Noncommutative Anandan quantum phase

E. Passos,[1,*] L. R. Ribeiro,[1,†] C. Furtado,[1,‡] and J. R. Nascimento[1,2,§]
[1]*Departamento de Física, Universidade Federal da Paraíba, Caixa Postal 5008, 58051-970, João Pessoa, PB, Brazil*
[2]*Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05315-970, São Paulo, São Paulo, Brazil*
(Received 10 November 2006; revised manuscript received 15 May 2007; published 18 July 2007)

In this work, we study the noncommutative nonrelativistic quantum dynamics of a neutral particle, which possesses permanent magnetic and electric dipole moments, in the presence of external electric and magnetic fields. We use the Foldy-Wouthuysen transformation of the Dirac spinor with a nonminimal coupling to obtain the nonrelativistic limit. In this limit, we study the noncommutative quantum dynamics and obtain the noncommutative Anandan geometric phase. We analyze the situation where the magnetic dipole moment of the particle is zero, and we obtain the noncommutative version of the He-McKellar-Wilkens effect. We demonstrate that this phase in the noncommutative case is a geometric dispersive phase. We also investigate this geometric phase by considering the noncommutativity in the phase space, and the Anandan phase is obtained.

DOI: 10.1103/PhysRevA.76.012113 PACS number(s): 03.65.Vf, 11.10.Nx, 13.40.Em

I. INTRODUCTION

In 1959, Aharonov and Bohm [1] demonstrated that a quantum charge circulating a magnetic flux tube acquires a quantum topological phase. This effect was observed experimentally by Chambers [2,3]. Aharonov and Casher showed that a particle with a magnetic moment moving in an electric field accumulates a quantum phase [4], which has been observed in a neutron interferometer [5] and in a neutral atomic Ramsey interferometer [6].

He and McKellar [7] and Wilkens [8] independently predicted the existence of a quantum phase acquired by an electric dipole while it is circulating around and parallel to a line of magnetic monopoles. A simple practical experimental configuration to test this phase was proposed by Wei *et al.* [9], where the electric field of a charged wire polarizes a neutral atom and a uniform magnetic field is applied parallel to the wire.

In a recent paper, a topological phase effect was proposed by Anandan [10], which describes a unified and fully relativistic covariant treatment of the interaction between a particle with permanent electric and magnetic dipole moments and an electromagnetic field. This problem has been investigated in nonrelativistic quantum mechanics by Anandan [11] and Furtado and de Lima Ribeiro [12].

Recently, the study of physics in noncommutative space has attracted much interest in several areas of physics [13]. Noncommutative field theories are related to *M* theory [14], string theory [15], and the quantum Hall effect [16–18]. In quantum mechanics, a great number of problems have been investigated in the case of noncommutative space-time. Some important results obtained are related to geometric phases, such as the Aharonov-Bohm effect [19–23], the Aharonov-Casher effect [24,25], and Berry's quantum phase [26,27], and others involve the dynamics of dipoles [28]. In this paper, we analyze the noncommutative quantum topological phase effect proposed by Anandan for a quantum particle with permanent magnetic and electric dipole moments in the presence of external electric and magnetic fields, and study the appearance of a geometrical quantum phase in their dynamics. We investigate the nonrelativistic geometric phase, proposed by Anandan, for a quantum particle with permanent magnetic and electric dipole moments in the presence of external electric and magnetic fields in noncommutative quantum mechanics. We also investigate the He-McKellar-Wilkens phase in noncommutative space.

This paper is organized in the following way. In the next section, we discuss the quantum dynamics of a neutral particle in the presence of an external electromagnetic field. In Secs. III and IV, the noncommutative nonrelativistic quantum dynamics of an quantum dipoles in the presence of an external field is investigated. In Sec. V, we study the noncommutative Aharonov-Casher effect; in Sec. VI, we extend the study for the He-McKellar-Wilkens phase in a noncommutative space-time. In Sec. VII, we discuss this quantum phase, considering momentum-momentum noncommutativity or phase space noncommutativity. Finally, in Sec. VIII, the conclusions are presented.

