Dipole binding in a cosmic string background due to quantum anomalies

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We propose quantum dynamics for the dipole moving in cosmic string background and show that the classical scale symmetry of a particle moving in cosmic string background is still restored even in the presence of dipole moment of the particle. However, we show that the classical scale symmetry is broken due to inequivalent quantization of the nonrelativistic system. The consequence of this quantum anomaly is the formation of a bound state in the interval $\xi \in (-1,1)$. The inequivalent quantization is characterized by a one-parameter family of self-adjoint extension parameter Σ. We show that within the interval $\xi \in (-1,1)$, a cosmic string with zero radius can bind the dipole and the dipole does not fall into the singularity.

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Anomaly [1,2] is a breaking of classical symmetry due to quantization of the system, which occurs in various problems in physics. It is one of the three possible symmetry breakings [3], i.e., spontaneous, explicit, and anomalous symmetry breaking, which are extremely important due to their consequences in different physical processes. Chiral anomaly [4,5] is one such important example in high energy physics. In quantum field theory the concept of anomaly has been used successfully [1,6]. The other important area of physics is string theory [7], where anomaly has also been used successfully. In quantum mechanics, anomaly can be understood as follows. An operator, which is the generator of the symmetry in a classical system, becomes anomalous when it does not keep the domain of the Hamiltonian invariant. By this definition of anomaly, it has been shown that in molecular physics [8] (in quantum mechanical context), there exists an interesting scaling anomaly. For example, interaction of an electron in the field of a polar molecule is a simple example of an anomaly, where the classical scaling symmetry of the system is broken once it goes inequivalent quantization [9]. An obvious consequence of this scaling anomaly in molecular physics is the occurrence of bound state and the dependence of momentum in the phase shift of the S matrix.

In cosmic string scenario, scaling anomaly has been observed [11] for particles moving in it, where the induced potential is $1/r^2$ in nature. Inverse square potential appears in various situations in physics starting from molecular physics to black hole [8–10]. The anomaly, shown in Ref. [11], leads to a bound state for the particle. In this paper we consider quantum dynamics of a particle with dipole moment D moving in cosmic string background. This problem has been discussed [12] for a large negative coupling constant of the inverse square potential, where the particle falls into the center [13] due to the formation of an infinite number of bound states with ground state energy being negative infinite. However, fall to the center has been avoided [12] by considering a finite radius for the cosmic string. We will, however, consider a particular portion of the coupling constant of the inverse square potential which will allow us to obtain a nontrivial boundary condition. This nontrivial boundary condition will break the scaling symmetry by introducing a length scale in the form of a single bound state.

Quantum mechanics [14] in cosmic string background has received a lot of interest due to its analogy [15] with the Aharanov-Bohm effect [16]. In relativistic theory it has been shown [17] that a Dirac equation in cosmic string background needs a nontrivial boundary condition to be imposed on the spinor wave function at the origin. In language of mathematics the construction of a nontrivial boundary condition is usually called self-adjoint extensions [18]. The extensions can be characterized by independent parameters and different values of the parameters lead to inequivalent theories. It has been observed [19] that in cosmic string scenario the fermionic charge can be a nonintegral multiple of Higgs charge. Since the flux is quantized with respect to the Higgs charge it will lead to nontrivial Aharanov-Bohm scattering for fermion. The cross section increases due to this Aharanov-Bohm scattering in addition to gravitational scattering. In nonrelativistic theory [20], the consideration of inequivalent quantization is also inevitable in order to get a bound state for the particle moving in cosmic string background. In Ref. [21] gravitational scattering by particles of a spinning source in two dimensions has been studied. There it has been shown that the energy eigenvalue and corresponding eigenfunction of a particle in the field of a massless spinning source is equivalent to that in a background Aharanov-Bohm gauge field of an infinitely thin flux tube. This topological defect appears in astrophysics [22] and also in condensed matter physics [23].

This paper has been organized in the following way: first, we study the scaling symmetry of the classical system, which undergoes anomalous breaking upon quantization; second, we make an inequivalent quantization of the system, which is responsible for the anomaly and discuss its consequences.

