Quantum entanglement induced by dissipation

Taeseung Choi
Department of Physics, Konkuk University, Seoul 143-701, Korea

Hyuk-jae Lee*
Department of Physics, Kookmin University, Seoul 136-702, Korea
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We solve the master equation of a two-qubit system with the induced unitary term and the dissipation term coming from the system-environment coupling. We explain under this situation how entanglement between two qubits can be generated and persist at the asymptotic time.

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Entanglement is a genuine feature of the quantum world and one of the most valuable resources in quantum information and quantum computation [1]. It is well known that, in most case, quantum coherence is destroyed by the environment, and as a result a quantum state becomes a classical state very rapidly. It was expected that the decoherence by the environment also makes entanglement disappear [2]. This fragility has been thought of as an obstacle for the applications of entanglement, since one cannot avoid interactions with the environment.

Recently several authors [3–8] found the astonishing result that entanglement can be created and even increased in some dissipative systems. Braun [4] thought that the origin of entanglement generation comes from the induced interaction between two qubits. However, others [5,6] have shown that entanglement can be created through a purely noisy mechanism in the Markovian regime. In particular, Benatti *et al.* [5] have also discussed the general criteria for the possibility of entanglement creation by considering a common bath with only a noisy term. An *et al.* [7] showed that the entanglement dynamics of a quantum register interacting with a common environment can be analytically solved by the quantum jump method, and the entanglement generation gives rise due to a decoherence-free state included in the initial state. Jakóbczyk solved the master equation directly in an example with only a dissipation term due to the thermal heat bath [8]. Nevertheless, it is still an interesting question whether entanglement can be created only through purely dissipative dynamics even in the presence of an effective induced interaction. In general, it is almost impossible to turn off the unitary coupling term induced by the interaction with the bath and study only the dissipation term, because of the fluctuation-dissipation theorem [9]. We will discuss this question by exactly solving the master equation in the Markovian regime in a real physical system.

As is well known, the creation of entanglement by given interactions is very sensitive to the choice of an initial state as well as the interaction between qubits. For example, the initial state $|\uparrow\rangle\otimes|\uparrow\rangle$ cannot be entangled under the Ising-type interaction S_zS_z, but $|+\rangle\otimes|+\rangle$ $[|+\rangle=(1/\sqrt{2})(|\uparrow\rangle+|\downarrow\rangle)]$ can be entangled under the same interaction. We will show that the dissipative dynamics increases entanglement, while the effective induced interaction makes no change to the entanglement at an arbitrarily long time for the Werner-type initial state in a special representation. In this paper, we will find a physical model that creates entanglement solely by the dissipation term in the presence of the unitary term induced by the interaction with the bath, and consider several kinds of interactions with the bath and the initial states of system. In order to describe the full story, it is necessary to solve the master equation. First, we will solve the master equation of the model suggested by Benatti *et al.* [5] This provides the method to approach the entanglement generation in a purely noisy environment.

The evolution of the system interacting with an environment can be written using a master equation of Kossakowski-Lindblad form [10,11] under appropriate approximations:

$$\frac{d}{dt}\rho_s(t) = -i[\mathcal{H},\rho_s(t)] + \mathcal{L}(\rho_s(t)), \tag{1}$$

where ρ_s denotes the reduced density operator representing the state of the system, and $\mathcal{H}$ denotes the sum of the original system Hamiltonian and the unitary effective induced interaction term. For simplification, we consider only the induced interaction Hamiltonian and ignore other Hamiltonians. This simplification does not affect the general argument related to the entanglement creation, because the local Hamiltonian gives only a local unitary transformation, which does not change the entanglement.

The general form of the nonunitary term that describes the dissipation of the reduced density operator is

$$\mathcal{L}(\rho_s(t)) = \sum_{\alpha,\beta} D_{\alpha\beta}\{[L_\alpha,\rho_s(t)L_\beta^\dagger] + [L_\alpha\rho_s(t),L_\beta^\dagger]\}, \tag{2}$$

where L_α are generators of a dynamical semigroup [10]. If $D_{\alpha\beta}$ is positive definite, Eq. (1) guarantees the positive density operator which is essential for the density operator to be the physical operator.

