Entangled rings, matrix product states, and exact solutions of *XYZ* spin chains

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We show that the ground state of the Heisenberg spin-1/2 chain in an external magnetic field, can be exactly expressed as a matrix product state, provided that the coupling constants are constrained to be on a specific two dimensional surface. This ground state has a very interesting property: all the pairs of spins are equally entangled with each other. In this last respect, the results are of interest for engineering long-range entanglement in experimentally realizable finite arrays of qubits, where the ground state will act as the initial state of a quantum computer.

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I. INTRODUCTION

Combination of ideas from condensed matter physics and quantum information science, has in recent years resulted in fruitful and complementary methods in the study of quantum properties of many body systems. While the starting point in condensed matter physics is the Hamiltonian, it is the ground or the thermal state that is of particular interest in quantum information science, where thanks to the quantitative measures of entanglement developed in the past decade, one can now quantify exactly the nonclassical correlations inherent in such states [1,2]. These entanglement properties will then be of utmost importance, when we think of such states as the states of arrays of qubits in quantum computers, where unique quantum features are manipulated to exponentially surpass classical computers.

To put it in a picturesque way, in the road which connects the above two disciplines, one traditionally starts from a Hamiltonian in the condensed matter camp and ends at an approximate ground state in the quantum information camp. However, one can traverse the same road in reverse direction; start from a state with exactly known and predetermined properties and go on until one finds the Hamiltonian of which this is the ground state. This is the so-called matrix product formalism originated in the work on the Affleck-Kennedy-Lieb-Tasaki (AKLT) model [3] and finitely correlated states [4] and resumed in many recent works on various aspects of this interrelation [5]. This approach has led among other things, to a new insight on quantum phase transitions [6], where it has been shown that one can engineer classes of quantum phase transitions (QPT) with predetermined properties, which have specific similarities and differences with conventional QPT's.

However, this reverse direction has an obvious drawback: the Hamiltonian, which is found at the end of the road, may not be simple and of wide interest to condensed matter physicists, or may have couplings of a fine-tuned nature [3,8].

In the present work we show that by starting from a suitable ansatz, the ground state of the one dimensional Heisenberg spin-1/2 chain, the prototype of an interacting many body spin system, described by the following Hamiltonian

$$H=\sum_{i=1}^{N} J_x\sigma_{x,i}\sigma_{x,i+1}+J_y\sigma_{y,i}\sigma_{y,i+1}+J_z\sigma_{z,i}\sigma_{z,i+1}-B_x\sigma_{x,i}, \tag{1}$$

can be found exactly, on a hypersurface in the space of its parameters. This surface is defined as follows:

$$J_x=-J+\frac{1+g^2}{2},\quad J_y=-\eta J+g,$$

$$J_z=-\eta J-g,\quad B_x=\epsilon(1-g^2), \tag{2}$$

where $(\epsilon,\eta)=\pm(1,\pm 1)$ are two discrete parameters, and g and $(J>0)$ are two continuous parameters.

We will calculate the spin correlation functions exactly and show that singularities in the thermodynamic limit develop at $g=0$, a property which has been called the matrix-product-state (MPS)–quantum-phase transition (QPT) in [6], to distinguish them from known examples of QPT's [7]. We will also show that, the ground state of this system has a very interesting property: all the pairs of spins have equal entanglement with each other (Fig. 1). This is a very desirable situation for quantum computers: an array of spins, which at low temperatures have long range and experimentally controllable entanglement.

The structure of this paper is as follows: In Sec. II we briefly explain the matrix product formalism; in Sec. III we show that by removing an apparent restriction in this formalism one can indeed construct an exact solution for the ground state of the spin-1/2 Heisenberg chain in a certain subset of the parameter space. In this same section we also show that in an even more constrained subset the model is equivalent to a free-fermion model. In Sec. IV we consider the entanglement properties of the ground state of this model. Finally, we conclude the paper with a discussion.