First, scaling symmetry is associated with the transformation $\mathbf{r} \to \lambda \mathbf{r}$ and $t \to \lambda^2 t$, where λ is the scaling factor. In classical physics when the action is invariant under this transformation, then the corresponding system has scale symmetry. Since in nonrelativistic quantum theory, cosmic string induces a $V = \frac{(1-\alpha^2)D^2}{48\pi\alpha^2 r^2}\cos 2\Theta$ [24] potential to the particle with dipole moment D moving in its background, the relevant classical symmetry would be the scale symmetry. To be more specific classically, the dipole moving in cosmic

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string background can be described by the Lagrangian $L=\frac{M}{2}g_{ij}\dot{\mathbf{r}}^i\dot{\mathbf{r}}^j-V$. This Lagrangian L scales as $\frac{1}{\lambda^2}L$. So the action $\mathcal{A}=\int dtL$ will be scale invariant under the transformation $\mathbf{r}\rightarrow\lambda\mathbf{r}$ and $t\rightarrow\lambda^2 t$. The scale invariance of this action means if ψ is an eigenstate of the Hamiltonian $H=\frac{M}{2}g_{ij}\dot{\mathbf{r}}^i\dot{\mathbf{r}}^j+V$ with eigenvalue E, i.e., $H\psi=E\psi$, then $\psi_\lambda=\psi(\lambda\mathbf{r})$ will also be an eigenstate of the same Hamiltonian with energy E/λ^2. This essentially means that the system with scale symmetry does not have any lower bound in energy and therefore cannot have any bound state. Scale symmetry is, however, a part of a larger conformal symmetry formed by three generators: the Hamiltonian H, the Dilatation generator $\mathcal{D}=tH-\frac{1}{4}(\mathbf{r}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{r})$, and the conformal generator $K=Ht^2-\frac{1}{2}(\mathbf{r}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{r})+\frac{1}{2}M\mathbf{r}^2$. They form the SO(2,1) algebra: $[\mathcal{D},H]=-i\hbar H$, $[\mathcal{D},K]=i\hbar K$, and $[H,K]=2i\hbar\mathcal{D}$ [25]. We will show in our case that this scale symmetry will break once the classical system is quantized.

We consider a nonrelativistic particle of mass M, dipole moment D moving in the background field of a cosmic string. The background is described by the space-time metric in cylindrical coordinate (r,ϕ,z) as

$$ds^2=dt^2-dz^2-dr^2-\alpha^2r^2d\phi^2, \tag{1}$$

where $\alpha=1-4G\mu<1$ characterizes the string, with μ is the mass per unit length of the string. The constant α introduces an angular deficit of $2\pi(1-\alpha)$ in the Minkowski space time. The interaction between the dipole and the cosmic string background is described by the electromagnetic self-energy [24] of the dipole due to the nonflat geometry. The potential induced in the nonrelativistic system is $V=\frac{(1-\alpha^2)D^2}{48\pi\alpha^2r^2}\cos 2\Theta$ [24], where Θ is the angle between the string and the dipole moment. This potential transforms under the scale transformation $\mathbf{r}=\lambda\mathbf{r}$ and $t=\lambda^2t$ in such a way that the Schrödinger equation for the system becomes scale covariant. Due to cylindrical symmetry of the space, we can easily see that the motion of the particle in the z direction is basically a free particle motion, described by the wave function e^{ikz}, where k is a wave vector of the particle along the z direction. Since we are considering an infinite cosmic string along the z direction, it is enough to discuss the motion of the particle on the plane perpendicular to the z direction. The motion of the particle on the plane perpendicular to the z axis is described by the time independent Schrödinger equation (in $\hbar^2=M=1$ unit)

$$\left(-\frac{1}{2}\nabla^2+\frac{(1-\alpha^2)D^2}{48\pi\alpha^2r^2}\cos 2\Theta\right)\Psi=E\Psi. \tag{2}$$

The wave function can be separated as $\Psi(r,\phi)=R(r)\exp(im\phi)$ and Eq. (2) gives the radial equation

$$H_DR(r)\equiv-\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{\xi^2}{r^2}\right)R(r)=2ER(r), \tag{3}$$

where H_D is the radial Hamiltonian, with $\xi^2=\frac{1}{\alpha^2}\left(\frac{(1-\alpha^2)D^2}{24\pi}\cos 2\Theta-m^2\right)$ and $m=0,\pm1,\pm2,\ldots$. We will now discuss the solution of the Hamiltonian H_D.

