Solution of the one-dimensional spatially inhomogeneous cubic-quintic nonlinear Schrödinger equation with an external potential

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Properties of the one-dimensional spatially inhomogeneous cubic-quintic nonlinear Schrödinger equation (ICQNLSE) with an external potential are studied. When it is associated with the homogeneous CQNLSE, a general condition exists linking the external potential and inhomogeneous cubic and quintic (ICQ) nonlinearities. Besides for the nonpresence of an external potential, two classes of Jacobian elliptic periodic potentials are discussed in detail, and the corresponding ICQ nonlinearities are found to be either periodic or localized. Exact analytical soliton solutions in these cases are presented, such as the bright, dark, kink, and periodic solitons, etc. An appealing aspect is that the ICQNLSE can support bound states with any number of solitons when the ICQ nonlinearities are localized and an external potential is either applied or not.

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I. INTRODUCTION

During the past several years, there have been a great deal of theoretical and experimental investigations in models based on the nonlinear Schrödinger (NLS) or Gross-Pitaevskii (GP) equations with a spatially periodic potential, cubic and quintic nonlinearities [1–8]. The physical models of this type emerge in various nonlinear physical phenomena, such as pulse propagation in optical fibers and waveguides [9], light propagation in chalcogenide glasses [10], convection in pure and binary fluids [11], mode-locked lasers [12], plasma-laser interaction [13], pattern formation [14], and some organic materials [15].

The cubic-quintic nonlinear Schrödinger equations (CQNLSEs) with nonlinearity management presents practical interest, since it appears in diverse branches of physics such as in nonlinear optics [16] and in Bose-Einstein condensation (BEC) [17]. In effect, in the past decade, techniques for managing nonlinearity [18] have attracted considerable attention, for instance, nonlinearity management arises in optics for transverse beam propagation in layered optical media [19], as well as in atomic physics for the Feshbach resonance of the scattering length of interatomic interactions in BECs [20,21]. In these situations, one has to deal with the governing equations with the nonlinearity coefficients being functions of time [21,22], or equivalently, the variable representing the propagation distance [19,23]. Recently, spatially dependent nonlinear interactions are also receiving a great deal of attention, for instance, there are many studies on nonlinear waves in BECs with spatially inhomogeneous interactions including solitonic emission [24]. In a recent reference [25], by using the classical Lie group theory and canonical transformations, some general classes of nonlinear modulations and external potentials were found. Specifically, explicit soliton solutions for the cubic nonlinear Schrödinger equation with a spatially inhomogeneous nonlinearity have been reported. It turns out that localized nonlinearities can support bound states with an arbitrary number of solitons without any additional external potential.

The aim of the present paper is to study explicit stationary solutions of the spatially inhomogeneous cubic-quintic nonlinear Schrödinger equation (ICQNLSE) with an external potential, which can be written in the following dimensionless form:

$$i\psi_t + \psi_{xx} + g(x)|\psi|^2\psi + G(x)|\psi|^4\psi - V(x)\psi = 0. \qquad (1)$$

In the context of BECs, Eq. (1) can model a dilute BEC in the quasi-one-dimensional regime when both the two- and three-body interactions of the condensate are considered [26]. In this case, ψ is the macroscopic wave function of the condensate, $V(x)$ is an external potential, $g(x)$ and $G(x)$ are the inhomogeneous cubic and quintic nonlinear coefficients, corresponding to the two-body and three-body interactions, respectively. The signs of $g(x)$ and $G(x)$ can be positive or negative in the whole domain of x, indicating that the interactions are attractive or repulsive, respectively.

It is noted that exact solutions of the homogeneous CQNLSE, i.e., $g(x)$ and $G(x)$ in Eq. (1) are x independent, have been discussed in Ref. [27] with and without external potentials. In order to construct explicit stationary solutions of ICQNLSE (1), we can write the wave function in the form of $\psi = \phi e^{-i\lambda t}$, where λ is a constant and ϕ is an x-dependent function. The substitution of $\psi = \phi e^{-i\lambda t}$ into Eq. (1) leads to a time independent inhomogeneous nonlinear equation

$$\lambda\phi + \phi_{xx} + g(x)\phi^3 + G(x)\phi^5 - V(x)\phi = 0. \qquad (2)$$

It is remarkable that, similar to Ref. [25], Eq. (2) can also be transformed to the CQNLSE without an external potential and with homogeneous nonlinearities in the new canonical variables obtained from the classical Lie symmetry of Eq. (2). It is known that besides the classical and nonclassical Lie group approaches [28], the CK (Clarkson and Kruskal) direct method [29] is also powerful in finding solutions of nonlinear equations. Moreover, it has been revealed that all the solutions obtained by the CK direct method can have corresponding Lie symmetry explanations either for classical or nonclassical cases. Hence it is implied that the CK direct method might be applied to find transformations between spatial inhomogeneous equations and the corresponding homogeneous equations. This type of transformation is known as the non-auto-Bäcklund transformation (NABT), which has been widely used in integrable systems to connect solutions of two different equations [30].

In Sec. II of this paper, we first use the classical Lie group approach to obtain the classical Lie group symmetry of Eq. (2), from which the canonical variables are constructed to transform the inhomogeneous equation (2) to a CQNLSE with homogeneous nonlinearities. Then, following the idea of the CK direct method without using any group theory, we find a NABT between the inhomogeneous equation (2) and its corresponding homogeneous one. It is manifested that, for particular parameters, the NABT is equivalent to the canonical transformation. At the same time, a condition arises to link the potential and the inhomogeneous cubic and quintic (ICQ) nonlinearities. This condition can give rise to many sets of potentials and ICQ nonlinearities, by which solutions of ICQNLSE can be easily constructed from those of the CQNLSE. Two classes of solutions are presented in Sec. III. First, in the absence of an external potential, the corresponding ICQ nonlinearities can be either periodic or localized. In the periodic case, Eq. (2) can have bright and dark solitons with periodic tails. In the nonperiodic case, we find that Eq. (2) can support bound states with any number of solitons. Second, two types of Jacobian elliptic periodic potentials are introduced, when the ICQ nonlinearities are periodic with the same periods of the potentials. Some exact Jacobian elliptic periodic solutions of Eq. (2) are presented. Results in the limit of the moduli of the Jacobian elliptic functions approaching 1 are discussed. Some representative figures are plotted to show the profiles of the potentials, ICQ nonlinearities, and exact soliton solutions. In Sec. IV, the main results of the paper are briefly discussed and summarized.

