Entanglement fidelity and measurement of entanglement preservation in quantum processes

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The entanglement fidelity provides a measure of how well the entanglement between two subsystems is preserved in a quantum process. By using a simple model, we show that in some cases this quantity in its original definition fails in the measurement of entanglement preservation. On the contrary, the modified entanglement fidelity, obtained by using a proper local unitary transformation on a subsystem, is shown to exhibit behavior similar to that of the concurrence in quantum evolution.

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Quantum entanglement is a key element for applications of quantum communications and quantum information. A complete discussion of this has been given in Ref. [1]. Characterizing and quantifying the entanglement is a fundamental issue in quantum-information theory. For pure and mixed states of two qubits, this problem about the description of the entanglement has been well elucidated [2–7]. Recently, Jordan *et al.* [8] considered two entangled qubits, one of which interacts with a third qubit named the control qubit that is never entangled with either of the two entangled qubits. They found that the entanglement of these two qubits can be both increased and decreased by the action of the control qubit on just one of them. If we regard the control qubit as an environment and the state of the qubit interacting with the control qubit as the information source, this example is just a model for the time evolution of quantum information via a noisy quantum channel originating from the interaction with the control qubit. Schumacher [9] and Barnum *et al.* [10] have investigated a general situation where R and Q are two quantum systems and the joint system RQ is initially prepared in a pure entangled state $|\Psi^{RQ}\rangle$. The system R is dynamically isolated and has a zero internal Hamiltonian, while the system Q undergoes some evolution that possibly involves interaction with the environment. The evolution of Q might represent a transmission process via some quantum channel for the quantum information in Q. They introduced a fidelity $F_e=\langle\Psi^{RQ}|\rho^{RQ'}|\Psi^{RQ}\rangle$, which is the probability that the final state $\rho^{RQ'}$ would pass a test checking whether it agrees with the initial state $|\Psi^{RQ}\rangle$. This quantity is called the entanglement fidelity (EF). The EF can be defined entirely in terms of the initial state ρ^Q and the evolution of the system Q, so the EF is related to a process, specified by a quantum operation ε^Q, which we shall discuss later in more detail, acting on some initial state ρ^Q. Thus, the EF can be denoted by a function of the form $F_e(\rho^Q,\varepsilon^Q)$. The EF is usually used to measure how well the state ρ^Q is preserved by the operation ε^Q and to identify how well the entanglement of ρ^Q with other systems is preserved by the operation of ε^Q. A complete discussion of the EF can be seen in [9,11]. In the present work we will investigate the following question: Is EF a good measurement of entanglement preservation? Using the example of Jordan *et al.*, we find that in some cases the EF defined above completely fails at measuring entanglement preservation though it may be a good measurement of entanglement preservation in the case of slight noise. We also find that in order to make the EF indeed equivalent to an entanglement measure, the modified entanglement fidelity (MEF) should be used. Some detailed discussions about the MEF have been given in [9,12,13]. Recently, Surmacz *et al.* [14] have investigated the evolution of the entanglement in a quantum memory and showed that the MEF can be used to measure how well a quantum memory setup can preserve the entanglement between the qubit undergoing the memory process and an auxiliary qubit. For the example of Jordan *et al.*, we derive an analytic expression for the MEF and give a comparison of it with the concurrence.

The quantum operation ε^Q is a map for the state of Q,

$$\rho^{Q'}=\varepsilon^Q(\rho^Q). \tag{1}$$

Here ρ^Q is the initial state of the system Q, and after the dynamical process the final state of the system becomes $\rho^{Q'}$. Then the dynamical process is described by ε^Q. In the most general case, the map ε^Q must be a trace-preserving and positive linear map [15,16], so it includes all unitary evolutions. They also include unitary evolving interactions with an environment E. Suppose that the environment is initially in state ρ^E. The operator can be written as

$$\varepsilon^Q(\rho^Q)=\mathrm{Tr}_E U(\rho^Q\otimes\rho^E)U^\dagger=\mathrm{Tr}_E U\Big(\rho^Q\otimes\sum_i p_i|i\rangle\langle i|\Big)U^\dagger$$
$$=\sum_j E_j^Q\rho^Q E_j^{Q\dagger}, \tag{2}$$