II. MATRIX PRODUCT FORMALISM

Let us briefly review the matrix product state (MPS) formalism. On a ring of N sites of d-level particles, a state is called a matrix product state if there exist matrices A_i, $i=0,\ldots,d-1$ (of dimension D) such that

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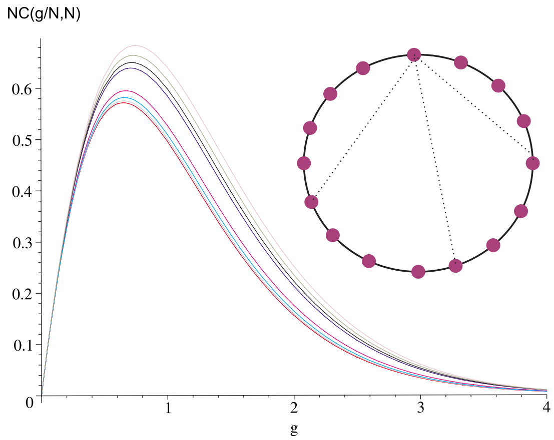

FIG. 1. (Color online) In the MPS ground state of Eq. (1), all the pairs of spins are equally entangled with each other. The curves show scaled concurrence for rings of size (from top to bottom) 6, 7, 8, 9, 10, 20, 30, 40, and 50.

$$\psi_{i_1,i_2,\ldots,i_N} = \frac{1}{\sqrt{Z}}\mathrm{tr}(A_{i_1}A_{i_2}\cdots A_{i_N}), \tag{3}$$

where Z is a normalization constant given by $Z=\mathrm{tr}(E^N)$ in which

$$E:=\sum_{i=0}^{d-1}(A_i^*\otimes A_i).$$

The state (3) is reflection symmetric if there exists a matrix Π such that $A_i^T=\Pi A_i\Pi^{-1}$ (where T means transpose) and time-reversal invariant if there exists a matrix V such that $A_i^*=VA_iV^{-1}$. All the correlation functions can be calculated exactly. For example, for a local observable O, one finds

$$\langle\Psi|O(k)|\Psi\rangle = \frac{\mathrm{tr}(E^{k-1}E_O E^{N-k})}{\mathrm{tr}(E^N)}, \tag{4}$$

where

$$E_O:=\sum_{i,j=0}^{d-1}\langle i|O|j\rangle A_i^*\otimes A_j.$$

In the thermodynamic limit ($N\to\infty$), only the eigenvector(s) corresponding to the eigenvalue $\lambda_{\max}$ of E with the largest absolute value matters and any level crossing in this eigenvalue leads to a discontinuity in correlation functions.

Given a matrix product state, the reduced density matrix of k adjacent sites is given by

$$\rho_{i_1\cdots i_k,j_1\cdots j_k} = \frac{\mathrm{tr}[(A_{i_1}^*\cdots A_{i_k}^*\otimes A_{j_1}\cdots A_{j_k})E^{N-k}]}{\mathrm{tr}(E^N)}.$$

This density matrix has at least d^k-D^2 zero eigenvalues. To see this, suppose that we can find complex numbers $c_{i_1\cdots i_k}$ such that

$$\sum_{j_1,\ldots,j_k=0}^{d-1} c_{j_1\cdots j_k}A_{j_1}\cdots A_{j_k}=0. \tag{5}$$

This is a system of D^2 equations for d^k unknowns, which has at least d^k-D^2 independent solutions. Any such solution gives a null eigenvector of ρ. Thus for the density matrix of k adjacent sites to have a null space, it is sufficient (but not necessary) that $d^k>D^2$. Let the null space of the reduced density matrix be spanned by the orthogonal vectors $|e_\alpha\rangle$, $\alpha=1,\ldots,s$, then we can construct the local Hamiltonian acting on k consecutive sites as

$$h:=\sum_{\alpha=1}^{s}J_\alpha|e_\alpha\rangle\langle e_\alpha|,$$

where J_α are positive constants. The total Hamiltonian on the chain will then be given by the positive operator

$$H=\sum_{l=1}^{N}h_{l,l+k},$$

where $h_{l,l+k}$ is the embedding of h into sites l to $l+k$ of the chain. The state $|\psi\rangle$ will then be a ground state of H. In the next section we show how this helps us to find an exact solution for the spin-1/2 Heisenberg chain in a certain range of parameter space of couplings.