Benatti *et al.* studied a model having the coefficients of the dissipation term given by

*lhjae@kookmin.ac.kr

$$D = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix}, \qquad (3)$$

which is defined by

$$A = C = \begin{pmatrix} 1 & -ia & 0 \\ ia & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where a and b are real constants. This model shows noticeable properties in the entanglement dynamics. The physical condition of the positivity of D describes the condition $a^2 + b^2 \leqslant 1$ which is inside the unit disk in Fig. 1. Benatti *et al.* showed that entanglement is created in the four portions between the outside of the embedded square $|a \pm b| = 1$ and the disk boundary. These regions are between the black and red lines in Fig. 1.

To investigate the entanglement dynamics of the model, let us solve the master equation straightforwardly. For an initial reduced state of the form $\rho(0) = |+\rangle\langle+| \otimes |+\rangle\langle+|$, the density matrix is easily obtained, but the expression is too long. So we write only the asymptotic expressions of the solutions as

$$\lim_{t\to\infty} \rho_w^{11}(t) = \frac{1 + a^2 - b^2 + 2a - 2ab^2}{4(1 - b^2)},$$

$$\lim_{t\to\infty} \rho_w^{22}(t) = \frac{1 - a^2 - b^2}{4(1 - b^2)},$$

$$\lim_{t\to\infty} \rho_w^{44}(t) = \frac{1 + a^2 - b^2 - 2a + 2ab^2}{4(1 - b^2)},$$

$$\lim_{t\to\infty} \rho_w^{14}(t) = -\frac{a^2 b}{2(1 - b^2)} = \lim_{t\to\infty} \rho_w^{41}(t).$$

The other components of the density matrix become zero. Since the reduced system is bipartite and a 2×2 system, we can check entanglement by using the concurrence. The concurrence of the above asymptotic state is $\lim_{t\to\infty} C(\rho_w(t)) = \lim_{t\to\infty}\{2[\rho_w^{14}(t) - \rho_w^{22}(t)]\} = -(2a^2 b + 1 - a^2 - b^2)/2(1 - b^2)$. The positivity of the concurrence explains why the system of two qubits has entanglement at asymptotic time. The persistence of entanglement on the asymptotic time is shown in the four regions between the green and black lines in Fig. 1. The system in these regions generates entanglement at the asymptotic time. The region of entanglement at the asymptotic time is smaller than that of Benatti *et al.* The parameters between the red and green lines satisfy the criterion of Benatti *et al.* only, but not the entanglement condition of the concurrence at the asymptotic time. The solution shows that entanglement between two qubits whose parameters are in this region is created for a short time and disappears with the lapse of time. That is, the entanglement generation in this region is transient, and the state of the system goes to a separable mixed state at the asymptotic limit. For the region between the green and black lines, the concurrence increases monotonically and the maximum value can be 1 for special parameter values.

This model has the further interesting behavior that any initial state approaches the same asymptotic state. This fact has no relevance to whether the initial state is pure or entangled or mixed. As the time gets large, all the states lead to

$$\rho_{asymp} = \begin{pmatrix} -\frac{1 + 2a + a^2 - b^2 - 2ab^2}{4(-1 + b^2)} & 0 & 0 & \frac{ba^2}{2(-1 + b^2)} \\ 0 & \frac{-1 + a^2 + b^2}{4(-1 + b^2)} & 0 & 0 \\ 0 & 0 & \frac{-1 + a^2 + b^2}{4(-1 + b^2)} & 0 \\ \frac{ba^2}{2(-1 + b^2)} & 0 & 0 & -\frac{1 + 2a + a^2 - b^2 - 2ab^2}{4(-1 + b^2)} \end{pmatrix}. \qquad (4)$$

This result comes from the fact that the density matrices in arbitrary time are always of Werner type. Since Markovian dynamics has no history dependence, these Werner-type states go to one asymptotic state of Eq. (4).