To discuss that we need to know some general properties of an operator, let us say A. In this paper, let us restrict ourselves to the case of an unbounded operator because the Hamiltonian we are discussing is unbounded. Now, for an unbounded operator A, one can define a domain $D(A)$, such that the domain is dense in the Hilbert space. From the information of A and $D(A)$, one can construct the adjoint operator A^* and the corresponding domain $D(A^*)$ by using the relation $\int_0^\infty\phi^*(r)A\chi(r)dr=\int_0^\infty[A^*\phi(r)]^*\chi(r)dr$, $\forall\chi(r)\in D(A)$. The condition for self-adjointness for the operator A is given by $D(A)=D(A^*)$. An alternative definition of self-adjointness is given in terms of deficiency indices, found by using von Neumann's method. According to von Neumann's method, the deficiency indices $n_\pm$ are defined by dimension of the kernel $\mathrm{Ker}(i\pm A^*)$. If $n_\pm=0$, then the operator A is essentially self-adjoint. If $n_+=n_-=n\neq0$, then A is not self-adjoint but admits self-adjoint extensions. Self-adjoint extensions can be characterized by n^2 parameters. Different values of the parameters give rise to different physics. For $n_+\neq n_-$, the operator A cannot have any self-adjoint extensions.

Let us now come back to the discussion of our operator of interest, which is H_D. The Hamiltonian H_D acts over the Hilbert space of square-integrable functions, given by the domain $\mathcal{L}^2[R^+,rdr]$. As discussed earlier, classically this system is scale invariant under the scale transformation $\mathbf{r}=\lambda\mathbf{r}$, $t=\lambda^2t$. It can also be understood from the fact that the coupling constant of the inverse square potential ξ is a dimensionless coefficient. Now, we need to see whether this scale symmetry is still restored after quantization. Since this kind of model (inverse square interaction) has been investigated extensively in the literature, we know that the Hamiltonian H_D is essentially self-adjoint for $\xi^2\geq1$. Since any system is defined by a Hamiltonian and its corresponding domain, in our case the Hamiltonian H_D for $\xi^2\geq1$ acts over the domain

$$\mathcal{D}_0=\{\psi\in\mathcal{L}^2(rdr),\psi(0)=\psi'(0)=0\}. \tag{4}$$

Let us now investigate another portion of the coupling constant ξ^2. In this paper we will not consider the strong region because it has been investigated earlier. So the remaining region left to be investigated is $\xi\in(-1,1)$. In this region the Hamiltonian is not essentially self-adjoint and therefore we need to make self-adjoint extensions of the original domain, so that the Hamiltonian becomes self-adjoint. For the moment we consider $\xi\neq0$ because the case $\xi=0$ should be treated separately. From now onward we confine our analysis to the zero angular momentum states, i.e., we set $m=0$ in the expression of ξ for simplicity. However, our analysis is valid as long as $0\leq\xi^2<1$. Since the quantum dynamics in the interval $\xi\in(-1,1)$ essentially depends on the behavior of the coupling constant ξ, we plot ξ as a function of the cosmic string parameter α in Fig. 1, and in Fig. 2 we plot the same thing as a function of the variable Θ. Note that for $\xi\in(-1,1)$, the deficiency indices are (1, 1), so the self-adjoint extensions are characterized by a one-parameter element $e^{i\Sigma}$. Now, the domain under which our Hamiltonian H_D should be self-adjoint is given by $\mathcal{D}_\Sigma$. Mathematically this domain can be represented by