II. SYMMETRIES, CANONICAL TRANSFORMATIONS, AND NON-AUTO-BÄCKLUND TRANSFORMATIONS

Generally, it is difficult to directly solve Eq. (2) with some particular external potentials and ICQ nonlinearities. Recently, Ref. [25] provided a way to treat this problem by transforming a variable coefficient equation to a corresponding constant coefficient equation by virtue of the canonical transformations obtained from the classical symmetries of the original equation. In Sec. II A, following Ref. [25], we apply the classical Lie group approach to Eq. (2) for obtaining its classical Lie group symmetries, from which we then obtain the canonical transformations, and thus a CQNLSE without an external potential and with homogeneous nonlinearities is deduced. In Sec. II B, we demonstrate that this problem can also be solved by employing the idea of the CK direct method, where a NABT between Eq. (2) and its corresponding homogeneous CQNLSE is obtained. It is found that for some particular parameters, the canonical transformations and the NABT are equivalent. Hence exact solutions of Eq. (2) can be easily constructed based on the solutions of the CQNLSE.

A. Symmetries and canonical transformations via the classical Lie group approach

The vector field

$$M = X\frac{\partial}{\partial x} + \Phi\frac{\partial}{\partial \phi}, \tag{3}$$

where X and Φ are functions of the variables (x,ϕ), is a symmetry of Eq. (2) if

$$pr^{(2)}M\, A|_{A=0} = 0, \tag{4}$$

where $A=\lambda\phi+\phi_{xx}+g(x)\phi^3+G(x)\phi^5-V(x)\phi$, and $\mathrm{pr}^{(2)}M = M+\Phi^x\partial/\partial\phi_x+\Phi^{xx}\partial/\partial\phi_{xx}$ is the second order prolongation of the vector field M. It means that Eq. (2) is invariant under the transformation $x\to x+\epsilon X$, $\phi=\phi+\epsilon\Phi$, where ϵ is an infinitesimal parameter. Solving Eq. (4), we find that the only Lie point symmetry (3) of Eq. (2) reads

$$M = a(x)\frac{\partial}{\partial x} + \left(\frac{1}{2}a'(x)+K\right)\phi\frac{\partial}{\partial \phi}, \tag{5}$$

with

$$g(x) = \frac{g_0}{a(x)^3}\exp\left[-2K\int_0^x \frac{1}{a(s)}ds\right],$$

$$G(x) = \frac{G_0}{a(x)^4}\exp\left[-4K\int_0^x \frac{1}{a(s)}ds\right], \tag{6}$$

and

$$a'''(x) - 2a(x)V'(x) - 4[V(x)-\lambda]a'(x) = 0, \tag{7}$$

where K is an arbitrary constant, and the prime stands for $\partial/\partial x$.

Using the fact that the invariance of the energy is associated with the translational invariance whose generator is of the form $M=\partial/\partial x$, we can define the canonical transformation related to the symmetry (5) as

$$\xi = p(x), \quad U = q(x)\phi, \tag{8}$$

where $p(x)$ and $q(x)$ can be determined by requiring that $M=\partial/\partial\xi$ exists in the canonical variables to preserve the energy conservation law. From Eqs. (5) and (8), we obtain

$$p(x) = \int_0^x \frac{1}{a(s)}ds, \quad q(x) = \frac{1}{\sqrt{a(x)}}\exp\left[-K\int_0^x \frac{1}{a(s)}ds\right]. \tag{9}$$

Therefore Eq. (2) is transformed to

$$\frac{d^2U}{d\xi^2} + 2K\frac{dU}{d\xi} + EU + g_0U^3 + G_0U^5 = 0, \tag{10}$$

where $E=[\lambda-V(x)]a(x)^2-\frac{1}{4}a'(x)^2+\frac{1}{2}a(x)a''(x)$ is a constant, because $\partial E/\partial x=0$ in light of Eq. (7).

It is noted that when $K=0$, Eq. (10) is just the CQNLSE without an external potential and with homogeneous CQ nonlinearities. Moreover, Eq. (10) with $K=0$ is also known as the ϕ^6 model, which has wide applications in solid state, condensed mater, quantum field theory, etc. [31], and many exact solutions of Eq. (10) have been reported [32]. Generally, we can find solutions of Eq. (10) to construct exact solutions of ICQNLSE (1). For simplicity and also for a direct connection between the ICQNLSE and CQNLSE, we take $K=0$. In this case, the solutions of Eq. (2) are given by

$$\phi = \sqrt{a(x)}U(\xi), \tag{11}$$

where $U(\xi) \equiv U$ satisfies Eq. (10) with $K=0$, and ξ is determined by

$$\xi = \int_0^x \frac{1}{a(s)} ds, \tag{12}$$

$V(x)$ is determined by Eq. (7), $g(x)$ and $G(x)$ are, respectively, given by

$$g(x) = \frac{g_0}{a(x)^3}, \quad G(x) = \frac{G_0}{a(x)^4}, \tag{13}$$

where $a(x)$ is an arbitrary function of the indicated argument, g_0 and G_0 are arbitrary constants.

B. Non-auto-Bäcklund transformations via the modified CK direct method

The CK direct method was first proposed to find similarity solutions of nonlinear partial differential equations [29]. It is remarkable that we cannot simply apply the original CK direct method to find the NABT. In this subsection, we show how to apply the idea of the CK direct method to construct a NABT between Eq. (2) and Eq. (10) with $K=0$, namely,

$$\frac{d^2U}{d\xi^2} + EU + g_0U^3 + G_0U^5 = 0, \tag{14}$$

where U is a function of the argument ξ, and E, g_0, and G_0 are constants. In effect, a similar idea has been used to find exact solutions of a general variable coefficient Korteweg–de Vries (KdV)-type equation from the standard constant coefficient KdV equation [33]. Moreover, the CK direct method has been modified or extended in several different manners to find conditional similarity reduction solutions [34] and non-Lie symmetry groups [35].

Hinted at by the CK direct method, the solutions of Eq. (2) can be written in the form

$$\phi = \alpha + \beta U(\xi) \equiv \alpha + \beta U, \tag{15}$$

where α, β, and ξ are functions of x which will be determined later. The differences from the standard CK direct method are twofold. First, the new function U in Eq. (15) has the same number of independent new variables as the function ϕ, while in the standard CK direct method, U is called a similarity reduction function which must have less independent new variables than ϕ. Second, in the standard CK direct method, we first substitute the similar assumption (15) into the original model, and then obtain the so called similarity reduction equation that U satisfies. Contrarily, now we first prescribe an equation that U satisfies, and then substitute the assumption (15) together with this prescribed equation into the original model. In our case, U satisfies Eq. (14).