Although Benatti *et al.*'s criterion to create entanglement give useful information about the interaction parameters, it is not good enough to use it as the physical criterion of entanglement for a practical system. This is because, for some values of the interaction parameters, the system shows only transient entanglement. It is better to define the physical criterion using the asymptotic behavior of the reduced system.

At this stage, we will ask what kinds of physical baths cause creation and persistence of entanglement, and also

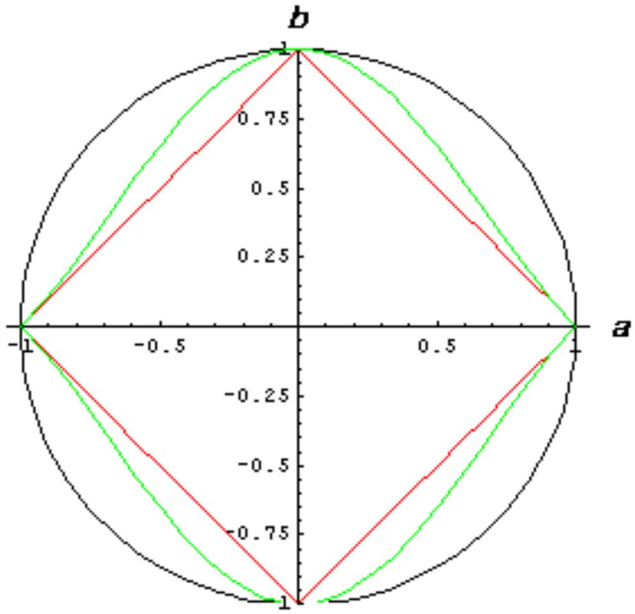

FIG. 1. (Color online) The black line (outer circle) represents the boundary of the physical region. The red line (inner square) represents the boundary of the region of entanglement obtained by Benatti *et al.* The green line (surrounding red square) represents the boundary of the parameters that give asymptotic entanglement.

study what is the source of entanglement creation. Which one of the unitary or the dissipation term, or both, gives entanglement creation? Another interesting question is whether it is universal that the entanglement at asymptotic time is independent of the initial states.

For these questions, let us consider the system interacting with a thermal bath, which consists of harmonic oscillators. The Hamiltonian of this model is given by

$$\mathcal{H} = \mathcal{H}_S + \mathcal{H}_R + \mathcal{H}_{SR}, \tag{5}$$

where

$$\mathcal{H}_S = \frac{\hbar}{2}\epsilon\sigma_1^z + \frac{\hbar}{2}\epsilon\sigma_2^z,$$

$$\mathcal{H}_R = \sum_j \hbar\omega_j r_j^\dagger r_j,$$

$$\mathcal{H}_{SR} = \sum_j \hbar(g_j^* r_j^\dagger \sigma_1^- + g_j r_j \sigma_1^+ + g_j^* r_j^\dagger \sigma_2^- + g_j r_j \sigma_2^+).$$

Here, $\sigma_1^i = \sigma^i \otimes \mathbb{1}$ and $\sigma_2^i = \mathbb{1} \otimes \sigma^i$. The interaction between the system and the bath is governed by the relaxation interaction.

The dynamics of the system is determined by the following master equation in the Schrödinger representation:

$$\begin{aligned}\frac{d\rho_s^1}{dt} = &-\frac{i}{2}(\epsilon\sigma_1^z + \epsilon\sigma_2^z, \rho_s^1) - i\Delta'(\sigma_1^z + \sigma_2^z, \rho_s^1)\\ &- i\Delta\left(\frac{1}{2}(\sigma_1^z + \sigma_2^z) + (\sigma_1^+\sigma_2^- + \sigma_2^+\sigma_1^-), \rho_s^1\right)\\ &+ \sum_{\alpha,\beta=1}^{6} D_{\alpha\beta}\left(\sigma_\alpha\rho_s\sigma_\beta - \frac{1}{2}\{\sigma_\beta\sigma_\alpha, \rho_s^1\}\right),\end{aligned} \tag{6}$$

