Robust controlled-NOT gate in the presence of large fabrication-induced variations of the exchange interaction strength

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We demonstrate how using two-qubit composite rotations a high fidelity controlled-NOT (CNOT) gate can be constructed, even when the strength of the interaction between qubits is not accurately known. We focus on the exchange interaction oscillation in silicon based solid-state architectures with a Heisenberg Hamiltonian. This method easily applies to a general two-qubit Hamiltonian. We show how the robust CNOT gate can achieve a very high fidelity when a single application of the composite rotations is combined with a modest level of Hamiltonian characterization. Operating the robust CNOT gate in a suitably characterized system means concatenation of the composite pulse is unnecessary, hence reducing operation time, and ensuring the gate operates below the threshold required for fault-tolerant quantum computation.

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I. INTRODUCTION

The ability to correct errors arising from the construction or operation of any quantum computing architecture is essential for a successful implementation. Without the ability to correct the random and/or systematic errors that arise throughout operation, the implementation of large scale quantum algorithms is hopelessly undermined. In a realistic device the threshold for fault-tolerant quantum computation is likely to be well below 10^{-4}, placing severe constraints on the tolerable magnitude of errors due to decoherence or lack of precision in quantum control. This work focuses on minimizing a particular type of *systematic* error, namely, uncertainty in the coupling strength of two-qubit devices as a result of imperfect fabrication, which causes systematic under- or over-rotations. We use recently developed two-qubit composite rotations to correct for this uncertainty in the strength of the electron spin exchange interaction in Si:P based architectures [1,2]. The exchange interaction is also common to other spin qubit systems [3–5]. Our results also apply more generally and could be used to correct this type of systematic error in a range of solid-state systems.

The strength of the exchange interaction coupling between donors in silicon based solid-state architectures is known to be highly sensitive to donor placement. The cause of this is the intervalley interference between the six degenerate conduction band minima of silicon, resulting in oscillations of the exchange coupling strength [6–9]. Exact positioning of donors to better than 2 to 3 sites is difficult [10] and therefore we expect significant uncertainty in the unbiased strength of the coupling between donors. The uncertainty in our knowledge of the coupling leads to error in gate operation. Systematic errors of this kind are correctable using composite rotations. Experimental applications already exist in a variety of quantum systems demonstrating the usefulness of composite rotations for ensuring robust operations [11–16]. Recently two-qubit composite rotations have been considered for systems with uncertainty in their coupling strength [17,18].

In this paper, we follow the method for creating a robust controlled-NOT (CNOT) gate developed in Ref. [18] and quantitatively study the performance of the robust CNOT gate using simulated exchange oscillation data. We specifically consider the global Si:P electron spin control case where the interaction is of Heisenberg type and gate times are in the O (10–100 ns) regime. This technique is readily generalizable to any two-qubit Hamiltonian, and for a full treatment, the reader is directed to Ref. [18].

Misplacement of donors by only one implantation site can lead to large variations in the exchange coupling strength, even in Si:P systems with voltage bias applied to top gates [8], meaning a single application of the composite rotations may not be enough to guarantee a high fidelity CNOT gate. Concatenating the pulse by feeding it back into itself can help to achieve correction to a higher level, however, performing multiple concatenations costs a large increase in time. In certain cases using composite rotations alone will not improve the fidelity of the operation above an uncorrected CNOT gate, as the composite rotations are designed to work within a specific uncertainty range. We show that in unison with Hamiltonian characterization [19,20], the process of experimentally determining a Hamiltonian, a single application of the composite rotations guarantees a high fidelity CNOT operation with an error rate below the fault-tolerant error threshold. Operating the CNOT gate this way helps remove the need for concatenation and strikes a balance between fully characterizing the system and using composite rotations to construct robust operations.

II. CONSTRUCTING ROBUST GATES USING COMPOSITE ROTATIONS

Composite rotations have been widely used in NMR experiments to correct for pulse length errors and off-resonance effects [21,22]. In the case of pulse length errors, a deviation of the field strength from its nominal value leads to systematic under- or over-rotations. Although originally designed

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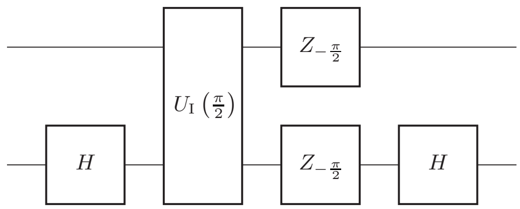

FIG. 1. Circuit diagram for a CNOT gate constructed from an Ising interaction, where H is a Hadamard gate, $Z_{-\pi/2}=\exp\left(i\frac{\pi}{4}\sigma_Z\right)$ and $U_{\mathrm{I}}\left(\frac{\pi}{2}\right)=\exp\left(-i\frac{\pi}{4}\sigma_Z\otimes\sigma_Z\right)$.

for applications involving single spin quantum systems, composite rotations may be extended to two-spin operations. In the context of quantum computation, only a certain class of composite rotations, sometimes referred to as fully compensating pulses, are applicable, as they work on any initial state. Using these fully compensating pulses, the application of composite rotations for constructing robust two-qubit gates against pulse length error has already been found for an Ising Hamiltonian [17], and a general two-qubit Hamiltonian [18].