Now, substituting Eq. (15) into Eq. (2), replacing all the terms $U_{n\xi}$ $(n \geqslant 2)$ with the help of Eq. (14), and setting to zero all the coefficients of U_ξ and different powers of U, we obtain a system of seven equations. To set to zero the coefficients of U^4, i.e., $G(x)\alpha\beta^4$, we must have $\alpha=0$. Hence the set of equations are reduced to the following four equations:

$$G(x)\beta^4 - G_0\xi'^2 = 0, \quad g(x)\beta^2 - g_0\xi'^2 = 0, \tag{16}$$

$$\beta\xi'' + 2\beta'\xi' = 0, \quad \beta'' - \beta E\xi'^2 + [\lambda - V(x)]\beta = 0. \tag{17}$$

Solving Eqs. (16) and (17), we determine the new variable

$$\xi = c_1 + c_2\int_0^x \frac{1}{\beta(s)^2} ds \tag{18}$$

and obtain three conditions

$$g(x) = \frac{c_2^2 g_0}{\beta^6}, \quad G(x) = \frac{c_2^2 G_0}{\beta^8}, \tag{19}$$

and

$$[\lambda - V(x)]\beta^4 + b^3\beta''' - c_2^2 E = 0, \tag{20}$$

where β is an arbitrary function of x, and c_1 and c_2 are arbitrary integration constants.

Therefore the NABT between Eq. (2) and Eq. (14) is found to be Eq. (15) with $\alpha=0$ and Eqs. (18)–(20). It is seen that when $c_1=0$, $c_2=1$, $\beta^2=a(x)$, this NABT is the same as the special canonical transformation Eq. (11) with Eqs. (12) and (13).

III. SYSTEMS WITH DIFFERENT EXTERNAL POTENTIALS

In this section, we investigate exact solutions of Eq. (2) with particular external potentials and ICQ nonlinearities, which are obtained from the transformation (11) with Eqs. (12), (13), and (7) with the help of the solutions of Eq. (14) [or Eq. (10) with $K=0$]. It is noted that many sets of external potentials and ICQ nonlinearities, which allow Eq. (2) to support explicit analytical solutions, can be generated from Eq. (7) by first assuming the form of $V(x)$ and then solving $a(x)$, or vice versa. In Ref. [25], only a zero potential and a quadratic trapping potential were discussed for the spatially inhomogeneous cubic NLS equation. Recently, many spatially periodic potentials have been considered for the GP and NLS equations, such as $V(x) \sim \cos(x)$ [3,4], $\cos^2(x)$ [5], and $\mathrm{sn}^2(x,k)$ [6–8]. In Sec. III A, we consider the case without an external potential, and Secs. III B and III C are devoted to the study of two types of Jacobian elliptic periodic external potentials, respectively.

A. Systems without an external potential

In the absence of an external potential, namely $V(x)=0$, Eq. (7) has solutions

$$a(x) = C_1 + C_2\cos(2\sqrt{\lambda}x + C_3) \tag{21}$$

for $\lambda > 0$, and

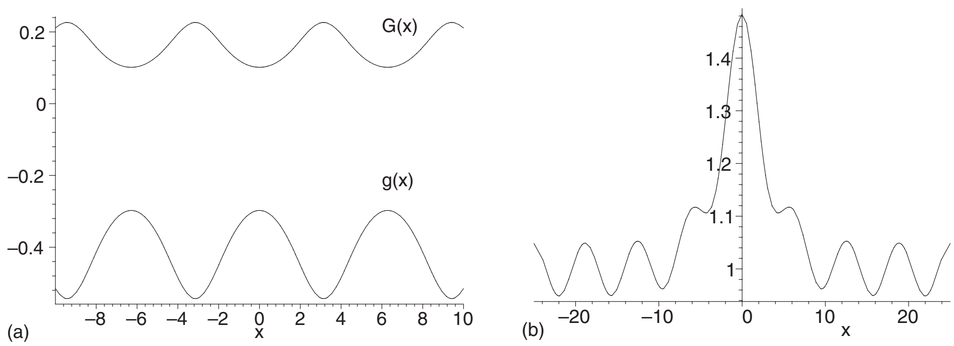

FIG. 1. (a) The profiles of the periodic ICQ nonlinearities given by Eq. (23) with Eq. (25) and $\lambda=1/4$, $a_1=C_1=1$, $C_2=0.1$, $C_3=0$. (b) The bright soliton solution of Eq. (2) with $V(x)=0$, corresponding to (a).

$$a(x)=C_1+C_2\exp(2\sqrt{-\lambda}x)+C_3\exp(-2\sqrt{-\lambda}x) \quad (22)$$

for $\lambda<0$, where the integration constants C_1, C_2, and C_3 must be chosen to ensure $a(x)$ positive. Accordingly, many exact solutions can be obtained for Eq. (2) with either periodic or exponentially localized ICQ nonlinearities. Here, we simply present some special exact soliton solutions for each case.

Case 1. For the case of Eq. (21), the periodic ICQ nonlinearities are in the form

$$g(x)=g_0[C_1+C_2\cos(2\sqrt{\lambda}x+C_3)]^{-3},$$

$$G(x)=G_0[C_1+C_2\cos(2\sqrt{\lambda}x+C_3)]^{-4}, \quad (23)$$

where the condition $|C_2|<|C_1|$ must be enforced for nonsingularities. Thus E in Eq. (14) has a positive value $\lambda(C_1^2-C_2^2)$.

Equation (14) with $E=\lambda(C_1^2-C_2^2)$ may have many possible exact solutions. For instance, it has the following possible localized soliton solutions:

$$U=\sqrt{a_1[\text{sech}(\mu\xi)\pm 1]}, \quad (24)$$

with

$$\mu^2=\frac{4}{5}\lambda(C_1^2-C_2^2), \quad g_0=\mp\frac{8\lambda(C_1^2-C_2^2)}{5a_1},$$

$$G_0=\frac{3\lambda(C_1^2-C_2^2)}{5a_1^2}, \quad (25)$$

and a_1 being an arbitrary constant.