II. THE NONRELATIVISTIC LIMIT

Now we consider the relativistic quantum dynamics of a single neutral spin-half particle with nonzero magnetic and electric dipoles moving in an external electromagnetic field, which is described by the following equation (we used $\hbar = c = 1$):

$$\left(i\gamma_{\mu}\partial^{\mu}+\frac{1}{2}\mu\sigma_{\alpha\beta}F^{\alpha\beta}-\frac{i}{2}d\sigma_{\alpha\beta}\gamma_5F^{\alpha\beta}-m\right)\psi=0, \qquad (1)$$

where μ is the magnetic dipole moment and d is the electric dipole moment. We use the following convention for field strength [29]:

$$F^{\mu\nu}=\{\vec{E},\vec{B}\}, \quad F^{\mu\nu}=-F^{\nu\mu},$$

$$F^{i0}=E^{i}, \quad F^{ij}=-\epsilon_{ijk}B^{k},$$

*passos@fisica.ufpb.br
†lrr@fisica.ufpb.br
‡furtado@fisica.ufpb.br
§jroberto@fisica.ufpb.br; jroberto@fma.if.usp.br

$$F^{\mu\nu}=\begin{bmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{bmatrix}, \tag{2}$$

$$\sigma_{0i}=\frac{i}{2}(\gamma_0\gamma_i-\gamma_i\gamma_0)=i\gamma_0\gamma_i=-i\alpha_i,$$

$$\sigma_{ij}=\frac{i}{2}(\gamma_i\gamma_j-\gamma_j\gamma_i)=i\gamma_i\gamma_j=\epsilon_{ijl}\Sigma_l, \tag{3}$$

and our γ_5 choice for convenience is

$$\gamma_5=\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{4}$$

where 0 and -1 are the corresponding 2×2 matrices [30]. Hence, we may write Eq. (1) as

$$[i\gamma^\mu\partial_\mu+\mu(i\vec{\alpha}\cdot\vec{E}-\vec{\Sigma}\cdot\vec{B})-id(i\vec{\alpha}\cdot\vec{E}-\vec{\Sigma}\cdot\vec{B})\gamma_5-m]\psi=0, \tag{5}$$

with the Dirac matrices given by

$$\hat{\beta}=\gamma^0=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad \gamma^j=\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix},\quad \vec{\alpha}=\hat{\beta}\vec{\gamma}=\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix},$$

$$\vec{\Sigma}=\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix},\quad \vec{\Pi}=\hat{\beta}\vec{\Sigma}=\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix},$$

where σ^j are the Pauli matrices obeying the relation $\{\sigma^i\sigma^j+\sigma^j\sigma^i\}=-2g^{ij}$. Then the Hamiltonian given by Eq. (5) is reduced to the form

$$i\frac{\partial}{\partial t}\psi=H\psi=(\vec{\pi}\cdot\vec{\alpha}+\mu\vec{\Pi}\cdot\vec{B}+d\vec{\Pi}\cdot\vec{E}+\hat{\beta}m)\psi, \tag{6}$$

where $\vec{\pi}=-i(\vec{\nabla}+\mu\hat{\beta}\vec{E}-d\hat{\beta}\vec{B})$.

From now on, we will use the Foldy-Wouthuysen method. This a very convenient method of description of the relativistic particle interaction with an external field; the transition to the semiclassical description is the Foldy-Wouthuysen (FW) transformation [31]. The Foldy-Wouthuysen representation provides the best opportunity for the transition to the classical limit of relativistic quantum mechanics. The Hamiltonian in Eq. (5) takes the form

$$\hat{H}=\hat{\beta}(m+\hat{\epsilon})+\hat{O}, \tag{7}$$

where $\hat{\epsilon}=\mu\vec{\Pi}\cdot\vec{B}+d\vec{\Pi}\cdot\vec{E}$ and $\hat{O}=\vec{\pi}\cdot\vec{\alpha}$ are the even and odd terms in the Hamiltonian. Hence we introduce the transformation