In Ref. [18], it was noted that for a general two-qubit Hamiltonian expanded in the Pauli basis,

$$H = \sum_{i,j=\{I,X,Y,Z\}} J_{ij}\sigma_i \otimes \sigma_j, \tag{1}$$

any interaction term can be effectively extracted using a technique called *term isolation* [23]. The isolation of a given term will in general not be exact but can be made arbitrarily accurate. This result is particularly useful and can be used to isolate the Ising coupling term, J_{ZZ}, such that we can construct a CNOT gate from this interaction as in Fig. 1. In the case of the Heisenberg interaction with isotropic couplings,

$$H_{\mathrm{H}} = J(\sigma_X \otimes \sigma_X + \sigma_Y \otimes \sigma_Y + \sigma_Z \otimes \sigma_Z), \tag{2}$$

the isolation of the J_{ZZ} term is exact,

$$\begin{aligned}\exp(-iJ_{ZZ}t\sigma_Z \otimes \sigma_Z) = &-(Z_\pi \otimes I)\exp(-iH_{\mathrm{H}}t)\\ &\times (Z_\pi \otimes I)\exp(-iH_{\mathrm{H}}t),\end{aligned} \tag{3}$$

where for single qubit gates Z_a is a rotation about the σ_Z axis by an angle a, and similarly for other operators, $J_{ZZ}=2J$, and the global phase factor is included.

We now consider constructing a robust CNOT gate using composite rotations, whereby we replace the interaction term with one created using composite rotations. Doing this compensates for any uncertainty in our knowledge of the exchange interaction coupling strength, J. In Fig. 2 the entire process of constructing a robust CNOT gate from composite rotations is demonstrated schematically.

In an ideal system with a perfectly characterized coupling strength, the evolution operator generated by the Ising interaction is

$$\theta_0 \equiv U_{\mathrm{I}}(\theta) = \exp\left(-i\frac{\theta}{2}\sigma_Z \otimes \sigma_Z\right). \tag{4}$$

Here, θ_0 is a two-qubit rotation by an angle θ about the $\sigma_Z \otimes \sigma_Z$ axis. In general, θ_a is a two-qubit rotation by an angle θ around an axis tilted from the $\sigma_Z\otimes\sigma_Z$ axis towards the $\sigma_Z \otimes \sigma_X$ axis by an angle a,