It is evident that Eq. (24) represents a bright soliton for the upper sign (where a_1 is positive) and a dark soliton for the lower sign (where a_1 is negative). In both cases, g_0 is negative while G_0 is positive, which leads to $g(x)$ negative and $G(x)$ positive in the whole region of x. Therefore the dilute BEC described by Eq. (1) has only repulsive two-body and attractive three-body interactions, respectively. Under some particular parameters, the periodic ICQ nonlinearities and the corresponding bright soliton solution with periodic tails are illustrated in Fig. 1.

Case 2. For the case of Eq. (22), the ICQ nonlinearities can be exponentially localized in the form

$$g(x)=g_0\,\text{sech}^3(2\sqrt{-\lambda}x), \quad G(x)=G_0\,\text{sech}^4(2\sqrt{-\lambda}x), \quad (26)$$

when in Eq. (22) $C_1=0$ and $C_2=C_3=1/2$. Hence $E=-\lambda$. A special solution of Eq. (14) with $E=-\lambda$ can be easily obtained. We have

$$U=\frac{\text{sn}(\mu\xi,m)}{\sqrt{a_0+a_1\text{dn}^2(\mu\xi,m)}}, \quad (27)$$

with the requirement $a_0>|a_1|$, m being the modulus of the Jacobian elliptic functions,

$$\mu^2=\sqrt{\frac{\lambda(a_0+a_1)}{2m^2a_1-m^2a_0-a_0-a_1}}, \quad (28)$$

and

$$g_0=\frac{2\lambda m^2(a_0^2-2a_1m^2a_0-a_1^2+m^2a_1^2)}{m^2a_0+a_0-2m^2a_1+a_1},$$

$$G_0=\frac{3\lambda a_1m^4a_0(a_0+a_1-m^2a_1)}{m^2a_0+a_0-2m^2a_1+a_1}. \quad (29)$$

Now let us show that with the localized ICQ nonlinearities given by Eq. (26), the ICQNLSE without an external potential can also support bound states with any number of solitons, which resembles Ref. [25]. In this particular case, from Eq. (12), one can obtain $\xi=\arctan(e^{2\sqrt{-\lambda}x})/\sqrt{-\lambda}$. Obviously, $0\leqslant\xi\leqslant\xi_1$ with $\xi_1=\pi/2\sqrt{-\lambda}$. Therefore we have to enforce $U(0)=U(\xi_1)=0$ to satisfy the boundary condition $\phi(\pm\infty)=0$. Evidently, $U(0)=0$ is satisfied for the solution (27). The requirement $U(\xi_1)=0$ can be easily met by introducing the condition $\mu\xi_1=2nK(m)$ $(n=1,2,\ldots)$, where μ is determined by Eq. (28) and $K(m)$ is the elliptic integral,

$$K(m)=\int_0^{\pi/2}\frac{1}{\sqrt{1-m^2\sin^2(x)}}dx. \quad (30)$$

Hence for every n, we can find a corresponding solution of m. For instance, when we choose the parameters $a_0=-\lambda$

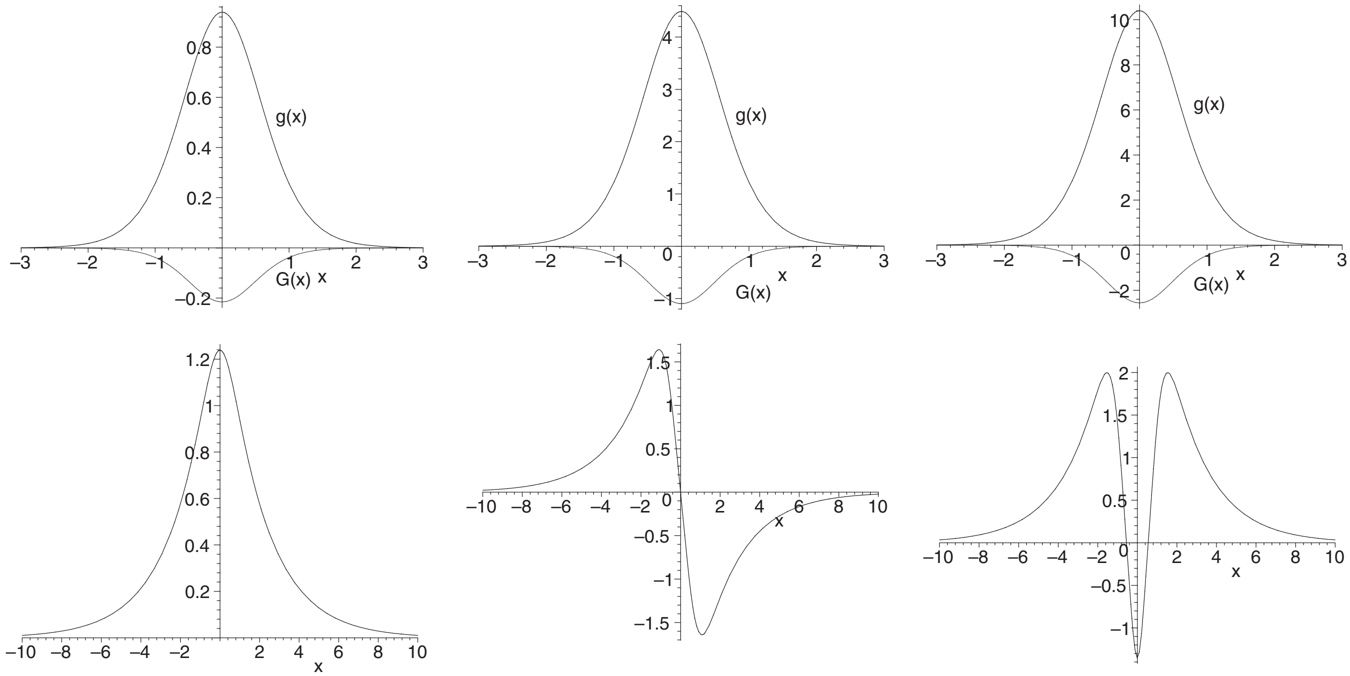

FIG. 2. The structures of the localized ICQ nonlinearities (top) given by Eq. (26) and the corresponding bound states (bottom) with one, two, and three solitons for Eq. (2) without an external potential are displayed with the parameters fixed by Eqs. (28) and (29), $a_0 = -\lambda = 1/4$, $a_1 = 1$, $m = 0.773\,382$ (left), $m = 0.829\,632$ (middle), and $m = 0.838\,431$ (right), respectively.