III. MATRIX PRODUCT STATE FOR THE HEISENBERG CHAIN

Equation ($d^k>D^2$) puts a stringent requirement on the dimensions of the matrices used in the construction of a matrix product state. When dealing with spin 1/2 with nearest-neighbor interactions, for which $d=2$ and $k=1$, it appears that the only admissible dimension for the matrices A_0 and A_1 is $D=1$, leading to a product state. However, it is crucial to note that the condition $d^2>D^2$ is only a sufficient and not a necessary condition for the density matrix ρ to have a null space. To proceed with our construction, we require that the state satisfy some natural symmetries, i.e., spin-flip symmetry which, in the language of matrix product formalism, means that there is a matrix X such that

$$XA_0X^{-1}=\epsilon A_1,\quad XA_1X^{-1}=\epsilon A_0,$$

where $\epsilon^2=1$. Working in the basis where $X=\sigma_z$, we find the general form of the matrices A_0 and A_1 as follows:

$$A_0=\begin{pmatrix} a & b\\ c & d\end{pmatrix},\quad A_1=\epsilon\begin{pmatrix} a & -b\\ -c & d\end{pmatrix}.$$

Although these two matrices are not symmetric, the state constructed from them is symmetric under parity, since there is a matrix $\Pi=\begin{pmatrix} b & 0\\ 0 & c\end{pmatrix}$ with the property

$$\Pi A_0^t\Pi^{-1}=A_0,\quad \Pi A_1^t\Pi^{-1}=A_1.$$

We now consider the matrix equation (5), which in the present case is

$$c_{00}A_0^2 + c_{01}A_0A_1 + c_{10}A_1A_0 + c_{11}A_1^2 = 0. \tag{6}$$

This is a set of linear equations for the four coefficients c_{ij}, which can be written as a matrix equation $MC=0$, leading to a nonzero solution when

$$\det(M) \equiv 16b^2c^2(a-d)^2(a+d)^2 = 0.$$

Thus we will find nontrivial models for $a=d$ or $a=-d$. The models with $b=0$ or $c=0$ are not symmetric under parity, since in these cases the matrix Π will not be invertible. We can always rescale the matrices by a constant factor without affecting the matrix product state, so we set $a=1$ and use a subsequent gauge transformation $A_i \to SA_iS^{-1}$ with $S = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$ to set $c=1$. Therefore we are left with the following four classes of models defined by the matrices:

$$A_0 = \begin{pmatrix} 1 & g \\ 1 & \eta \end{pmatrix}, \quad A_1 = \epsilon \begin{pmatrix} 1 & -g \\ -1 & \eta \end{pmatrix}, \tag{7}$$

where g is a continuous parameter and $(\epsilon, \eta) = \pm(1, \pm 1)$. The four types of models are distinguished by the values of the pair (ϵ, η).

In order to see how the Hamiltonian is constructed, we solve Eqs. (6), which in view of Eq. (7) take the form

$$(1+g)(C_{00}+C_{11}) + \epsilon(1-g)(C_{01}+C_{10}) = 0, \tag{8}$$

$$(1+\eta)(C_{00}-C_{11}) - \epsilon(1-\eta)(C_{01}-C_{10}) = 0. \tag{9}$$

It is easy to verify that the solution space is determined by the following two un-normalized vectors:

$$|e_1\rangle = (1+\eta)|\psi_-\rangle + (1-\eta)|\phi_-\rangle,$$

$$|e_2\rangle = (1+g)|\psi_+\rangle - \epsilon(1-g)|\phi_+\rangle,$$

where $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle \pm |1,0\rangle)$ and $|\phi_\pm\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle \pm |1,1\rangle)$ are Bell states. Under spin flip the above states transform as $|e_{1,2}\rangle \to \mp |e_{1,2}\rangle$. The final local Hamiltonian will be given by

$$h = J|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|,$$

where J is a non-negative parameter and we have used the freedom for rescaling the couplings of the Hamiltonian to set one of the parameters equal to 1. In view of the symmetry property of the vectors under spin flip, this Hamiltonian will be symmetric under spin flip. Expressing the above operator in terms of Pauli operators and subtracting a constant term $J + \frac{1+g^2}{2}$, we find the total Hamiltonian, which is written in Eqs. (1) and (2). This shows that the ground state energy of the Hamiltonian (1) is

$$\mathcal{E}_0 := -N\left(J + \frac{1+g^2}{2}\right). \tag{10}$$

Note that the models with $\epsilon = \pm 1$ (sign of the magnetic field) are related by local π rotations of spins around the z axis, where $(\sigma_{x,y} \to -\sigma_{x,y})$. Also, the models with $\eta = \pm$ are related to each other by simultaneous rotations $R_x(\frac{\pi}{2}) \otimes R_x(\frac{-\pi}{2})$ of spins on adjacent sites, under which we have $(\sigma_{z,i}\sigma_{z,i+1} \rightleftharpoons -\sigma_{y,i}\sigma_{y,i+1})$. This is, of course, possible only when N is even [12].