$$\hat{H}'=e^{i\hat{S}}(\hat{H}-i\partial_0)e^{-i\hat{S}}, \tag{8}$$

where $\hat{S}$ is a Hermitian matrix. The purpose is to minimize the odd part of the Hamiltonian, or even to make it vanish. Thus, we have

$$\hat{H}'=\hat{H}+i[\hat{S},\hat{H}]-\frac{1}{2}[\hat{S},[\hat{S},\hat{H}]]-\frac{1}{6}i[\hat{S},[\hat{S},[\hat{S},\hat{H}]]]+\frac{1}{24}[\hat{S},[\hat{S},[\hat{S},\hat{\beta}m]]]+\cdots, \tag{9}$$

where $\hat{\epsilon}$ and $\hat{O}$ above obey the relations $\hat{\epsilon}\hat{\beta}=\hat{\beta}\hat{\epsilon}$ and $\hat{O}\hat{\beta}=-\hat{\beta}\hat{O}$.

For nonrelativistic particles in an electromagnetic field, the FW transformation can be performed with the operator $\hat{S}=-(i/2m)\hat{\beta}\hat{O}$, such that we have

$$\hat{H}'=\hat{\beta}m+\hat{\epsilon}'+\hat{O}', \tag{10}$$

where $\hat{O}'$ is on the order of $1/2m$, and we calculate the second-order FW transformation with $\hat{S}'=-(i/2m)\hat{\beta}\hat{O}'$. This yields

$$\hat{H}''=\hat{\beta}m+\hat{\epsilon}'+\hat{O}'', \tag{11}$$

where $\hat{O}''\approx 1/m^2$. After that, the third FW approximation with $\hat{S}''=-(i/2m)\hat{\beta}\hat{O}''$ makes the odd part of the nonrelativistic expansion vanish; so finally we find the usual result

$$\hat{H}'''\cong\hat{\beta}m+\hat{\epsilon}'=\hat{\beta}\left(m+\frac{1}{2m}\hat{O}^2-\frac{1}{8m^3}\hat{O}^4\right)+\hat{\epsilon}-\frac{1}{8m^2}[\hat{O},[\hat{O},\hat{\epsilon}]]. \tag{12}$$

After replacing $\hat{\epsilon}$ and $\hat{O}$ in (12), we will consider only terms up to order $1/m$. We obtain the following Hamiltonian:

$$\hat{H}'''\approx\hat{\beta}\left(m-\frac{1}{2m}[\vec{\nabla}-i\mu\hat{\beta}(\vec{\Sigma}\times\vec{E})+id\hat{\beta}(\vec{\Sigma}\times\vec{B})]^2-\frac{\mu^2E^2}{2m}-\frac{d^2\vec{B}^2}{2m}-\frac{\mu\hat{\beta}}{2m}\vec{\nabla}\cdot\vec{E}+\frac{d\hat{\beta}}{2m}\vec{\nabla}\cdot\vec{B}\right)+\mu\vec{\Pi}\cdot\vec{B}+d\vec{\Pi}\cdot\vec{E}. \tag{13}$$

Equation (13) is the nonrelativistic quantum Hamiltonian for four-component fermions. However, for several applications at low energies in nonrelativistic quantum mechanics, the two-component spinor field is considered; we may write (13) for two-component fermions in the form

$$\hat{H}'''\approx m-\frac{1}{2m}[\vec{\nabla}-i(\vec{\mu}\times\vec{E})+i(\vec{d}\times\vec{B})]^2-\frac{\mu^2E^2}{2m}-\frac{d^2\vec{B}^2}{2m}-\frac{\mu}{2m}\vec{\nabla}\cdot\vec{E}+\frac{d}{2m}\vec{\nabla}\cdot\vec{B}+\vec{\mu}\cdot\vec{B}+\vec{d}\cdot\vec{E}, \tag{14}$$

where $\vec{\mu}=\mu\vec{\sigma}$, $\vec{d}=d\vec{\sigma}$, and $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$; and σ_i ($i=1,2,3$) are the 2×2 Pauli matrices. The Hamiltonian (14) describes the system formed by a neutral particle, which possesses permanent electric and magnetic dipole moments, in the presence of electric and magnetic fields. Several topological and geometrical effects may be investigated by changing the field-dipole configuration [7–9].