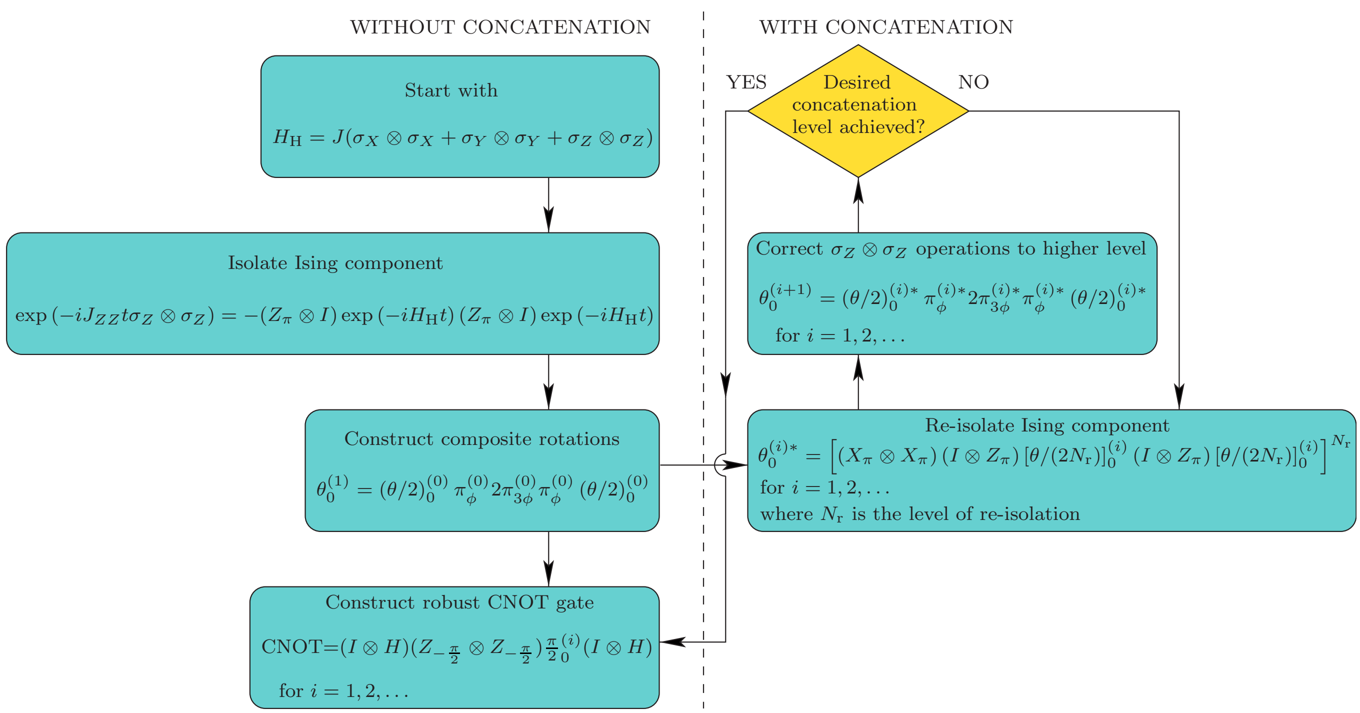

FIG. 2. (Color online) Procedural flowchart for constructing a robust CNOT gate using composite rotations, and concatenating to higher implementation levels.

$$\theta_a = \exp\left[-i\frac{\theta}{2}(\sigma_Z \otimes \sigma_Z \cos a + \sigma_Z \otimes \sigma_X \sin a)\right]. \quad (5)$$

This two-qubit rotation is achievable via

$$\theta_a = (I \otimes Y_a)\theta_0(I \otimes Y_{-a}). \quad (6)$$

We make the assumption that all single qubit unitaries are error free, but note that single qubit operations may also be made robust using existing techniques developed in the context of NMR.

In reality a fractional error, Δ, in the two-qubit operation will be present due to the uncertainty in our knowledge of the actual coupling strength, J_{ZZ},

$$\Delta = \frac{J_{ZZ}}{J_P} - 1. \quad (7)$$

Here, J_P is our prediction of the Ising coupling strength based on the targeted donor separation. Therefore the actual rotation performed will be

$$\theta_0^{(0)} \equiv U(\theta) = \exp\left[-i\frac{\theta}{2}(1+\Delta)\sigma_Z \otimes \sigma_Z\right]. \quad (8)$$

The superscript of $\theta_a^{(b)}$ in the above equation indicates the *implementation level* of the actual (nonideal) rotation, with "(0)" being an uncorrected implementation and higher levels signifying subsequent corrections from composite rotations. The *implementation level* should not be confused with *concatenation level*, (e.g., the second implementation level is the first concatenation level).

It has been previously noted that single qubit composite rotations can be extended to two-qubit composite rotations for use in quantum computation [17,18] using fully compensating pulses. A class of these composite rotations known as BB1 [22,24] is particularly useful for applications involving quantum computation [25]. Replacing the pulse $\theta_0^{(0)}$ with the symmetrized BB1 class composite pulse

$$\theta_0^{(1)} = (\theta/2)_0^{(0)}\pi_\phi^{(0)}2\pi_{3\phi}^{(0)}\pi_\phi^{(0)}(\theta/2)_0^{(0)}, \quad (9)$$

where ϕ=arcos$(-\theta/4\pi)$, will result in a higher fidelity operation. The fidelity of an operation is defined as

$$\mathcal{F} = \frac{|\mathrm{Tr}[U^\dagger(\theta)U_{\mathrm{I}}(\theta)]|}{\mathrm{Tr}[U_{\mathrm{I}}^\dagger(\theta)U_{\mathrm{I}}(\theta)]}. \quad (10)$$

We may reisolate the Ising component J_{ZZ} again to arbitrary accuracy as in Fig. 2. The reisolated Ising component can then be used to correct to even higher order by passing this pulse back into each of the constituents of Eq. (9) (see Fig. 2). In principle there is no limit to how often this concatenation can be done, however, the increase in gate time means that in practice this process will be limited by the decoherence time of the system in which the CNOT gate is being implemented. In Fig. 3 the performance of the uncorrected CNOT gate is compared to the robust gate for various implementation levels, as originally calculated in Ref. [18]. Notice each subsequent implementation level performs better over a larger range of the fractional error, Δ.