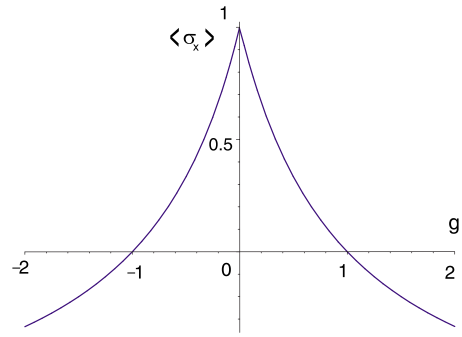

FIG. 2. (Color online) The magnetization in the x direction as a function of g $(\epsilon=1)$.

The correlation functions can be derived from Eq. (4). The eigenvalues of the matrix $E = A_0 \otimes A_0 + A_1 \otimes A_1$ are $2(\eta \pm g)$, $2(1 \pm g)$ showing an MPS-QPT at $g=0$. Consider for definiteness, the case $\eta=1$. The magnetization per site is found from Eq. (4) to be

$$\langle \sigma_y \rangle = \langle \sigma_z \rangle = 0, \quad \langle \sigma_x \rangle = \epsilon u \frac{1+u^{N-2}}{1+u^N},$$

where $u := \frac{1-g}{1+g}$ and the correlation functions $G_a(1,r) := \langle \sigma_{a,1}\sigma_{a,r} \rangle$ are similarly found to be as follows:

$$G_x(1,r) = \frac{u^2 + u^{N-2}}{1+u^N},$$

$$G_y(1,r) = \frac{u^{N-2}(u^2-1)}{1+u^N}, \quad G_z(1,r) = \frac{1-u^2}{1+u^N}. \tag{11}$$

These correlation functions satisfy the following relations:

$$G_x + G_y + G_z = 1, \quad (1-G_z)(1-G_y) = \langle \sigma_x \rangle^2. \tag{12}$$

In the thermodynamic limit $(N \to \infty)$, discontinuities develop in these correlation functions at $g=0$. For example, the magnetization per site is given by

$$\langle \sigma_x \rangle = \epsilon \frac{1-|g|}{1+|g|},$$

which shows a cusplike singularity at $g=0$ (Fig. 2).

The explicit form of such a ground state can also be determined. For $\eta=1$, the matrices A_0 and A_1 commute. By a similarity transformation, which does not change the state (3), both the matrices are made diagonal,

$$A_0 = \begin{pmatrix} 1+\sqrt{g} & 0 \\ 0 & 1-\sqrt{g} \end{pmatrix}, \quad A_1 = \epsilon \begin{pmatrix} 1-\sqrt{g} & 0 \\ 0 & 1+\sqrt{g} \end{pmatrix},$$

and the MPS state (3) will be given by

$$|\Psi\rangle_{\eta=1} = \frac{1}{\sqrt{Z}}(|\phi_+\rangle^{\otimes N} + |\phi_-\rangle^{\otimes N}), \tag{13}$$

where

$$|\phi_\pm\rangle = (1 \pm \sqrt{g})|0\rangle + \epsilon(1 \mp \sqrt{g})|1\rangle,$$

and $Z=2^{N+1}[(1+g)^N+(1-g)^N]$. These expressions are valid for all values of g, provided that we replace $\sqrt{g}\to i\sqrt{-g}$ when we consider negative values of g. Note that $\langle\phi_+|\phi_-\rangle=2(1-g)$. One can indeed check that the separate product states are ground states of H. This was first shown in [13] by showing that the local Hamiltonian acting on two adjacent sites, when added by a suitable constant, annihilates $|\phi_\pm\rangle^{\otimes 2}$. However, the advantage of the matrix product state $|\phi_+\rangle^{\otimes N}+|\phi_-\rangle^{\otimes N}$ (or the other one obtained by subtracting them) is that they are invariant under spin-flip transformation $\sigma_x^{\otimes N}$. Thus even if the couplings of the Hamiltonian are not fine-tuned as in Eq. (2), first order perturbation theory guarantees that one of the entangled states and not the product states, will be the unique grounds state of H.