$=1/4$ and $a_1 = 1$, then we get $m = 0.773\,382$ for $n = 1$, $m = 0.829\,632$ for $n = 2$, and $m = 0.838\,431$ for $n = 3$, respectively. In this manner, an infinite number of solutions ϕ are obtained in the form of Eq. (27) with exactly $n-1$ zeros. With the above parameters, three sets of figures are plotted in Fig. 2 to display the configurations of the ICQ nonlinearities and the localized exact soliton solutions of Eq. (2) without an external potential.

B. Systems with the first type of periodic potentials

In this subsection, a class of periodic potentials of the form $V(x) \sim \mathrm{sn}^2(x,k)$ [6–8] are studied. From Eq. (7), we obtain a set of solutions

$$V(x) = 2k^2 \mathrm{sn}^2(x,k), \quad a(x) = \frac{\lambda - 1 - k^2}{k^2} + \mathrm{sn}^2(x,k), \quad (31)$$

where k is the modulus of the Jacobian elliptic sine function. Hence the ICQ nonlinearities are given by

$$g(x) = \frac{g_0 k^6}{[\lambda - 1 - k^2 + k^2 \mathrm{sn}^2(x,k)]^3},$$

$$G(x) = \frac{G_0 k^8}{[\lambda - 1 - k^2 + k^2 \mathrm{sn}^2(x,k)]^4}, \quad (32)$$

and thus E in Eq. (14) reads

$$E = \frac{1}{k^4}[\lambda^3 - 2(k^2+1)\lambda^2 + (k^4 + 3k^2 + 1)\lambda - k^2(k^2+1)]. \quad (33)$$

It is noted that the amplitude of the external potential depends on the modulus k. In the limit $k \to 1$, $V(x)$ becomes a hyperbolic secant potential well, and the ICQ nonlinearities become localized similar to case 2 in the previous subsection. However, the limit $k \to 0$ is not allowed since it will lead not only the potential to be zero, but the ICQ nonlinearities vanish as well.

Now, we solve Eq. (14) with Eq. (33). It is not difficult to verify that

$$U = \sqrt{-\frac{2m^2\mu^2}{g_0}} \frac{\mathrm{sn}(\mu\xi, m)}{\sqrt{1 + \mathrm{dn}(\mu\xi, m)}}, \quad (34)$$

with m being the modulus of the Jacobian elliptic functions and

$$\mu^2 = \frac{2[\lambda^3 - 2(k^2+1)\lambda^2 + (k^4 + 3k^2 + 1)\lambda - k^2(k^2+1)]}{k^4(m^2+4)}, \quad (35)$$

is a solution of Eq. (14) with Eq. (33) and

$$G_0 = \frac{3g_0^2}{16\mu^2}. \quad (36)$$

In this case, we must have $g_0 < 0$ in order to ensure the solution (34) to be real and $G_0 > 0$ from Eq. (36), which means that the system (1) has only repulsive two-body and attractive three-body interactions, respectively. Moreover, $\lambda > 1 + k^2$ must be enforced to ensure $a(x)$ positive. It is also a manifestation from Eq. (34) that we cannot take the limit $k \to 0$, since it will introduce singularities.

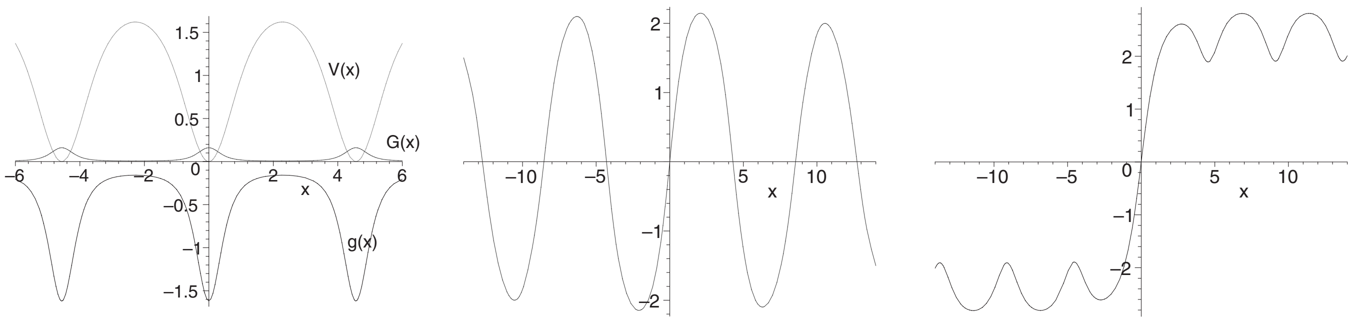

FIG. 3. The configurations of the periodic external potential given by Eq. (37) and the periodic ICQ nonlinearities given by Eq. (32) with Eqs. (35) and (36), $\lambda=5/2$, $g_0=-1$, and $k=0.9$ are shown in the left panel. The corresponding periodic soliton solution of Eq. (2) is shown in the middle panel for $m=0.9$, which will become a kink with periodic tails in the limit $m\to 1$ as shown in the right panel.

Therefore soliton solutions of Eq. (2) corresponding to Eq. (34) can be written straightforwardly. In Fig. 3, the left panel shows the configurations of the periodic external potential and ICQ nonlinearities, the middle panel exhibits the corresponding periodic solution of Eq. (2), which may become a kink with periodic tails when the modulus m approaches 1, as shown in the right panel. Figure 4 is related to Fig. 3 in the limit $k\to 1$, which reveals that the ICQ nonlinearities become localized, the external potential becomes a sech-type trapping well, and the periodic tails of the kink disappears. It is seen from Figs. 3 and 4 that the amplitude of the negative cubic nonlinearity is larger than that of the positive quintic one, which is rightly coherent to the fact that the repulsive two-body interaction in the dilute BEC is stronger than the attractive three-body interaction.

It is noted that with the parameters used in Fig. 4, Eq. (12) becomes $\xi=\int_0^x 2/[1+2\tanh^2(s)]ds$, and hence $-\infty<\xi<\infty$. Therefore in this case, quite different from case 2 in the last subsection, we cannot obtain bound states with an infinite number of solitons supported by Eq. (2) in the form of Eq. (34).