where $\Sigma_i p_i|i\rangle\langle i|$ is the spectral decomposition of ρ^E, with $\{|i\rangle\}$ being a base in the Hilbert space $\mathcal{H}_E$ of the environment E, and $E_j^Q=\Sigma_i\sqrt{p_i}\langle j|U|i\rangle$. Now we can use Eq. (2) to get the intrinsic expression for $\langle\Psi^{RQ}|\rho^{RQ'}|\Psi^{RQ}\rangle$, i.e., $F_e(\rho^Q,\varepsilon^Q)$. Because

$$\rho^{RQ'}=\mathcal{I}^R\otimes\varepsilon^Q(\rho^{RQ})=\sum_j(1^R\otimes E_j^Q)\rho^{RQ}(1^R\otimes E_j^Q)^\dagger, \tag{3}$$

one has

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$$
\begin{aligned}
F_e &= \langle \Psi^{RQ} | \rho^{RQ'} | \Psi^{RQ} \rangle \\
&= \sum_j \langle \Psi^{RQ} | (1^R \otimes E_j^Q) | \Psi^{RQ} \rangle \\
&\quad \times \langle \Psi^{RQ} | (1^R \otimes E_j^Q)^\dagger | \Psi^{RQ} \rangle \\
&= \sum_j (\mathrm{Tr}\, \rho^Q E_j^Q)(\mathrm{Tr}\, \rho^Q E_j^{Q\dagger}). \qquad (4)
\end{aligned}
$$

If systems R and Q both have zero internal Hamiltonian and there is no interaction between R and Q, the operation ε^Q entirely originates from the interaction between Q and the environment. In this sense the example of Jordan *et al.* is a special case of this situation.

We consider two entangled qubits A and B, and suppose that qubit A interacts with a control qubit C. Then A, B, and C, respectively, correspond to systems Q, R, and the environment E that we have just mentioned. We suppose that the initial states of the three qubits are

$$W = \rho_\pm^{AB} \otimes \frac{1}{2} 1_c, \qquad (5)$$

where

$$\rho_\pm^{AB} = \frac{1}{4}(1 \pm \sigma_1^A \sigma_1^B \pm \sigma_2^A \sigma_2^B - \sigma_3^A \sigma_3^B), \qquad (6)$$

with $\sigma_i^{A\ (B)}$, $i=1,2,3$, being Pauli matrices for qubit A (B). ρ_+^{AB} and ρ_-^{AB} are two Bell states, representing maximally entangled pure states for the combined system of qubits A and B. The total spins of states ρ_-^{AB} and ρ_+^{AB} are 0 and 1, respectively.

We suggest an interaction between qubits A and C described by the unitary transformation

$$U = e^{-itH}, \qquad (7)$$

where

$$H = \frac{\lambda \sigma_3^A}{2}(|\alpha\rangle\langle\alpha| - |\beta\rangle\langle\beta|), \qquad (8)$$

λ is the strength of the interaction, and $|\alpha\rangle$ and $|\beta\rangle$ are two orthonormal vectors for system C. Then the changing density matrix for the combined system of qubits A and B can be calculated as

$$
\begin{aligned}
\rho_\pm^{AB'} &= \mathrm{Tr}_c[(U \otimes 1^B) W (U \otimes 1^B)^\dagger] \\
&= \frac{1}{4}[1 \pm (\sigma_1^A \sigma_1^B + \sigma_2^A \sigma_2^B)\cos(\lambda t) - \sigma_3^A \sigma_3^B] \\
&= \rho_\pm^{AB} \cos^2\left(\frac{\lambda t}{2}\right) + \rho_\mp^{AB} \sin^2\left(\frac{\lambda t}{2}\right). \qquad (9)
\end{aligned}
$$