III. NONRELATIVISTIC QUANTUM DYNAMICS OF DIPOLES

We consider the nonrelativistic quantum dynamics of a particle corresponding to the Hamiltonian (13), which describes several physical situations such as the Aharonov-Casher effect for $\mu \neq 0$ and $d=0$, the He-McKellar-Wilkens effect for $\mu=0$ and $d\neq 0$, and the Anandan phase in the general case $\mu\neq 0$ and $d\neq 0$. It is obvious that all these effects occur in specific field-dipole configurations. We analyze the quantum dynamics of a particle governed by the Hamiltonian (14). We consider that the electric and magnetic fields in which the particle is immersed are cylindrically radial [7–12]. The electric and magnetic dipoles are aligned in the z direction. The Hamiltonian (14) that describe the electric and magnetic dipoles in external electric and magnetic fields can be written in the following way:

$$H=-\frac{1}{2m}(\vec{\nabla}-b_{\mu})^2+b_0; \tag{15}$$

here, the interaction with the electric and magnetic fields is similar to that of a particle minimally coupled to a non-Abelian gauge field with potential b_μ, where

$$b_0=-\frac{\mu^2E^2}{2m}-\frac{d^2\vec{B}^2}{2m}-\frac{\mu}{2m}\vec{\nabla}\cdot\vec{E}+\frac{d}{2m}\vec{\nabla}\cdot\vec{B}+\vec{\mu}\cdot\vec{B}+\vec{d}\cdot\vec{E}, \tag{16}$$

and $b_i=(\vec{\mu}\times\vec{E})+(\vec{d}\times\vec{B})$. The first two terms in b_0 can be considered as an external potential and do not contribute in the study of the geometric phase [9]. Observe that the potential b_0, which depends on E^2 and B^2, represents a local influence on the wave function. We are interested in studying the asymptotic states for the dynamics. Thus we will not consider this term, because it represent a local effect [9]. Also, Anadan [11] has demonstrated that terms of the order $O(E^2)$ and $O(B^2)$ can be neglected in the study of the geometric phase. The last four terms in the potential b_0 in the Hamiltonian (15) give null contributions to the dynamics of this dipole in the field configuration, since the dipoles are aligned with the z direction [11,10,12]. We assume that the particle moves in the plane x-y, in the presence of the external electric and magnetic fields [9]. We also suppose that the fields generated by the source are radially distributed in the space. Now, we consider the commutative space version of this geometric phase. Thus the only terms that contribute to the geometric phase in (15) are

$$\hat{H}=-\frac{1}{2m}[\vec{\nabla}-i(\vec{\mu}\times\vec{E})+i(\vec{d}\times\vec{B})]^2-\frac{\mu^2E^2}{2m}-\frac{d^2\vec{B}^2}{2m}; \tag{17}$$

the other terms of (13) do not contribute to the quantum phase because of the choice of specific dipole-field configurations. The terms in the dynamical part of the Hamiltonian yield no force on the particle, while in quantum mechanics they affect the wave function of the particles by attaching to it a nondispersive geometric phase. The momentum operator can be written as

$$k_i=mv_i=[p_i-(\vec{\mu}\times\vec{E})_i+(\vec{d}\times\vec{B})_i]. \tag{18}$$

The Schrödinger equation for this problem takes the form

$$\left(-\frac{1}{2m}[\vec{\nabla}-i(\vec{\mu}\times\vec{E})+i(\vec{d}\times\vec{B})]^2-\frac{\mu^2E^2}{2m}-\frac{d^2\vec{B}^2}{2m}\right)\Psi=E\Psi. \tag{19}$$