We now apply the robust CNOT gate to the Si:P architecture with large fabrication induced variations (and hence uncertainty) in the exchange interaction strength.

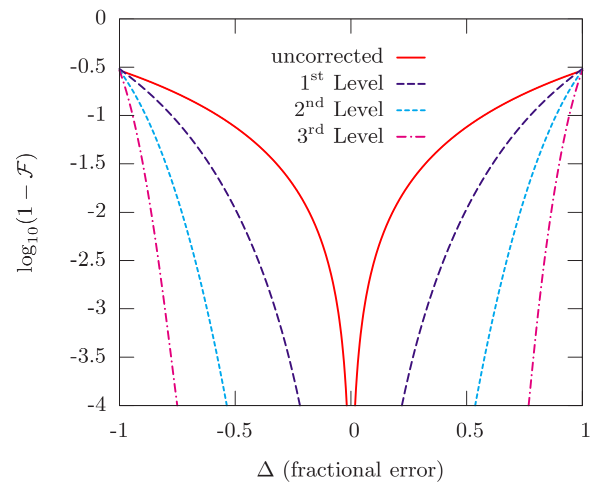

FIG. 3. (Color online) CNOT error $(1-\mathcal{F})$ as a function of the fractional error in our knowledge of the coupling strength, Δ, for various implementation levels. These composite rotations provide improvement over an uncorrected implementation for $\Delta \in (-1,1)$. The fidelity of a CNOT gate constructed from the Heisenberg interaction using composite rotations was originally calculated in Ref. [18].

III. CORRECTING FOR UNKNOWN EXCHANGE INTERACTION STRENGTH

Systematic errors arising from imperfections in the fabrication process are correctable. In Kane type architectures [1,2] where phosphorus donors are implanted into an isotopically pure ^{28}Si matrix, two fabrication processes are being pursued concurrently [26]. The *top down* approach uses ion beam implantation of phosphorus ions incident on the silicon substrate. Precise placement of phosphorus donors is limited in this approach due to scattering off the silicon atoms, in a process known as *straggling*. State-of-the-art top down fabrication results in placement uncertainties of O (10 nm) [27]. The *bottom up* approach offers atomically precise fabrication using a phosphine gas. The gas is applied to a hydrogen terminated silicon substrate, where scanning tunneling microscopy has removed individual hydrogen atoms from the hydrogen monolayer at the desired implantation sites. Once the phosphorus is integrated into the substrate, the monolayer is removed and overgrown with silicon. Small deviations from target implantation of O (1 nm) (approximately 2 to 3 sites) can still occur during the annealing process [10].

The exchange coupling J of the Heisenberg Hamiltonian [see Eq. (2)] is highly sensitive to donor electron wave function overlap. This means that even small deviations from the targeted implantation sites can lead to large variations in the exchange coupling between donors [6,7]. Calculated variations in the strength of J for small deviations from the targeted donor separation in an unbiased, $J(V=0)$, system are shown in Fig. 4. This calculation was performed using the Heitler-London formalism, where the wave functions for the phosphorus donors in silicon were expressed in Kohn-Luttinger effective mass form, with Bloch states explicitly

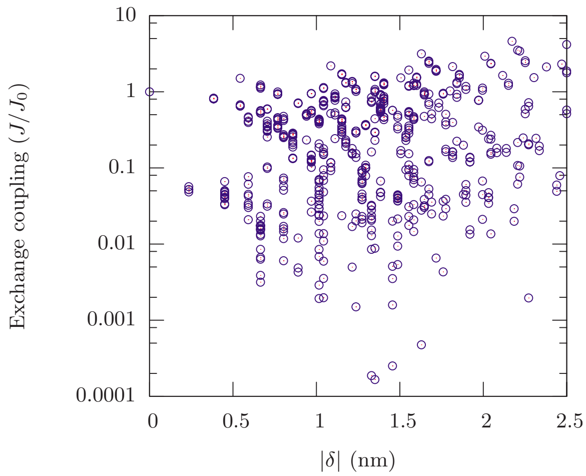

FIG. 4. (Color online) Exchange couplings in an unbiased, $J(V=0)$, system for donors at fcc lattice sites misplaced by a distance δ in all directions from the target separation of 20.6 nm (in the [100] direction). The exchange coupling strengths are given as a fraction of the target coupling strength, $J_0 \approx 0.1$ μeV.