The changing density matrix $\rho_\pm^{AB'}$ usually represents a mixed state. In order to quantify its entanglement we use the Wootters concurrence [5] defined as

$$C(\rho) \equiv \max[0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}], \qquad (10)$$

where ρ is the density matrix representing the investigated state of the combined system of A and B, λ_1, λ_2, λ_3, and λ_4 are the eigenvalues of $\rho\sigma_2^A\sigma_2^B\rho^*\sigma_2^A\sigma_2^B$ in decreasing order, and ρ^* is the complex conjugate of ρ. From Eq. (9) we can obtain

$$C(\rho_\pm^{AB'}) = |\cos \lambda t|. \qquad (11)$$

It is found that at time $\lambda t = \pi/2$ the state $\rho_\pm^{AB'}$ is changed from a maximally entangled state at $t=0$ to a separable state and at time $\lambda t = \pi$ the state $\rho_\pm^{AB'}$ returns to the maximally entangled state. The explicit calculation of $\rho^{AB'}$ and $C(\rho_\pm^{AB'})$ can be seen in [8].

Now we adopt the EF to investigate this example. Using Eqs. (2), (5), (7), and (8), we obtain the quantum operation on qubit A,

$$
\begin{aligned}
\varepsilon^A(\rho^A) &= \mathrm{Tr}_C U(\rho^A \otimes \rho^C) U^\dagger \\
&= \mathrm{Tr}_C U\left[\rho^A \otimes \left(\frac{1}{2}(|\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta|)\right)\right] U^\dagger \\
&= \frac{1}{2} e^{-i\sigma_3^A(\lambda t/2)} \rho^A e^{+i\sigma_3^A(\lambda t/2)} + \frac{1}{2} e^{+i\sigma_3^A(\lambda t/2)} \rho^A e^{-i\sigma_3^A(\lambda t/2)}.
\end{aligned}
\qquad (12)
$$

So $E_\alpha^A = (1/\sqrt{2}) e^{-i\sigma_3^A(\lambda t/2)}$ and $E_\beta^A = (1/\sqrt{2}) e^{+i\sigma_3^A(\lambda t/2)}$. Substituting them into Eq. (4) and noting that $\rho^A \equiv \mathrm{Tr}_B(\rho_\pm^{AB}) = \frac{1}{2}1$, we can get the EF as

$$
\begin{aligned}
F_e &= \sum_j (\mathrm{Tr}\, \rho^A E_j^A)(\mathrm{Tr}\, \rho^A E_j^{A\dagger}) \\
&= \left\{ \frac{1}{\sqrt{2}} \mathrm{Tr}\left[\begin{pmatrix} e^{-i\lambda t/2} & 0 \\ 0 & e^{+i\lambda t/2} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \right\}^2 \\
&\quad + \left\{ \frac{1}{\sqrt{2}} \mathrm{Tr}\left[\begin{pmatrix} e^{+i\lambda t/2} & 0 \\ 0 & e^{-i\lambda t/2} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \right\}^2 \\
&= \left(\cos \frac{\lambda t}{2} \right)^2. \qquad (13)
\end{aligned}
$$

We can easily find the disagreement between the evolutions of F_e and $C(\rho_\pm^{AB'})$. At $\lambda t = \pi$, the state $\rho_\pm^{AB'}$ returns to the maximally entangled state as can be seen from the concurrence, but its entanglement fidelity is zero ($F_e=0$). On the contrary, the initial maximally entangled state has been changed to a separable state at $\lambda t = \pi/2$, but the EF at this time is not zero. The evolutions of the EF F_e and concurrence $C(\rho_\pm^{AB'})$ are depicted in Fig. 1.

In fact, $F_e(\rho^Q, \varepsilon^Q) = F_s^2(\rho^{RQ}, \rho^{RQ'})$, where $F_s(\rho^{RQ}, \rho^{RQ'})$ is the static fidelity [11]. The static fidelity satisfies $0 \le F_s(\rho^{RQ}, \rho^{RQ'}) \le 1$, where the first symbol $\le$ becomes an equality if and only if ρ^{RQ} and $\rho^{RQ'}$ have orthogonal support, and the second symbol becomes an equality if and only if $\rho^{RQ} = \rho^{RQ'}$. When $\lambda t = \pi$, from Eq. (9) we can see that $\rho_\pm^{AB'} = \rho_\mp^{AB}$. The $\rho_\pm^{AB}$ are two different Bell states and correspond, respectively, to eigenstates of total spin one and total spin zero of the combined system of qubits A and B. So they have orthogonal support in the Hilbert space $\mathcal{H}^A \otimes \mathcal{H}^B$. This is the reason that $F_e(\rho^A, \varepsilon^Q) = F_s^2(\rho^{AB}, \rho^{AB'}) = 0$ at $\lambda t = \pi$.