C. Systems with the second type of periodic potentials

Actually, Eq. (7) can give many sets of solutions for $V(x)$ and $a(x)$. For the first type of periodic potentials presented above, it turns out that the modulus k of the Jacobian elliptic function cannot take the limit $k\to 0$. Moreover, in the limit $k\to 1$, though the ICQ nonlinearities turn out to be localized, unlike the case without an external potential, an infinite number of soliton solutions of Eq. (2) are not obtainable. In order to improve these facts, we find another set of Jacobian elliptic periodic solutions from Eq. (7). It is easy to check that

$$V(x)=\frac{1}{4}\mathrm{dn}^2(x,k)+\frac{3}{4}(k^2-1)\mathrm{dn}^{-2}(x,k),\quad a(x)=\mathrm{dn}^{-1}(x,k), \tag{37}$$

where k is the modulus of the Jacobian elliptic dn function, solve Eq. (7) for $\lambda=(k^2-2)/4$. Hence the ICQ nonlinearities read

$$g(x)=g_0\mathrm{dn}^3(x,k),\quad G(x)=G_0\mathrm{dn}^4(x,k). \tag{38}$$

Due to Eqs. (37) with $\lambda=(k^2-2)/4$, E in Eq. (14) becomes zero.

Obviously, in the limit $k\to 1$, $V(x)$ turns into a hyperbolic secant potential barrier, and the ICQ nonlinearities again become localized. While in the limit $k\to 0$, both the external potential and the ICQ nonlinearities assume constant values so that we are back to the homogeneous nonlinear case.

The similar solution forms of Eqs. (27) and (34) are also applicable to Eq. (2) with $E=0$ and different parameter conditions. In fact, Eq. (2) possesses abundant exact solutions related to the different combinations of the different Jacobian elliptic functions. It is easy to prove that

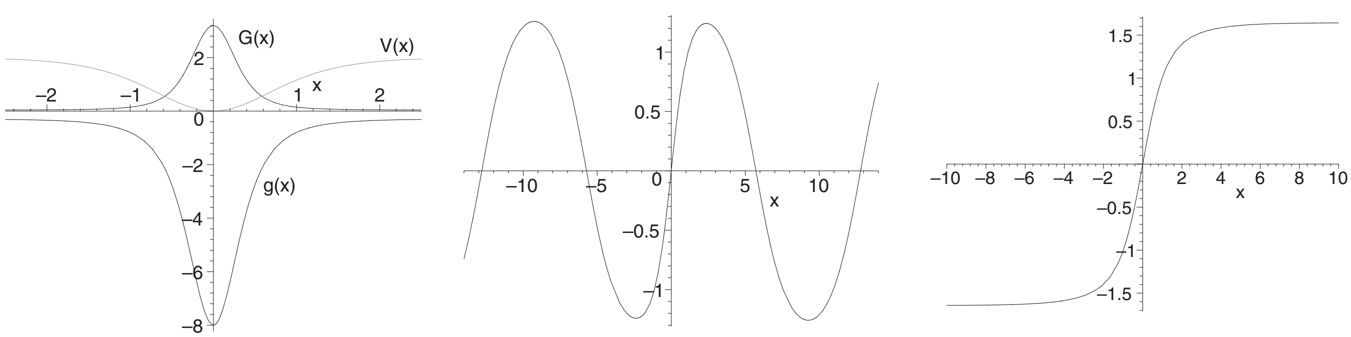

FIG. 4. The configurations of the external sech-type trapping potential given by Eq. (37) and the localized ICQ nonlinearities given by Eq. (32) with Eqs. (35) and (36), $\lambda=5/2$, $g_0=-1$, and $k=1$ are shown in the left panel. The corresponding exact periodic soliton solution of Eq. (2) is shown in the middle panel for $m=0.9$, which becomes a kink in the limit $m\to 1$ as shown in the right panel.

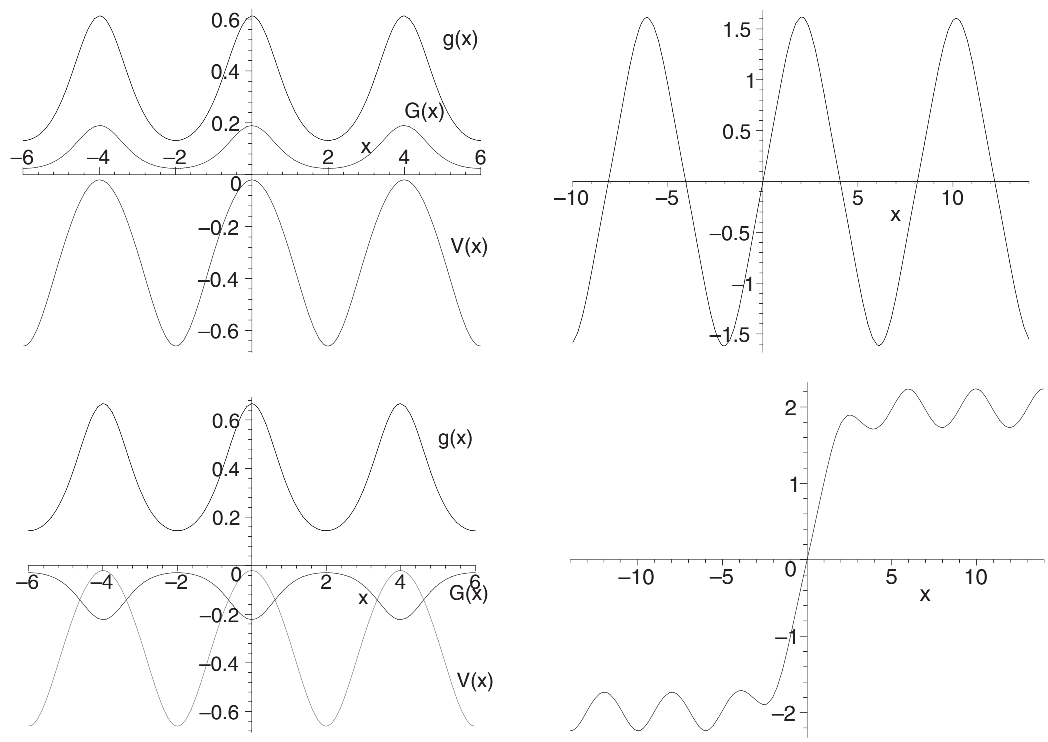

FIG. 5. The left panels are the profiles of the periodic potentials given by Eq. (37) and the ICQ nonlinearities given by Eq. (38), and their corresponding exact soliton solutions of Eq. (2) are shown in the right panels, with the parameters given by Eq. (40), $k=0.8$, $a_0=1$, $\mu=1$, $m=0.3$ (top), and $m=1$ (bottom), respectively.