where $\sigma_\alpha = \sigma_\alpha^1$ for $\alpha = 1,2,3$ and $\sigma_\alpha = \sigma_{\alpha-3}^2$ for $\alpha = 4,5,6$. Δ and Δ' are defined in the continuum limit as

$$\Delta' \equiv \mathrm{P}\int_0^\infty d\omega \frac{\mathcal{D}(\omega)|g(\omega)|^2 \bar{n}(\omega,T)}{\omega_0 - \omega}, \tag{7}$$

$$\Delta \equiv \mathrm{P}\int_0^\infty d\omega \frac{\mathcal{D}(\omega)|g(\omega)|^2}{\omega_0 - \omega}, \tag{8}$$

where P means the principal value. $\mathcal{D}(\omega)$ is the density of states and $g(\omega)$ the frequency-dependent interaction strength for the thermal bath in the continuum limit. $\bar{n}(\omega_j)$ is the average occupation number of each mode ω_j for the thermal bath, i.e., $\bar{n}(\omega_i) = \mathrm{Tr}_R(\rho_{bath} r_j^\dagger r_j) = \Sigma_{n_j}\{n_j \exp(-\hbar\omega_j n_j/k_B T)/[1-\exp(-\hbar\omega_j/k_B T)]\}$. ρ_{bath} is the density matrix of the thermal bath and T is the temperature. D is represented by the 6×6 matrix $\left(\begin{smallmatrix} A & B \\ B^\dagger & C\end{smallmatrix}\right)$, where

$$A = B = C = \begin{pmatrix} \frac{\gamma}{4} + \frac{\gamma}{2}\bar{n} & i\frac{\gamma}{4} & 0 \\ -i\frac{\gamma}{4} & \frac{\gamma}{4} + \frac{\gamma}{2}\bar{n} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

γ is defined by $2\pi\mathcal{D}(\epsilon)g(\epsilon)$.

The Δ'- and Δ-proportional terms represent the Hamiltonian induced by the thermal bath. Both terms renormalize the system Hamiltonian, but the Δ-proportional term has an additional term corresponding to the indirect interaction between two qubits. This induced interaction is a planar-type interaction between two system qubits.

In order to investigate the effect of the unitary planar-type interaction and the dissipative dynamics on entanglement of two system qubits clearly, we consider the dissipation term and the unitary term separately. To study the pure dissipative dynamics first we set the unitary term to zero. Under this situation, the master equation gives the asymptotic solution for the initial Werner state as

$$\rho^1_{asymp} = \begin{pmatrix} \frac{3a^2+6ab+3b^2+a^2r+2abr+b^2r}{4(a^2+3b^2)} & 0 & 0 & 0 \\ 0 & -\frac{a^2-3b^2+a^2r+b^2r}{4(a^2+3b^2)} & \frac{-2a^2+2b^2r}{4(a^2+3b^2)} & 0 \\ 0 & \frac{-2a^2+2b^2r}{4(a^2+3b^2)} & -\frac{a^2-3b^2+a^2r+b^2r}{4(a^2+3b^2)} & 0 \\ 0 & 0 & 0 & \frac{3a^2-6ab+3b^2+a^2r-2abr+b^2r}{4(a^2+3b^2)} \end{pmatrix} \quad (10)$$

where $a=\gamma/4$ and $b=\gamma/4+\gamma/2\bar{n}$. We note that the general solution is easily calculated for the systems we deal with. However, the solution is too lengthy, so we will discuss the behavior of the solutions at the interesting time, i.e., the asymptotic time. The initial Werner state is defined by $\rho_W = r|\Phi^+\rangle\langle\Phi^+| + [(1-r)/4]I_4$, where $|\Phi^+\rangle$ is the Bell state; $(1/\sqrt{2})(1,0,0,1)^T$ in the representation using the eigenstates of $\sigma_z \otimes \sigma_z$.