To obtain the quantum phase we use the following ansatz:

$$\Psi=\Psi_0e^{\phi}, \tag{20}$$

where Ψ_0 is the solution of the equation

$$\left(-\frac{1}{2m}\nabla^2-\frac{\mu^2E^2}{2m}-\frac{d^2\vec{B}^2}{2m}\right)\Psi_0=E\Psi_0, \tag{21}$$

and the phase ϕ is given by

$$\phi=i\oint[(\vec{\mu}\times\vec{E})-(\vec{d}\times\vec{B})]dr; \tag{22}$$

this phase was studied by Anandan [10]. It is a nondispersive effect due to the independence of particle velocity [33]. Considering $d=0$ in (22), we have the Aharonov-Casher geometric phase. On the other hand, in the case $d\neq 0$ and $\mu=0$ in (22) we have the He-McKellar-Wilkens phase,

$$\phi_{HMW}=i\oint[-(\vec{d}\times\vec{B})]dr, \tag{23}$$

which is usually known as a topological phase, but really is a geometric phase [39].

IV. NONCOMMUTATIVE QUANTUM DYNAMICS OF DIPOLES

The usual noncommutative space canonical variables satisfy the following commutation relations:

$$[\hat{x}_i,\hat{x}_j]=i\Theta_{ij},\quad [\hat{p}_i,\hat{p}_j]=0,\quad [\hat{x}_i,\hat{p}_j]=i\delta_{ij}, \tag{24}$$

where $\hat{x}_i$ and $\hat{p}_i$ are the momentum and coordinate operators in a noncommutative space. The time-independent Schrödinger equation in the noncommutative (NC) space can be written in the form

$$H(x,p)\star\psi=E\psi, \tag{25}$$

where $H(x,p)$ is the usual Hamiltonian and the Moyal-Weyl product (or star product) is given by

$$(f\star g)(x)=\exp\left(\frac{i}{2}\Theta_{ij}\partial_{x_i}\partial_{x_j}\right)f(x_i)g(x_j). \tag{26}$$

Here $f(x)$ and $g(x)$ are arbitrary functions. On the NC space, the Moyal-Weyl product may be replaced by a Bopp shift [32], i.e., the Moyal-Weyl product can be changed into the ordinary product by replacing $H(x,p)$ with $H(\hat{x},\hat{p})$. This approach has been used by Li *et al.* [23]. Hence, the Schrödinger equation can be written in the form

$$H(\hat{x}_i,\hat{p}_i) = H\left(x_i - \frac{1}{2}\Theta_{ij}p_i, p_i\right)\psi = E\psi, \tag{27}$$

where x_i and p_i are the generalized position and momentum coordinates in the usual quantum mechanics. Therefore, Eq. (27) is then actually defined on the commutative space, and the NC effect may be calculated from the terms that contain Θ. Note that Θ in quantum mechanics may be taken as a perturbation, considering that $\Theta_{ij} \ll 1$.

When we have the presence of electric and magnetic fields as in (19), Eq. (25) becomes

$$\left(-\frac{1}{2m}[\vec{\nabla} - i(\vec{\mu}\times\vec{E}) + i(\vec{d}\times\vec{B})]^2 - \frac{\mu^2E^2}{2m} - \frac{d^2\vec{B}^2}{2m}\right)\star\Psi = E\Psi. \tag{28}$$

To map Eq. (28) from NC space to commutative space, we replace x_i and p_i by a Bopp shift [32], as well as the fields E_i and B_i, which will be replaced by a shift of the form

$$(\vec{\mu}\times\vec{E}) \to (\vec{\mu}\times\vec{E}) + \frac{i}{2}\Theta_{lm}[\vec{\kappa} - (\vec{\mu}\times\vec{E})]_l\partial_m(\vec{\mu}\times\vec{E}) \tag{29}$$

and

$$(\vec{d}\times\vec{B}) \to (\vec{d}\times\vec{B}) + \frac{i}{2}\Theta_{lm}[\vec{\kappa} - (\vec{d}\times\vec{B})]_l\partial_m(\vec{d}\times\vec{B}), \tag{30}$$