$$U=\frac{3\,\mathrm{sn}(\mu\xi,m)}{\sqrt{3a_0[3-(m^2+1)\mathrm{sn}^2(\mu\xi,m)]}} \tag{39}$$

is an exact solution of Eq. (2) with $E=0$ and

$$g_0=\frac{2}{3}a_0\mu^2(m^4-m^2+1),$$

$$G_0=\frac{1}{9}a_0^2\mu^2(m^2-2)(2m^2-1)(m^2+1), \tag{40}$$

where a_0 is an arbitrary constant and m is the modulus of the Jacobian elliptic sine function. Equation (39) requires $a_0>0$, which thus gives $g_0>0$ for any m. From the second equation in Eq. (40), one can find that G_0 is positive for $0\leqslant m<1/\sqrt{2}$ and negative for $1/\sqrt{2}<m\leqslant 1$. An exceptional point is remarkable for $m=1/\sqrt{2}$ since it gives $G_0=0$ and thus the quintic nonlinearity identically vanishes. Therefore in this case, the cubic nonlinearity in the ICQNLSE (1) can only be attractive, while the quintic one can be either attractive or repulsive, depending on m.

The representative structures of a class of periodic external potentials and ICQ nonlinearities and their corresponding soliton solutions of Eq. (2) are exhibited in Fig. 5. It is seen that with a small m, the periodic ICQ nonlinearities are both positive and the cubic possesses a larger amplitude. While in the limit $m\to 1$, the periodic quintic nonlinearity becomes negative still with a smaller amplitude, whereas both amplitudes are a little larger than those in the small m. Moreover, in these two cases, the periodic external potential with a negative amplitude remains unchanged, and the periods of the external potential and the ICQ nonlinearities are of the same. In addition, in the limit $m\to 1$, the periodic solution of Eq. (2) becomes a kink with periodic tails, sharing a similar structure as shown in Fig. 3 where the periodic external potential has a positive amplitude, the cubic nonlinearity is negative, and the quintic is positive.

It has been pointed out that in the limit $k\to 1$, the periodic external potential becomes a hyperbolic secant potential barrier, and the ICQ nonlinearities are localized. It is interesting that, in this case, we can also obtain bound states of Eq. (2) with an arbitrary number of solitons, as the case without an external potential. From Eq. (12) with Eq. (37) and $k=1$, we find $\xi=2\arctan(e^x)$, and hence $0<\xi<\pi$. Therefore $U(0)=U(\pi)=0$ must be satisfied to meet the boundary condition $\phi(\pm\infty)=0$. For this solution (39), $U(0)=0$ is obviously satisfied. In order to meet the requirement $U(\pi)=0$, we can restrict μ as $\mu=2nK(m)/\pi$ $(n=1,2,3\ldots)$ with $K(m)$ defined by Eq. (30). Here, instead of finding a value of m from the constraint condition for μ, which has been done for the case without an external potential, it is more simple to find a value of μ for given m and n. In this way, Eq. (39) can represent an n-soliton solution. The bound states with one, two, and three solitons are displayed in Fig. 6, together with their corresponding external potentials and localized ICQ nonlinearities, respectively. It is observed from Fig. 6 that the external potential has an invariant positive amplitude acting as a potential barrier and the localized attractive cubic nonlinearity is stronger than the quintic. In addition, the amplitudes of the ICQ nonlinearities become stronger with the

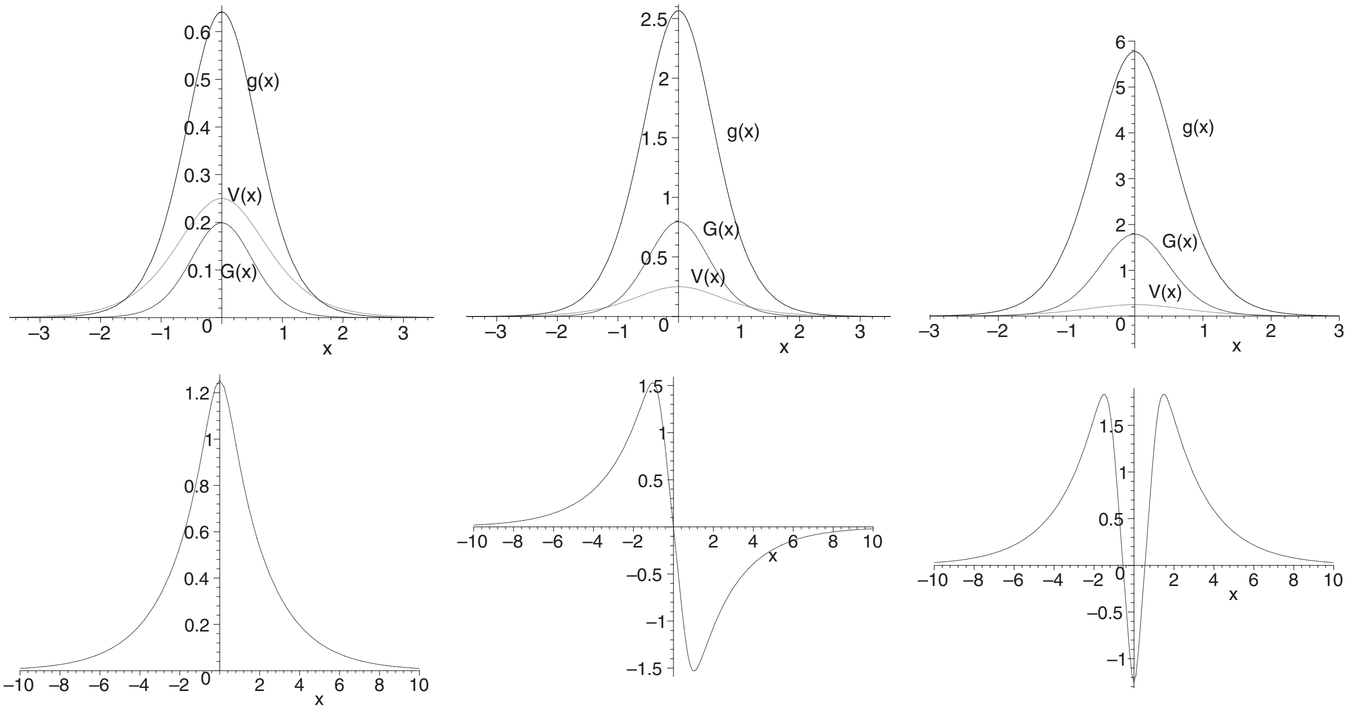

FIG. 6. The upper panels are about the configurations of the external potential barriers given by Eq. (37) and the localized ICQ nonlinearities given by Eq. (38), and the lower panels display the structures of the bound states with one, two, and three solitons for Eq. (2) corresponding to the upper, when the parameters are fixed by Eq. (40), $\mu=2nK(m)/\pi$, $a_0=k=1$, $m=0.3$, and $n=1$ (left), $n=2$ (middle), and $n=3$ (right), respectively.