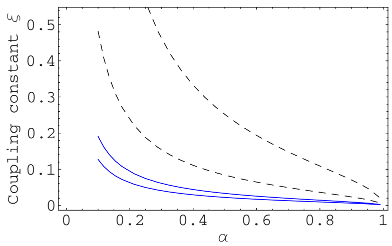

FIG. 1. (Color online) A plot of the coupling constant ξ as a function of the cosmic string parameter α. Dotted graphs correspond to $\Theta=\pi/8$ and from top to bottom $D=0.5$ and 1.5, respectively. Solid graphs (blue) correspond to $\Theta=\pi/5$ and from top to bottom $D=0.2$ and 0.3, respectively.

$$\mathcal{D}_\Sigma = \{\mathcal{D}_0 + \phi_+ + e^{i\Sigma}\phi_-\}, \tag{5}$$

where the deficiency space solutions $\phi_\pm$ are

$$\phi_+ = K_\xi(re^{-i\pi/4}), \quad \phi_- = K_\xi(re^{+i\pi/4}), \tag{6}$$

where K_ξ is the modified Bessel function [26]. The behavior of any function, belonging to the domain $\mathcal{D}_\Sigma$, near singularity $r\to 0$ can be found from the behavior of $\phi_+ + e^{i\Sigma}\phi_-$ at short distance, because near singularity, functions belonging to the domain $\mathcal{D}_0$ go to zero. Therefore

$$\phi_+ + e^{i\Sigma}\phi_- \simeq \mathcal{A}_+\left(\frac{r}{2}\right)^\xi + \mathcal{A}_-\left(\frac{2}{r}\right)^\xi, \tag{7}$$

where

$$\mathcal{A}_\pm = -\frac{\pi i}{\sin(\pi\xi)}\frac{\cos\left(\frac{\Sigma}{2}\pm\frac{\pi\xi}{4}\right)}{\Gamma(1\pm\xi)}.$$

Let us now solve the eigenvalue problem (3). Since for $\xi^2 \geq 1$ the Schrödinger equation does not have any normalizable solutions, we cannot have any bound state, because bound state solutions must be normalized in quantum mechanics. On the other hand it can be shown that [10] for $\xi \in (-1,1)$, there is exactly one bound state with energy $2E$, and eigenfunction $R(r)$:

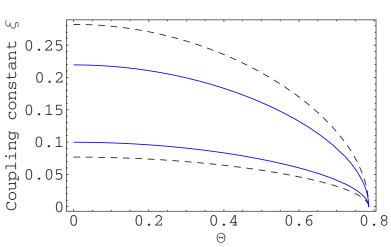

FIG. 2. (Color online) A plot of the coupling constant ξ as a function of Θ. Dotted graphs correspond to $D=0.5$ and from top to bottom $\alpha=0.2$ and 0.6, respectively. Solid graphs (blue) correspond to $\alpha=0.5$ and from top to bottom $D=0.5$ and 1.1, respectively.

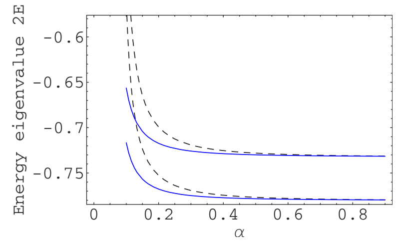

FIG. 3. (Color online) A plot of the energy eigenvalue $2E$ as a function of the cosmic string parameter α. Dotted graphs correspond to $\Theta=\pi/8$ and $D=1.6$ and from top to bottom $\Sigma=\pi/8$ and $\pi/10$, respectively. Solid graphs (blue) correspond to $\Theta=\pi/12$, $D=1$ and from top to bottom $\Sigma=\pi/8$ and $\pi/10$, respectively.

$$2E = -\sqrt[\xi]{\frac{\cos\frac{1}{4}(2\Sigma+\xi\pi)}{\cos\frac{1}{4}(2\Sigma-\xi\pi)}}, \quad R(r) = K_\xi(\sqrt{2E}r). \tag{8}$$

The bound state energy $2E$ in Eq. (8) as a function of the cosmic string parameter α has been plotted in Fig. 3. In Fig. 4 the eigenvalue $2E$ has been plotted as a function of the dipole moment D.