where κ_l is the eigenvalue of the momentum operator in the presence of the electric or magnetic field on NC space, and is defined as

$$[p_i - (\vec{\mu}\times\vec{E}) + (\vec{d}\times\vec{B})]\star\psi = \kappa_i\psi, \tag{31}$$

where $\kappa_i = mv_i$ and v_i is the ordinary gradient. The relations (29) and (30) may be obtained in the same form by Taylor expansion up to first order of (26); for example, let us take the magnetic dipole case

$$[(\vec{\mu}\times\vec{E})\star\psi](x) = \exp\left[\frac{i}{2}\Theta_{ij}\partial_{x_i}\partial_{x_j}\right](\vec{\mu}\times\vec{E})(x_i)\psi(x_j) = (\vec{\mu}\times\vec{E})\psi + \frac{i}{2}\Theta_{ij}\partial_i(\vec{\mu}\times\vec{E})\partial_j\psi. \tag{32}$$

From (31) we have

$$\partial_i\psi = [\kappa - (\vec{\mu}\times\vec{E})]_i\psi. \tag{33}$$

Therefore, using (33) in (31), we obtain (29). In the same way, we may obtain (30). Thus, the NC equation (28) mapped on commutative space is

$$-\frac{1}{2m}\left(\vec{\nabla} - i(\vec{\mu}\times\vec{E}) - \frac{i}{2}\Theta_{lm}[\kappa_l - (\vec{\mu}\times\vec{E})_l]\partial_m(\vec{\mu}\times\vec{E}) + i(\vec{d}\times\vec{B}) + \frac{i}{2}\Theta_{lm}[\kappa_l - (\vec{d}\times\vec{B})_l]\partial_m(\vec{d}\times\vec{B})\right)^2\psi = E\psi. \tag{34}$$

In the same way as in the usual quantum mechanics, the solution for (34) may be written as

$$\psi = \psi_0\exp(\phi), \tag{35}$$

where ψ_0 is a solution of the Schrödinger equation in the absence of electric and magnetic fields, and ϕ is the Anandan geometric phase given in the form

$$\phi = i\oint[(\vec{\mu}\times\vec{E}) - (\vec{d}\times\vec{B})]\cdot d\vec{r} + \frac{i}{2}\Theta_{lm}\times\oint\{[\kappa - (\vec{\mu}\times\vec{E})]_l\partial_m(\vec{\mu}\times\vec{E}) - [\kappa - (\vec{d}\times\vec{B})]_l\times\partial_m(\vec{d}\times\vec{B})\}\cdot d\vec{r}. \tag{36}$$

The first term of the integral in Eq. (36) is the usual Anandan phase in commutative quantum mechanics. The other terms are the corrections due to NC effects. In the three-dimensional commutative space, we define the vector $\theta = (\theta_1, \theta_2, \theta_3)$ with $\Theta_{ij} = \epsilon_{ijk}\theta_k$. Thus we rewrite the total phase (36) in the form

$$\phi = i\oint[(\vec{\mu}\times\vec{E}) - (\vec{d}\times\vec{B})]\cdot d\vec{r} + \frac{i}{2}m\oint\vec{\theta}\cdot[\vec{v}\times\vec{\nabla}(\vec{\mu}\times\vec{E})_i]dr_i - \frac{i}{2}m\oint\vec{\theta}\cdot[(\vec{\mu}\times\vec{E})\times\vec{\nabla}(\vec{\mu}\times\vec{E})_i]dr_i - \frac{i}{2}m\oint\vec{\theta}\cdot[\vec{v}\times\vec{\nabla}(\vec{d}\times\vec{B})_i]dr_i + \frac{i}{2}m\oint\vec{\theta}\cdot[(\vec{d}\times\vec{B})\times\vec{\nabla}(\vec{d}\times\vec{B})_i]dr_i. \tag{37}$$

The phase (37) is a noncommutative version of the nonrelativistic quantum Anandan phase. Notice the dependence on phase in the electric and magnetic fields. A further property of Eq. (37) is that the geometric phase depends on the velocity of the particle. The noncommutativity of space introduces this dependence in the phase. In the next section, we will discuss some special limits of this geometric phase.

