sqrt{\mu_i(\hat{\Gamma}_2\hat{\Gamma}_2^\dagger)}\leqslant|\alpha|\sqrt{\mu_i(\Phi_2\Phi_2^\dagger)}+|\beta|\sqrt{\mu_n(\Psi_2\Psi_2^\dagger)}. \tag{34}$$

Substituting Eqs. (34) into Eq. (12) we have

$$\begin{aligned}\|\alpha|\Phi_2\rangle+\beta|\Psi_2\rangle\|^2\mathcal{N}(\alpha\Phi_2+\beta\Psi_2)&\leqslant 2|\alpha|^2\tilde{\mathcal{N}}(\Phi_2)\\&\quad-\frac{\|\alpha|\Phi_2\rangle+\beta|\Psi_2\rangle\|^2}{2}+|\alpha|^2.\end{aligned} \tag{35}$$

Likewise, if we replace the two matrices $|\alpha|^2\Phi_2\Phi_2^\dagger$ with $|\beta|^2\Psi_2\Psi_2^\dagger$ in Eq. (32), we can obtain

$$\begin{aligned}\|\alpha|\Phi_2\rangle+\beta|\Psi_2\rangle\|^2\mathcal{N}(\alpha\Phi_2+\beta\Psi_2)&\leqslant 2|\beta|^2\tilde{\mathcal{N}}(\Psi_2)\\&\quad-\frac{\|\alpha|\Phi_2\rangle+\beta|\Psi_2\rangle\|^2}{2}+|\beta|^2.\end{aligned} \tag{36}$$

Combining Eqs. (35) and (36) gives Eq. (29). Thus the proof is completed.

Since there exists an extra term of the maximal eigenvalue in the second inequality in Eq. (33), generally it is difficult to find a universal formula for the lower bound of the negativity in this case. But it is our interest in future work. Note that the lower bound of the von Neumann entropy of superposition states was not yet offered [10].

In conclusion, we have shown that if the entanglement is quantified by the concurrence, two pure states of high fidelity to one another still have nearly the same entanglement, and we obtained an inequality that can guarantee that the concurrence is a continuous function even in infinite dimensions. However, the question of whether a similar property can apply to the negativity case is still open. The bounds on the negativity of superposed states in terms of negativities of the states being superposed were obtained. For the superposition states, in addition to the bounds of the well-studied measures of entanglement such as the von Neumann entropy [10], the concurrence [12], and the witnessed entanglement [13], in this paper we have presented the bounds for the case of negativity which is also one of the well-accepted measures of entanglement. In view of the concurrence being directly accessible in laboratory experiments [20], these bounds can find useful applications in estimating the amount of the entanglement of a given pure state.

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