As pointed out before, according to scale symmetry there should not be any bound state solution; but in our system we get a bound state solution for the nontrivial boundary condition. The bound state eigenvalue can be considered as a scale in the system, which has emerged due to the nontrivial boundary condition. Thus the classical scale symmetry is destroyed after quantization of the system. Let us now discuss the scaling anomaly in terms of operators over the Hilbert space. In quantum mechanics there is an operator called scaling operator, which encodes the features of scaling symmetry. The scaling operator is given by

$$\Lambda = \frac{1}{2}(rp+pr), \quad \text{where} \quad p = -i\frac{d}{dr}. \tag{9}$$

One can check that this scaling operator Λ is symmetric on the domain $\mathcal{D}_0$ of the Hamiltonian H_D. It can also be checked that for $\xi^2 \geq 1$, the domain of the Hamiltonian H_D remains invariant when Λ acts on it. For $\xi^2 \in (-1,1)$, $\Lambda\phi = -\frac{i}{2}(\phi + 2r\phi')$, where ϕ is any element, belonging to the domain $\mathcal{D}_\Sigma$. The behavior of the function $\Lambda\phi$ near singularity $(r\to 0)$ can be found as

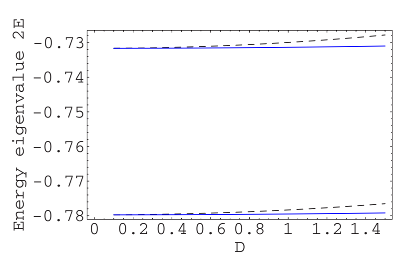

FIG. 4. (Color online) A plot of the energy eigenvalue $2E$ as a function of the dipole moment D. Solid graphs (blue) correspond to $\Theta=\pi/10$ and $\alpha=0.8$ and from top to bottom $\Sigma=\pi/8$ and $\pi/10$, respectively. Dotted graphs correspond to $\Theta=\pi/12$ and $\alpha=0.5$ and from top to bottom $\Sigma=\pi/8$ and $\pi/10$, respectively.

$$\Lambda\phi \simeq -\frac{i}{2}\left[(1+2\xi)\mathcal{A}_+\left(\frac{r}{2}\right)^{\xi} + (1-2\xi)\mathcal{A}_-\left(\frac{2}{r}\right)^{\xi}\right], \quad (10)$$

where the constants $\mathcal{A}_\pm$ are defined above.

Comparing the expressions (5) and (10), we see that $\Lambda\phi$ does not leave the domain of the Hamiltonian invariant, due to the two different terms $(1+2\xi)$ and $(1+2\xi)$ in the expression (10). Scaling symmetry is thus broken anomalously. The reason for this anomaly is the inequivalent quantization of the system, by making a self-adjoint extension of the initially non-self-adjoint Hamiltonian. Note that not all values of the self-adjoint extensions parameter Σ give rise to scaling anomaly. There are some values of the parameters for which scale symmetry is restored even after quantization. For example, for $\Sigma=(1\pm\frac{\xi}{2})\pi$ there is no bound state. One can also check from Eq. (10) that for $\Sigma=(1\pm\frac{\xi}{2})\pi$, Λ leaves the domain of the Hamiltonian invariant.

The case for $\xi=0$ can be handled in a similar fashion. The bound state energy and the wave function in this case are given by

$$2E = -\exp\left[\frac{\pi}{2}\cot\frac{\Sigma}{2}\right], \quad R(r) = K_0(\sqrt{-2E}r), \quad (11)$$

respectively, where K_0 [26] is the modified Bessel function. Here also the existence of a bound state implies breaking of scale symmetry.

In conclusion, we have shown that the presence of dipole moment of a particle, moving in cosmic string background, does not break the classical scale symmetry, which was present without the dipole moment. However, scale symmetry is anomalously broken by the inequivalent quantization of the system. The inequivalent quantization is characterized by one parameter family of self-adjoint extensions. The consequence of this anomaly is the existence of a bound state for the dipole and the scale is provided by the bound state energy. We have shown that scale symmetry can be restored for $\Sigma=(1\pm\frac{\xi}{2})\pi$ even after quantization.

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