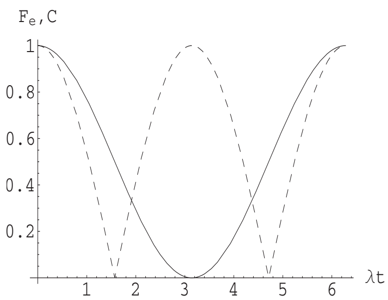

FIG. 1. Evolutions of the EF F_e (solid line) and the concurrence C (dashed line). We take $\hbar=1$ so λt is dimensionless.

The concept of the EF arises from the mathematical description for the purification of mixed states. Any mixed state can be represented as a subsystem of a pure state in a larger Hilbert space. The entanglement of a pure state may cause the states of subsystems to be mixed. The EF is usually used to measure how faithfully a channel maintains the purification, or, equivalently, how well the channel preserves the entanglement. In the above simple example, however, we found that, except for some special cases, only in the case of slight noise, i.e., $\lambda t - \to 0$, the does EF approximately agree with the concurrence. This means that this quantity may not be a good measurement for the evolution of the entanglement in the processes of interaction with the environment.

In fact, Schumacher [9] has noted that the EF can be lowered by a local unitary operation but the entanglement cannot be so. From this consideration he defined the MEF

$$F_e' = \max_{U^Q} \langle \Psi^{RQ} | (1^R \otimes U^Q) \rho^{RQ'} (1^R \otimes U^Q)^\dagger | \Psi^{RQ} \rangle, \quad (14)$$

where U^Q is any unitary transformation acting on Q. It is clear that $F_e' \geq F_e$. Since by using a proper local unitary operation we can make the Bell state $\rho_\pm^{AB}$ become the Bell state $\rho_\mp^{AB}$, we can find that in the above example $F_e'=1$ at time $\lambda t=\pi$ whereas $F_e=0$ at this time. So at $\lambda t=\pi$, the MEF equals the concurrence. By using the quantum operation that we discussed above, we can get the intrinsic expression for the MEF,

$$\begin{aligned} F_e' &= \max_{U^Q} \sum_j \langle \Psi^{RQ} | (1^R \otimes U^Q E_j^Q) | \Psi^{RQ} \rangle \\ &\quad \times \langle \Psi^{RQ} | (1^R \otimes U^Q E_j^Q)^\dagger | \Psi^{RQ} \rangle \\ &= \max_{U^Q} \sum_j (\mathrm{Tr}\, \rho^Q U^Q E_j^Q)[\mathrm{Tr}\, \rho^Q (U^Q E_j^Q)^\dagger]. \end{aligned} \quad (15)$$

For this example we can derive an analytic expression for F_e'. Suppose U is an arbitrary unitary operation on a single qubit. Then it can be written as [11]

$$\begin{aligned} U &= e^{-i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta) \\ &= e^{-i\alpha} \begin{pmatrix} e^{i(-\beta/2-\delta/2)} \cos \frac{\gamma}{2} & -e^{i(-\beta/2+\delta/2)} \sin \frac{\gamma}{2} \\ e^{i(+\beta/2-\delta/2)} \sin \frac{\gamma}{2} & e^{i(+\beta/2+\delta/2)} \cos \frac{\gamma}{2} \end{pmatrix}, \end{aligned}$$

where α, β, γ, and δ are real numbers, and $R_{y(z)}$ is the rotation operator about the y (z) axis. We have