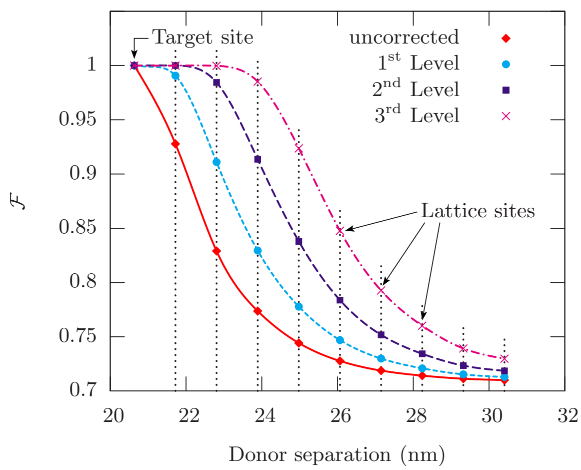

FIG. 5. (Color online) CNOT fidelity as a function of donor separation in the [100] direction for various implementation levels. The resulting fidelities are determined based on a target donor separation of 20.6 nm. Note that interpolating curves between lattice sites indicate donor separation scenarios for a given implementation, and vertical dotted lines guide the eye between implementations.

computed using the pseudopotential fit to the band structure. Details can be found in Ref. [7]. These Kohn-Luttinger calculations are representative only. It is now known through a more sophisticated band minima basis analysis including core correction and strain that the extreme oscillations arising in the Kohn-Luttinger treatment are strongly tempered [28]. Small values of the coupling are also enhanced, with further improvements expected in the biased, $J(V\neq 0)$, system [7,8]. Importantly, this type of systematic error is correctable using the composite rotations described above. The case $J=0$ can never be corrected, however, as Fig. 4 shows there are no instances of J exactly zero.

In an uncharacterized system we assume that the exchange interaction strength is J_0 and will be determined by the target donor separation and bias on the control gates. Fabrication-induced donor misplacement will cause the true exchange interaction strength, J, to be quite different from J_0. The fractional error in our knowledge of the coupling strength is

$$\Delta_0 = \frac{J}{J_0} - 1. \tag{11}$$

These composite rotations will only provide an improvement over an uncorrected implementation for $|\Delta_0| < 1$. For $|\Delta_0| > 1$ these composite rotations are actually outperformed by the uncorrected implementation, so if $J > 2J_0$ then the composite rotations provide a less robust operation. We address this point in Sec. IV, where we show how one may always remain within the correctable range of the composite rotations using a systematic two-qubit interaction characterization procedure.

Implementing the gate based on the target coupling strength J_0, the fidelity of the resulting CNOT operation will be determined by the size of the fractional error Δ_0 in the actual coupling strength. As an example in Fig. 5 we demonstrate the resulting CNOT fidelity for a number of donor separations in the [100] direction when the target separation is 20.6 nm. The results show that using composite rotations improves the fidelity of operation for the CNOT gate. For example, if the actual separation is 21.7 nm, one application of the composite pulsing scheme improves the fidelity from ~0.93 to ~0.99, while a second application brings the fidelity above 0.9999. The successive improvements due to the various levels of pulse concatenation do, however, come at the expense of operation time. We examine this issue in the following sections.

A. Gate count

The robust CNOT gate outperforms the uncorrected CNOT gate given an error in the targeted coupling strength, J_0, for $|\Delta_0| < 1$. Each level of concatenation provides further improvement, however, the cost of this improvement is an exponential increase in the total number of gates required. An unavoidable consequence of this is an increase in the time required to perform these robust operations. To be of use for quantum computation we need to be able to perform many precise operations within the decoherence time of the system. We show how to minimize the time taken to perform a robust CNOT gate using Hamiltonian characterization in Sec. IV. Below, we consider the actual time costs of concatenated composite pulse correction.

An uncorrected CNOT gate requires only six single qubit gates and two two-qubit gates. In comparison, a raw gate count for the number of single qubit gates required in constructing the robust Ising interaction for the CNOT gate yields

$$n_1 = 16,$$

$$n_i = 10N_r(n_{i-1} + 2) + 6, \quad i = 2,3, \ldots, \tag{12}$$

where n_i is the number of single qubit gates required for the ith implementation level, and N_r, which we assume to be

TABLE I. CNOT gate times for various pulse implementation levels in the electron spin solid-state quantum computing architecture.

Implementation level	Gate times (ns)		
	Single qubit	Two-qubit	Total
0	180	4	184
1	716	35	751
2	53257	2544	55801

constant, quantifies how much we reisolate the Ising term for pulse concatenation. Constructing a robust CNOT gate requires an additional four single qubit gates, such that the total number of single qubit gates required, $n_i^{1\mathrm{q}}$, is

$$n_i^{1\mathrm{q}} = n_i + 4, \quad i = 1,2, \ldots . \tag{13}$$

The total number of two-qubit gates needed in the robust CNOT construction is

$$n_i^{2\mathrm{q}} = 10^i N_{\mathrm{r}}^{i-1}, \quad i = 1,2, \ldots , \tag{14}$$

again assuming the same N_{r} for each level of concatenation. We may be able to reduce the total number of single qubit operations by compounding gates, however, this is not possible for the two-qubit operations. The viability of using multiple concatenation for constructing robust two-qubit gates lies in tenuous balance between the ability to perform the large number of operations required quickly, and adequate pulse timing control over the small two-qubit rotations which arise from the reisolating of the Ising component. The strength of the exchange coupling of our system will determine whether these conditions can be satisfied.