An interesting point is $J=\frac{1+g^2}{2}$, where the model reduces to an XY model in transverse magnetic field. More explicitly (after setting $\epsilon=\eta=1$ for definiteness and relabeling the axes $x\to z, y\to x, z\to y$) the Hamiltonian (1) will be given by

$$H=-\sum_{i=1}^N \frac{1}{2}(1-g)^2\sigma_{x,i}\sigma_{x,i+1}+\frac{1}{2}(1+g)^2\sigma_{y,i}\sigma_{y,i+1} + (1-g^2)\sigma_{z,i}. \tag{14}$$

According to our results, the ground state energy of this model will be $\mathcal{E}_0=-N(1+g^2)$. This is indeed the energy, which one obtains after turning the above Hamiltonian to a model of free fermions. Therefore at this particular point the free fermion solution is given by a matrix product state. To see this let us fix for definiteness the discrete parameters as $\epsilon=\eta=1$. As we have discussed before, models with different discrete parameters have equal ground state energy.

According to Eq. (10) the ground state energy of this Hamiltonian will be equal to

$$\mathcal{E}_0=-N(1+g^2). \tag{15}$$

On the other hand the anisotropic XY model in a transverse magnetic field is defined by the Hamiltonian

$$H(\lambda,\gamma)=-\sum_{i=1}^N \frac{\lambda}{2}(1+\gamma)\sigma_{x,i}\sigma_{x,i+1}+\frac{\lambda}{2}(1-\gamma)\sigma_{y,i}\sigma_{y,i+1}+\sigma_{z,i}. \tag{16}$$

The combination of a Jordan-Wigner transformation, followed by Bogoliubov transformation turns this Hamiltonian into a free fermion model with the following Hamiltonian:

$$H(\lambda,\gamma)=2\sum_q \omega_q\eta_q^\dagger\eta_q-\sum_q\omega_q, \tag{17}$$

where η_q are fermionic oscillators $q\in\{-\frac{N}{2},-\frac{N}{2}+1,\dots,\frac{N}{2}-1\}$, and

$$\omega_q=\sqrt{(\gamma\lambda\sin\phi_q)^2+(1+\lambda\cos\phi_q)^2}, \tag{18}$$

in which

$$\phi_q=\frac{2\pi q}{N}. \tag{19}$$

We want to see if our results agree with these well-known facts or not. From Eq. (1) we readily see that

$$H=-(1-g^2)\sum_{i=1}^N \frac{1-g}{2(1+g)}\sigma_{x,i}\sigma_{x,i+1}+\frac{1+g}{2(1-g)}\sigma_{y,i}\sigma_{y,i+1}+\sigma_{z,i}, \tag{20}$$

or

$$H=(1-g^2)H(\lambda,\gamma),$$

where λ and γ are obtained from the following comparison:

$$\frac{1-g}{2(1+g)}=\frac{\lambda}{2}(1+\gamma),\quad \frac{1+g}{2(1-g)}=\frac{\lambda}{2}(1-\gamma). \tag{21}$$

Solving these equations for γ and λ we find

$$\gamma=\frac{-2g}{1+g^2},\quad \lambda=\frac{1+g^2}{1-g^2}. \tag{22}$$

Hence for $g^2<1$, we have

$$\mathcal{E}_0=(1-g^2)E_0(\gamma,\lambda)=(1-g^2)\left[-\sum_q\omega_q\right], \tag{23}$$

where γ and λ are given as above and $E_0(\gamma,\lambda)$ is the ground state energy of the XY model. For $g^2>1$,

$$\mathcal{E}_0=(1-g^2)E_m(\gamma,\lambda)=(1-g^2)\sum_q\omega_q, \tag{24}$$

where $E_m(\gamma,\lambda)$ is the maximum energy of the XY model. Equations (23) and (24) can be combined in a single formula

$$\mathcal{E}_0=-|(1-g^2)|\sum_q\omega_q. \tag{25}$$

In the thermodynamic limit $N\to\infty$, where $\frac{2\pi q}{N}\to x$ and $\frac{2\pi}{N}\to dx$ we have

$$\sum_q\omega_q=N\int_{-\pi}^{\pi}\frac{dx}{2\pi}\sqrt{(\gamma\lambda\sin x)^2+(1+\lambda\cos x)^2}, \tag{26}$$

which is simplified to

$$\sum_q\omega_q=N\int_{-\pi}^{\pi}\frac{dx}{2\pi}\sqrt{\lambda^2(1-\gamma^2)\cos^2 x+(1+\lambda^2\gamma^2)+2\lambda\cos x}. \tag{27}$$