The concurrence of the density matrix at the asymptotic time is calculated as

$$C(\rho^1_{asymp}) = \max\left(\frac{1-r}{4} - \frac{3(b^2-a^2)(3+r)}{4(a^2+3b^2)}, 0\right). \quad (11)$$

This value can be greater than 0. Hence the dissipation term can generate entanglement between two qubits. We note that for high temperature, $b \gg a$, the above concurrence at the asymptotic time becomes $-(1+r)/2$. Since this is always negative, there is no entanglement induced by the high-temperature bath. For the zero-temperature bath, $a=b$, the concurrence at asymptotic time becomes $(1-r)/4$. This is always positive except for $r=1$. Then the zero-temperature bath can always generate entanglement between two qubits coupled with it. Furthermore, this result suggests that the initial Bell states $(r=1)$ become separable, and the fully separable initial states $(r=0)$ become most entangled at the asymptotic time. This result is unexpected. This also means that entanglement creation is very sensitive to the initial state, unlike the model of Eq. (3).

Next, we consider the effect of the unitary term by turning off the dissipative dynamics for the same initial state. The general solution for this case has a complicated form, but the asymptotic solution is the same as for the initial state, except for the oscillatory term $(r/2)\exp[-i(4c+2d)t]$ of ρ_{14}, where $c=\Delta'+\epsilon/2$ and $d=\Delta$. However, the oscillatory behavior disappears in the calculation of the concurrence. The concurrence at any time does not change from the initial value. This means that the unitary term, which has a planar-type interaction, cannot change the entanglement for the initial Werner states. Since it is well known that the planar-type interaction can entangle two qubits in general, this result shows that the entanglement generation at asymptotic time is also sensitive to the initial states.

Finally, let us consider the model with both the unitary and the dissipation terms. We can easily show that for this case the concurrence at asymptotic time is the same as that in the dissipation-only case. Hence, for this kind of thermal bath, interaction, and initial states, it shows that the entanglement is generated only by the dissipation.

Now we consider the system and the bath with a different interaction, called dephasing, as

$$\mathcal{H}_{SR} = \sum_j \hbar(g_j^* r_j^\dagger \sigma_z^1 + g_j r_j \sigma_z^1 + g_j^* r_j^\dagger \sigma_z^2 + g_j r_j \sigma_z^2). \quad (12)$$

This bath also induces unitary and dissipation terms in the system. The unitary term induced by Eq. (12) forms an Ising-type interaction between two qubits. Entanglement of initial Werner states is not created by the induced Ising-type interactions. This model has a purely dephasing interaction, and the asymptotic state becomes a separable mixed state under both the unitary and dissipation terms.

In a thermal bath, we have observed special features in the Werner state. When planar-type interactions for two qubits are induced by the thermal bath, the dissipation term instead of the unitary term gives the entanglement, contrary to our general belief. The dissipation term determines the entanglement generation in the asymptotic region despite the fact that the unitary term could not be turned off. This situation arises from the physical models that have a z-representation Werner state prepared initially, and a relaxation interaction with a thermal bath. Furthermore, entanglement survives to the asymptotic time. Hence, one can measure the entanglement induced solely by the dissipation term without turning off the unitary term for the initial Werner state at asymptotic time. This means that the dissipation can be a useful tool in quantum computers and quantum-information processing. It is also shown that there is no entanglement creation for high enough temperature. Another interesting feature is that entanglement is highly dependent on the initial states. The dephasing interaction between the system and the bath induces an Ising-type interactions between two qubits. The induced Ising-type interactions can create entanglement in general. However, it cannot create entanglement for some initial Werner states.

Until now, we have studied the relation of the entanglement creation and the physical situations of the system and the bath. However, we do not know the reasons why the system has entanglement through the dissipative interaction with the thermal bath. An *et al.* [7] showed that the entanglement generation occurs due to the decoherence-free state included in the initial state. Their opinion seems to explain well the case of Eq. (6). We can show that the entanglement generation in the system of Eq. (6) grows from an initial state including a decoherence-free state. The model of Benatti *et al.* cannot be explained by this method, since the asymptotic state goes to the same density state independent of the initial states. We will study the origin of the entanglement generation in further work.

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