B. Gate time

Each level of concatenation increases the time taken for the robust CNOT operation significantly. In a working quantum computer this may be problematic as the decoherence time of the system sets an upper limit on how long operations may take. For phosphorus donors in Si the coherence time, T_2, of donor electron spins has been measured to be $T_2 > 60$ ms at 7 K [29]. We calculate the total time taken for the robust CNOT gate for various implementation levels based on gate times using global control methods [30]. The results for this appear in Table I. As in Ref. [18], we assume that single qubit rotations by an angle π take 40 ns to perform as does the Hadamard gate. We also assume that two-qubit rotations by $\pi/4$ take approximately 2 ns if the coupling strength is given by $J_0 \approx 0.1$ μeV, taken from the calculated unbiased exchange data [7]. Actual time will decrease under the application of a J-gate bias [7,8], however, we assume a worst case scenario here.

As Table I demonstrates, operation time grows appreciably with concatenation. Furthermore, Fig 5 shows that the success of the robust CNOT gate is dependent on how accurately we can estimate the exchange coupling strength based on expectations of the fabrication process alone. In such an uncharacterized system we have shown that a sensible choice can be made based upon the target separation, yielding J_0. Large variations in the exchange interaction strength due to donor misplacement, and the additional time cost for multiple concatenation, means composite rotations alone cannot always guarantee a feasible, robust CNOT gate. However, we will now show that composite pulses at the lowest level coupled with a systematic two-qubit interaction characterization procedure allows for precise CNOT gate construction.

IV. ROLE OF TWO-QUBIT HAMILTONIAN CHARACTERIZATION

Using a combination of system identification and composite rotations, we may always construct a high fidelity robust CNOT gate. While many methods of system identification exist, we choose the procedure of Hamiltonian characterization because it provides direct knowledge of the Hamiltonian (which we require) in an efficient manner. This approach strikes a balance between the need for multiple concatenation and precision Hamiltonian characterization, and may be particularly useful for systems whose Hamiltonian parameters require recharacterization over time due to drift.

Recent work shows how characterization of a two-qubit Hamiltonian can be achieved via entanglement mapping of the squared concurrence relation [19,20]. The identification of the Hamiltonian coefficients amounts to determining the oscillation frequency of this entanglement function for different input states. The only requirements are an accurately characterized Hadamard gate and measurement on both qubits. An important result from the work in Ref. [19] is the fractional uncertainty in a frequency determination

$$\frac{\delta f}{f} \geq \frac{4}{N_t \sqrt{N_e}}, \tag{15}$$

where N_t is the number of discrete time points at which N_e projective measurements are made. An equivalent result can also be found in the earlier work of Huelga *et al.* in the context of Ramsey spectroscopy [31]. To accurately determine the frequency, the time over which the system is observed, t_{ob}, should be maximized, however, this process is limited by the decoherence time of the system. An accurate frequency determination is still possible in the presence of decoherence by allowing t_{ob} to be relatively large and performing two measurements at N_t time points. The uncertainty in the frequency can then be reduced by evolving the system for a suitably long time before measuring at two final time points. This process is repeated N_e times to estimate the phase of the oscillation. The total number of required measurements is then $N=2(N_t+N_e)$. Characterizing the system in this way results in the scaling of Eq. (15).

To characterize the Heisenberg Hamiltonian with isotropic couplings requires determining the oscillation frequency of three different input states, meaning $N=6(N_t+N_e)$ total measurements are needed. The fractional uncertainty in the characterized exchange coupling, J_c, as a function of N for a given N_t is

$$\frac{\delta J_c}{J_c} \equiv \frac{\delta f}{f} \geq \frac{4\sqrt{6}}{N_t\sqrt{N-6N_t}}. \quad (16)$$

To illustrate the effect of composite rotations we consider a modest amount of characterization by choosing $N_t=10$. Increasing the number of time points results in higher precision characterization.

In an uncharacterized system we assumed the coupling between donors, J_0, to be determined by the target donor separation. Donor misplacement as a result of fabrication uncertainties leads to variations in the coupling strength, J, from the target J_0. We have seen how the robust CNOT gate for an uncharacterized system performs in Fig. 5. We now consider the performance of a robust CNOT gate in a characterized system.