increased number of solitons, and so does the positive amplitude of the solitons, which shares the similar properties as shown in Fig. 2. However, it is noted that the n-soliton solutions revealed in Figs. 2 and 6 are related to two different physical systems described by Eq. (1) for the reason that in the former case, there is no external potential and the quintic nonlinearity is negative, while there is a hyperbolic secant potential barrier and the quintic nonlinearity is positive in the latter case. For instance, in the context of BECs, the former solutions can describe matter waves in a dilute BEC without an external potential when the two-body interaction is attractive and the three-body repulsive, while in the latter case we have matter waves in a dilute BEC with a potential barrier when the two-body and three-body interactions are both attractive.

IV. SUMMARY AND EXTENSIONS

The properties of the one-dimensional spatially inhomogeneous cubic-quintic nonlinear Schrödinger equation (ICQNLSE) with an external potential have been studied analytically. By using a canonical transformation obtained from the classical Lie group approach, this model is successfully transformed to a homogeneous cubic-quintic nonlinear Schrödinger equation (HCQNLSE). A similar transformation, also known as the non-auto-Bäcklund transformation, between the ICQNLSE and HCQNLSE is found by applying the idea of the CK direct method. A direct link between the ICQNLSE and HCQNLSE exists, when a specific condition associated with the external potential and the ICQ nonlinearities is satisfied. Consequently, only with the special classes of the external potentials and ICQ nonlinearities can one obtain exact solutions of the ICQNLSE from those of the HCQNLSE.

The investigation of the ICQNLSE without an external potential is carried out first, and then we go on further reporting two classes of Jacobian elliptic periodic potentials which have attracted a great deal of attention recently [6–8] for the NLS equations with homogeneous nonlinearities. In all these cases, the explicit expressions of the ICQ nonlinearities are obtained, either periodic or localized in different ranges of the parameters. In every class of the external potentials and ICQ nonlinearities, we present different types of exact solutions, such as the bright, dark, kink, and periodic soliton solutions. In particular, bound states with an arbitrary number of solitons can be supported by the ICQNLSE in the presence of the localized ICQ nonlinearities, and either without an external potential or with a potential barrier, with a distinct feature that the cubic nonlinearity in both situations is positive; however, the quintic is negative in the former while positive in the latter.

Figures 1–6 have one property in common that no matter positive or negative, the amplitude of the cubic nonlinearity is always larger than that of the quintic. It just means that the cubic nonlinearity is stronger than the quintic, which is quite reasonable because the quintic nonlinearity is a higher order nonlinear correction and thus should be smaller than the lower order cubic nonlinearity.

As an application, the physical meanings of the solutions obtained in Sec. III have been interpreted in the context of

BECs. Actually, our results for the ICQNLSE with the periodic and localized external potentials and ICQ nonlinearities might also be useful for understanding the nonlinear waves in nonlinear optics with spatially inhomogeneous potentials, but they are also relevant in plasma physics, in condensed matter physics, in nuclear physics, etc. For instance, the interest for considering ICQ nonlinearities in nonlinear optics can stem from a nonlinear correction to the medium refractive index and a correction due to the inhomogeneity of the medium as well. In this context, for instance, solutions shown in Fig. 2 can be used to describe solitons in a non-Kerr inhomogeneous optical media with a competition between self-focusing occurring at low intensities and self-defocusing taking over at high intensities.

It is known that only stable (or weakly unstable) solitary waves can be observed experimentally. Therefore the stability of the solitary waves against small perturbations is a crucial issue. Recently, Ref. [36] deals with the existence and stability of localized solutions of a one-dimensional discrete CQNLS equation without an external potential. Here we go a little further to generally analyze the stability of our solutions under small perturbations. In a linear stability analysis, we perturb these solutions by

$$\psi = [\phi(x) + \phi_1(x,t)]e^{-i\lambda t}, \tag{41}$$

where $\phi_1(x,t)$ represents a small perturbation. Substituting Eq. (41) into Eq. (1) and linearizing the resulting equation, then considering solutions of the form $\phi_1(x,t) = \hat{\phi}_1(x)\exp(\Omega t)$, we find that the real and imaginary parts $(v(x),w(x)) = (\mathrm{Re}[\hat{\phi}_1],\mathrm{Im}[\hat{\phi}_1])$ satisfy the following linear eigenvalue equations:

$$\frac{\partial^2 w}{\partial x^2} + [g(x)\phi^2 + G(x)\phi^4 - V(x) + \lambda]w + \Omega v = 0, \tag{42}$$

and

$$\frac{\partial^2 v}{\partial x^2} + [3g(x)\phi^2 + 5G(x)\phi^4 - V(x) + \lambda]v - \Omega w = 0. \tag{43}$$

A solitary wave solution of Eq. (1) is stable if none of the eigenmodes of the linear eigenvalue problem given by Eqs. (42) and (43) grows exponentially. In other words, if there is at least one eigenvalue Ω with a positive real part, then instability results. The presence of the inhomogeneous nonlinearities, the inhomogeneous external potential, and the complex forms of the solutions ϕ makes it difficult to find an exact analytic solution of this linear eigenvalue problem. Therefore numerical studies have to be carried out to address this problem. Furthermore, as is often used to investigate the stability of soliton solutions [6,36,37], we can also conduct numerical simulations of these solutions with perturbations initially implanted to see whether their propagations are stationary or not.

Since the object of this work is to obtain some exact analytical soliton solutions of the nonlinear Schrödinger equation with the inhomogeneous competing cubic-quintic nonlinearities in the presence of periodical external potentials, a thorough analysis of the stability of our new solutions is beyond the scope of the present paper and is thus left for further studies.

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