$$\begin{aligned} &\sum_j (\mathrm{Tr}\, \rho^A U E_j^A)[\mathrm{Tr}\, \rho^A (U E_j^A)^\dagger] \\ &= \frac{1}{2}\left[\frac{1}{2}\mathrm{Tr}\begin{pmatrix} e^{i(-\beta/2-\delta/2-\lambda t/2)}\cos\frac{\gamma}{2} & 0 \\ 0 & e^{i(\beta/2+\delta/2+\lambda t/2)}\cos\frac{\gamma}{2}\end{pmatrix}\right]^2 + \frac{1}{2}\left[\frac{1}{2}\mathrm{Tr}\begin{pmatrix} e^{i(-\beta/2-\delta/2+\lambda t/2)}\cos\frac{\gamma}{2} & 0 \\ 0 & e^{i(\beta/2+\delta/2-\lambda t/2)}\cos\frac{\gamma}{2}\end{pmatrix}\right]^2 \\ &= \frac{1}{2}\cos^2\left(\frac{\gamma}{2}\right)\cos^2(\beta/2+\delta/2+\lambda t/2) + \frac{1}{2}\cos^2\left(\frac{\gamma}{2}\right)\cos^2(\beta/2+\delta/2-\lambda t/2). \end{aligned} \quad (16)$$

We should find a unitary operator U that makes $\Sigma_j(\mathrm{Tr}\, \rho^A U E_j^A)[\mathrm{Tr}\, \rho^A (U E_j^A)^\dagger]$ take its maximum value. Since $\cos^2(\beta/2+\delta/2+\lambda t/2) \geq 0$ and $\cos^2(\beta/2+\delta/2-\lambda t/2) \geq 0$, we can take $\gamma=0$. So one obtains

$$\sum_j (\mathrm{Tr}\, \rho^A U E_j^A)[\mathrm{Tr}\, \rho^A (U E_j^A)^\dagger] = 1 + \cos^2(\beta/2+\delta/2)[2\cos^2(\lambda t/2) - 1] - \cos^2(\lambda t/2). \quad (17)$$

When $2\cos^2(\lambda t/2) - 1 \geq 0$ we take $\cos^2(\beta/2+\delta/2)=1$ and get $F_e'=\cos^2(\lambda t/2)$; when $2\cos^2(\lambda t/2)-1<0$ we take $\cos^2(\beta/2+\delta/2)=0$ and get $F_e'=1-\cos^2(\lambda t/2)$.

The evolutions of the MEF F_e' and the concurrence $C(\rho_\pm^{AB'})$ are depicted in Fig. 2. We can see that the MEF and the concurrence exhibit a similar behavior, although their values do not exactly agree with each other at all moments. When the state $\rho_\pm^{AB'}$ returns to the maximally entangled state, the MEF is equal to 1. The maximal difference between them comes at the separable states where the MEF is equal to $1/2$ while the concurrence is zero.

We have mentioned that the EF equals 1 if and only if

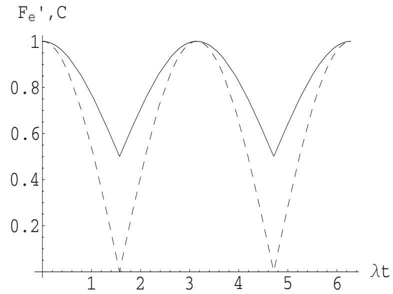

FIG. 2. Evolutions of the modified entanglement fidelity F_e' (solid line) and the concurrence C (dashed line).

$\rho^{RQ}=\rho^{RQ'}$. This means that the EF can be used to measure the difference between a quantum channel and the identity channel. If the concern is about entanglement preservation in an evolution process, however, one has to use the MEF because the EF can be lowered by a local unitary operation in this process, but the entanglement cannot be so. If a quantum channel is just a unitary operator, the entanglement is certainly invariant and the MEF always equals 1 in the quantum process. In this sense the MEF can be used to measure the difference between a quantum channel and an arbitrary unitary operator.

In summary, for the example of Jordan *et al.*, we have derived the analytic expressions of both the EF and the MEF, and show comparisons of them with the concurrence. From these we find that the MEF may admirably reflect the entanglement preservation in a quantum process.

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