Characterization of the Hamiltonian can be performed to any level of precision at the expense of extra measurements, with the uncertainty given by Eq. (16). In a characterized system, the estimated coupling strength is set to the characterized coupling strength, J_c (with uncertainty bounds $\pm\delta J_c$), rather than J_0. The fractional error in this case is

$$\Delta_c = \frac{J}{J_c} - 1, \quad (17)$$

where in general the characterized coupling strength J_c will be much closer to the true value of J than the target value J_0 is to J. This guarantees that we remain well within the correctable bounds of composite rotations, hence ensuring a high fidelity operation with fewer levels of concatenation, independent of donor misplacement direction.

Given that the total gate time increases so sharply with increased concatenation, operating with a single application of the composite rotations is preferential. For a one site deviation from the target separation, we show the resulting CNOT fidelity as a function of pulse implementation in a system characterized to the 10% level ($\delta J_c/J_c=0.1$) in Fig. 6. Characterization to this level would require at least 156 measurements assuming the previous parameters. We take $J_c \approx 0.9J$ to be the characterized value of the exchange coupling strength, as it corresponds to an extremal bound value. The results in Fig. 6 demonstrate that it is possible to construct a very high fidelity CNOT gate using one level of robust pulsing, provided a suitable amount of characterization is first performed.

The total number of characterization measurements needed to achieve a given fidelity can also be determined as a function of the implementation level. These results appear in Fig. 7. In reality the fidelity may be substantially higher than the results of Fig. 7 indicate, as they provide a lower bound for the corresponding number of measurements. These results show the clear benefit in using a single level of composite rotations and characterization to construct a robust CNOT gate. The improvements expected beyond this do not seem to warrant concatenation.

Any quantum computation proposal requires that many operations be performed within the dephasing time, T_2, of the system. The 10^{-4} level is widely assumed to be the fault-tolerant threshold for both environmentally induced and systematic errors [32], however, more rigorous bounds [33] recently calculated, suggest it could be closer to 10^{-5}. Figure 7 shows that it is possible to construct a CNOT gate to this precision level in the presence of significant fabrication-induced uncertainties, using either multiple concatenation of the composite rotations or a combination of the composite

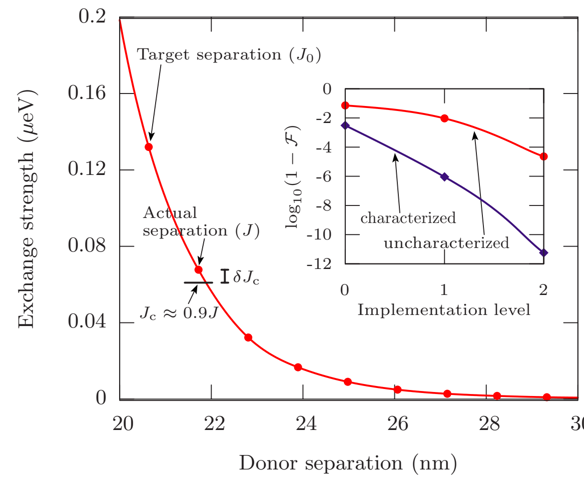

FIG. 6. (Color online) Exchange interaction strength as a function of donor separation along the [100] direction, showing a large variation in the coupling strength with donor misplacement (dots indicate actual site separations). For an uncharacterized system the coupling is set to the fabrication target J_0, with the actual placement giving coupling J. The resulting CNOT error, $(1-\mathcal{F})$, for a one site deviation ($\Delta_0 \approx -0.49$) from the target separation can be seen on the inset plot as a function of implementation level. In the characterized system the coupling is set to J_c. The CNOT error for a system characterized to the 10% level ($\delta J_c/J_c=0.1$), taking $J_c \approx 0.9J$ ($\Delta_c=0.1$), is shown as a function of implementation level in the inset also. Note that all curves are included purely to guide the eye.