However, from Eq. (22) we find

$$\lambda^2(1-\gamma^2)=1,\quad \to 1+\gamma^2\lambda^2=\lambda^2. \tag{28}$$

Thus Eq. (27) will simplify further to

$$\begin{aligned}\sum_q\omega_q&=N\int_{-\pi}^{\pi}\frac{dx}{2\pi}\sqrt{\cos^2 x+\lambda^2+2\lambda\cos x}\\&=\pm\frac{N}{2\pi}\int_{-\pi}^{\pi}(\lambda+\cos x)dx=\pm\frac{N}{2\pi}\int_{-\pi}^{\pi}\lambda dx=N|\lambda|\\&=N\frac{1+g^2}{|1-g^2|},\end{aligned} \tag{29}$$

where in the last step we have used the fact that $|\lambda|\geq 1$. Inserting this in Eq. (25) we find

$$\mathcal{E}_0 = -N(1+g^2),$$

which is exactly what we have obtained.

We should emphasize that for finite N, it is our result (15) and not Eq. (25), which gives the ground state energy of the XY model (for these particular parameters). The two answers differ by a term, which becomes smaller as N becomes larger and vanishes only in the thermodynamic limit. The reason is when the Jordan-Wigner transformation is applied to the original problem, which has periodic boundary condition $(a_0=a_N)$, $\left(a_i := \frac{\sigma_{x,i}+i\sigma_{y,i}}{2}\right)$ called the a-cyclic problem in [9], the new problem, which is written in terms of the fermion oscillators, is no longer strictly periodic, i.e., $a_N^\dagger a_1 = -c_N^\dagger c_1 \exp(i\pi\Sigma_{i=1}^{N-1} a_N^\dagger c_1) \neq c_N^\dagger c_1$ as it should if $c_0 \equiv c_N$. In the thermodynamic limit one can ignore this difference and treat the c-cyclic problem $(c_N=c_0)$ [9] as equivalent to the original a-cyclic problem. For finite values of N, Eq. (15) is the solution of the c-cyclic problem and not the original a-cyclic one.

IV. ENTANGLEMENT PROPERTIES OF THE GROUND STATE

We now come to the entanglement properties of the state (13). At $g=1$ when $\langle\phi_+|\phi_-\rangle=0$, the state becomes a standard Greenberger-Horne-Zeilinger (GHZ) state, $\frac{1}{\sqrt{2}}(|0\cdots 0\rangle + |1\cdots 1\rangle)$. For other values of g, when $|\phi_+\rangle$ and $|\phi_-\rangle$ are no longer orthogonal, it can be named a generalized GHZ state. Obviously such a state induces equal entanglement between any two spins regardless of their distance. To calculate this entanglement we determine the reduced two particle density matrix and use Wootters formula [10], with the result [11] as follows:

$$C = \frac{4|g|}{|(1+g)^N + (1-g)^N|}\left|1-|g|\right|^{N-2}.$$

Thus although the ring is not totally connected, the mutual entanglement of all pairs are equal and independent of their distances. Looking at the $N \gg 1$ limit, one can obtain the relation

$$NC\left(\frac{g}{N},N\right) \approx \frac{2|g|e^{-|g|}}{\cosh g}. \tag{30}$$

One can interpret the left hand side as the total mutual entanglement of a spin with all the other spins, and the above equation as a universal scaling relation for this total entanglement.