V. NONCOMMUTATIVE AHARONOV-CASHER EFFECT

First we consider the case in the expression (36) where $d=0$. In this case, we obtain the Aharonov-Casher (AC) phase given by

$$\phi_{AC} = i\oint\left(\vec{\mu}\times\vec{E} + \frac{1}{2}\Theta_{lm}[\kappa_l - (\vec{\mu}\times\vec{E})_l]\partial_m(\vec{\mu}\times\vec{E})\right)\cdot d\vec{r}. \tag{38}$$

The first term in the integral in Eq. (38) is the usual AC phase in commutative quantum mechanics. The second term is the NC correction to the AC phase. In the three-dimensional commutative space, we define the vector $\theta=(\theta_1,\theta_2,\theta_3)$ with $\Theta_{ij}=\epsilon_{ijk}\theta_k$. Thus, we rewrite the total phase (38) in the form

$$f=i\oint(\vec{\mu}\times\vec{E})\cdot d\vec{r}+\frac{i}{2}m\oint\vec{\theta}\cdot[\vec{v}\times\vec{\nabla}\cdot(\vec{\mu}\times\vec{E})]\cdot d\vec{r}-\frac{i}{2}m\oint\vec{\theta}\cdot[(\vec{\mu}\times\vec{E})\times\vec{\nabla}\cdot(\vec{\mu}\times\vec{E})]\cdot d\vec{r}. \quad (39)$$

This geometric phase is the same obtained in [23,24] in the relativistic case. Note that this is a dispersive geometric phase that depends on the velocity of the particle [33].

VI. NONCOMMUTATIVE HE-McKELLAR-WILKENS EFFECT

Now let us analyze the particular case of (22) in the noncommutative situation. This case is the noncommutative He-McKellar-Wilkens quantum phase in the nonrelativistic limit. He and McKellar [7] and Wilkens [8] independently demonstrated that the quantum dynamics of an electric dipole in the presence of a radial magnetic field exhibits a geometric phase. The method to obtain the NC He-McKellar-Wilkens (HMW) effect is similar to the AC case. We take $\mu=0$ in the Pauli term in Eq. (13). Hence, we find the NC Schrödinger equation for the electric dipole in the presence a magnetic field. Applying this limit in the phase (36) we obtain the following expression

$$\phi_{HMW}=-i\oint\left(\vec{d}\times\vec{B}+\frac{1}{2}\Theta_{lm}[\kappa_l-(\vec{d}\times\vec{B})_l]\partial_m(\vec{d}\times\vec{B})\right)\cdot d\vec{r}. \quad (40)$$

The first term in (40) is the usual commutative HMW quantum phase. The second term is the NC correction to the HMW phase. In the same way as in the AC case, in the three-dimensional commutative space we define the vector $\theta=(\theta_1,\theta_2,\theta_3)$ with $\Theta_{ij}=\epsilon_{ijk}\theta_k$. Thus we rewrite the total phase (40) in the form

$$\phi_{HMW}=-i\oint(\vec{d}\times\vec{B})\cdot d\vec{r}-\frac{i}{2}m\oint\vec{\theta}\cdot[\vec{v}\times\vec{\nabla}(\vec{d}\times\vec{B})]\cdot d\vec{r}+\frac{i}{2}m\oint\vec{\theta}\cdot[(\vec{d}\times\vec{B})\times\vec{\nabla}\cdot(\vec{d}\times\vec{B})]\cdot d\vec{r}. \quad (41)$$

This equation gives the expression for the noncommutative version of the He-MacKellar-Wilkens effect.