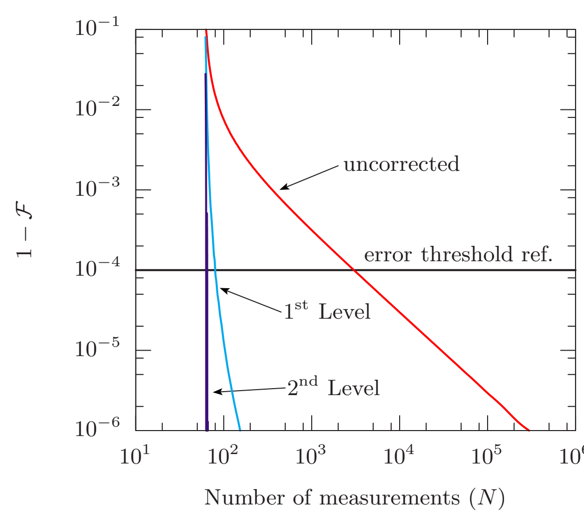

FIG. 7. (Color online) CNOT error $(1-\mathcal{F})$ as a function of the total number of characterization measurements required to achieve a given fidelity for various implementation levels. The results demonstrate the usefulness of combining composite rotations with Hamiltonian characterization when constructing a robust CNOT gate. The threshold reference line at 10^{-4} error rate is shown.

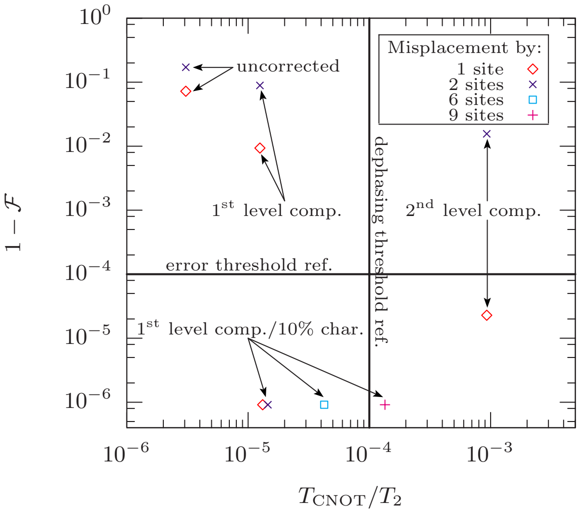

FIG. 8. (Color online) CNOT error $(1-\mathcal{F})$ as a function of the total gate time for an unbiased, $J(V=0)$, system (T_2=60 ms assumed). Results are shown for a range of separations in the [100] direction, larger than the targeted 20.6 nm separation. We consider various CNOT gate constructions, namely an uncorrected CNOT, one constructed from both a single and two applications of composite rotations, and finally a CNOT constructed using composite rotations in conjunction with characterization to the 10% level ($\delta J_c/J_c$ =0.1) taking $J_c \approx 0.9J$. Only for this final method has more than two sites been included, as for other methods results will clearly be worse. Threshold reference lines at 10^{-4} error rates are shown.

rotations and a modest level of characterization.

Assuming the system has been characterized to a modest level beforehand, we now show that in order to remain below the threshold for environmentally induced errors also, the robust CNOT should be constructed using a single application of composite rotations and characterization. In Fig. 8, these results are shown for a system with an unbiased J-gate, $J(V=0)$, based on the 60 ms dephasing time in isotopically pure ^{28}Si at 7 K [29], and for characterization to the 10% level, again assuming the extremal bound value of $J_c \approx 0.9J$. In a biased system, the exchange coupling is stronger. Calculations suggest that for donors separated by ~20 nm in the [100] direction, a 1 V bias applied to the control gates can strengthen the coupling by over two orders of magnitude [7,8]. A robust CNOT gate comprising characterization as described above could therefore operate at close to the 10^{-7} level for environmentally induced errors. Performing additional measurements to characterize the system to the 1% level would lower the systematic error level to well below 10^{-7} also, bringing it well within more rigorous threshold bounds [33].

For systems whose Hamiltonian parameters are not well-known due to fabrication uncertainties, or may drift over time, this is an important result, suggesting that operating the CNOT gate in this way can guarantee that the error rate remains below the fault-tolerant error threshold. For the case of Si:P quantum computer architectures Fig. 8 suggests that this may be fabrication uncertainties within up to six sites of the target site, or ~6.5 nm in the unbiased case, however, in the J-gate biased case this allowance may be much greater. The tradeoff for operating in this manner is the need for periodic recharacterization, however, the cost of this should be minimal as the number of required measurements is small.

V. CONCLUSIONS

The performance of a robust CNOT gate constructed using two-qubit composite rotations has been examined. Multiple concatenation of the composite rotations results in a high fidelity CNOT gate provided the fractional uncertainty in J lies within the correctable range. Large variations in the exchange interaction coupling with donor separation means this is not always the case. Furthermore, multiple concatenation of composite rotations requires long overall gate times with respect to the decoherence time of the system and results in gate operation which exceeds the current error threshold required for fault-tolerant quantum computation. As an effective fix to this problem, we demonstrated how, in a system with large variations in the qubit coupling strength, a high fidelity CNOT gate which operates below this error threshold can be constructed from a single level of composite rotations in conjunction with Hamiltonian characterization.

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