For the case $\eta=-1$, we use the transformation $U := R_x\left(\frac{\pi}{2}\right) \otimes R_x\left(\frac{-\pi}{2}\right)$ on adjacent sites, which transforms $H(\eta=1)$ to $H(\eta=-1)$ and act on Eq. (13) by $U^{\otimes N/2}$ to obtain the ground state as

$$|\Psi\rangle_{\eta=-1} = \frac{1}{\sqrt{Z}}(|\chi_+\rangle|\chi_-\rangle)^{\otimes N/2} + (|\chi_-\rangle|\chi_+\rangle)^{\otimes N/2}, \tag{31}$$

where

$$|\chi_\pm\rangle = (1+\sqrt{g})|y,\pm\rangle \pm i\epsilon(1-\sqrt{g})|y,\mp\rangle,$$

in which $|y,\pm\rangle$ denote the eigenstates of σ_y. An alternative way for deriving this ground state and indeed the reason for its simple structure, is to note that for $\eta=-1$, although the matrices A_0 and A_1 do not commute, the pairs of matrices corresponding to Bell states, defined by

$$\Phi_{mn} := \frac{1}{\sqrt{2}}[A_0A_m + (-1)^n A_1 A_{1+m}], \quad m,n = 0,1,$$

commute with each other. The reader can verify that the states $|\chi\rangle_\pm|\chi_\mp\rangle$ are indeed linear combinations of the Bell states $\phi_{00} \equiv \phi_+$, $\phi_{01} \equiv \psi_-$, and $\phi_{10} \equiv \psi_+$, making the state (31) a linear superposition of strings of various Bell states on adjacent sites.

V. CONCLUSION

We have shown that the matrix product formalism can indeed give the exact ground state of a two-parameter family of the spin-1/2 XYZ Heisenberg chain in a magnetic field. This adds a new example to the interesting AKLT [3] or Majumdar-Ghosh models [8], with an important difference, i.e., these models have no free coupling constants. Our model undergoes an MPS–quantum-phase transition [6] as one of the parameters passes a critical point, which stimulates further exploration of the MPS–quantum-phase transition in a set of important exactly solvable models. Moreover, the ground states have an interesting property that all the pairs of spins are equally entangled with each other, making them good candidates for engineering long-range entanglement in experimentally realizable arrays of qubits or spin systems.

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[1] K. M. O'Connor and W. K. Wootters, Phys. Rev. A **63**, 052302 (2001).

[2] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature **416**, 608 (2002); T. J. Osborne and M. A. Nielsen, Phys. Rev. A **66**, 032110 (2002); M. C. Arnesen S. Bose, and V. Vedral, Phys. Rev. Lett. **87**, 017901 (2001); M. Asoudeh and V. Karimipour, Phys. Rev. A **70**, 052307 (2004); A. R. Its, B.-Q. Jin, and V. E. Korepin, J. Phys. A **38**, 2975 (2005).

[3] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. Lett. **59**, 799 (1987).
[4] M. Fannes, B. Nachtergaele, and R. F. Werner, Europhys. Lett. **10** 633 (1989); A. Klumper, A. Schadschneider, and J. Zittartz, J. Phys. A **24** L955 (1991); **87**, 281 (1992).
[5] F. Verstraete, M. A. Martin-Delgado, and J. I. Cirac, Phys. Rev. Lett. **92**, 087201 (2004); F. Verstraete, M. Popp, and J. I. Cirac, *ibid.* **92**, 027901 (2004); F. Verstraete, J. I. Cirac, J. I. Latorre, E. Rico, and M. M. Wolf, *ibid.* **94**, 140601 (2005).
[6] M. M. Wolf, G. Ortiz, F. Verstraete, and J. I. Cirac, Phys. Rev. Lett. **97**, 110403 (2006).
[7] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 1999).
[8] C. K. Majumdar, J. Phys. C **3**, 911 (1970).
[9] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. **16**, 407 (1961).
[10] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[11] With conditions (12), the matrix $\rho\tilde{\rho}$ has only two nonzero eigenvalues given by $\lambda_{1,2}=\frac{1}{4}(G_y \pm G_z)^2$, which gives $C=|\sqrt{\lambda_1}-\sqrt{\lambda_2}|$.
[12] For $\eta=-1$ and $N=$odd such an MPS does not exist, i.e., Ψ is identical to zero. One can prove this by induction using the facts that $\mathrm{tr}(A_0)=\mathrm{tr}(A_1)=0$, and A_0^2, A_1^2, and $A_0A_1+A_1A_0$ are all proportional to identity.
[13] J. Kurmann, H. Thomas, and G. Müller, Physica A **112**, 235 (1982); G. Müller and R. E. Shrock, Phys. Rev. B **32**, 5845 (1985).