VII. NONCOMMUTATIVE DYNAMICS OF DIPOLES IN PHASE SPACE

In the previous section, we discussed the noncommutative version of the geometric phase in the quantum dynamics of a neutral particle that possesses permanent electric and magnetic dipole moments. Now we will discuss the case where we take into account momentum-momentum noncommutativity. The Bose-Einstein statistics in noncommutative quantum mechanics requires both space-space and momentum-momentum noncommutativity [23,25,34–36]. This formulation has been called phase-space noncommutativity. In this case, the momentum commutation relation in (24) is replaced by

$$[\hat{p}_i,\hat{p}_j]=i\bar{\Theta}_{ij}, \quad (42)$$

where $\bar{\Theta}$ is the antisymmetric matrix; its elements represent the noncommutativity of the momenta. Thus the Schrödinger equation (25) is written in the form

$$-\frac{1}{2m}[\nabla-i(\vec{\mu}\times\vec{E})+i(\vec{d}\times\vec{B})]^2\star\Psi=E\Psi. \quad (43)$$

In noncommutative phase space, the star product can be replaced by a generalized Bopp shift [32]; in this way, the star product can be changed into an ordinary product by shifting coordinates x_μ and momenta p_μ by

$$\hat{x}_i=\lambda x_i-\frac{1}{2\lambda}\Theta_{ij}p_j \quad (44)$$

and

$$\hat{p}_i=\lambda p_i-\frac{1}{2\lambda}\bar{\Theta}_{ij}x_j, \quad (45)$$

where the scale factor λ is an arbitrary constant parameter. The fields in the equation change according to the formula (27) and assume the following form:

$$(\vec{\mu}\times\vec{E})\rightarrow\lambda(\vec{\mu}\times\vec{E})+\frac{i}{2\lambda}\Theta_{lm}[\kappa_l-(\vec{\mu}\times\vec{E})_l]\partial_m(\vec{\mu}\times\vec{E}), \quad (46)$$

and the magnetic field term changes to the form

$$(\vec{d}\times\vec{B})\rightarrow\lambda(\vec{d}\times\vec{B})+\frac{i}{2\lambda}\Theta_{lm}[\kappa_l-(\vec{d}\times\vec{B})_l]\partial_m(\vec{d}\times\vec{B}). \quad (47)$$

Now the Schrödinger equation for the neutral particle becomes

$$-\frac{1}{2m}\left(\lambda\vec{\nabla}+\frac{i}{2\lambda}\bar{\Theta}_{ij}x_i-i\lambda(\vec{\mu}\times\vec{E})+\frac{1}{2\lambda}\Theta_{lm}[\kappa_l-(\vec{\mu}\times\vec{E})_l]\partial_m(\vec{\mu}\times\vec{E})+i\lambda(\vec{d}\times\vec{B})-\frac{1}{2\lambda}\Theta_{lm}[\kappa_l-(\vec{d}\times\vec{B})_l]\partial_m(\vec{d}\times\vec{B})\right)^2\psi=E\psi. \quad (48)$$

We can rewrite the Schrödinger equation in the following form:

$$-\frac{1}{2m'}\left(\vec{\nabla}+\frac{i}{2\lambda^2}\bar{\Theta}_{ij}x_i-i(\vec{\mu}\times\vec{E})+\frac{1}{2\lambda^2}\Theta_{lm}[\kappa_l-(\vec{\mu}\times\vec{E})_l]\partial_m(\vec{\mu}\times\vec{E})+i(\vec{d}\times\vec{B})-\frac{1}{2\lambda^2}\Theta_{lm}[\kappa_l-(\vec{d}\times\vec{B})_l]\partial_m(\vec{d}\times\vec{B})\right)^2\psi=E\psi, \quad (49)$$

where $m' = m/\lambda$. In the same way as in the usual quantum mechanics, the solution for (49) may be written as

$$\psi = \psi_0 \exp(\phi_{PS}), \tag{50}$$

where ψ_0 is the solution of the Schrödinger equation for a particle of mass m' in the absence of an electromagnetic field, and ϕ_{PS} is the Anandan geometric phase in noncommutative phase space given in the form

$$\begin{aligned}\phi_{PS} = i \oint \left(\vec{\mu} \times \vec{E} - \frac{1}{2\lambda^2}\Theta_{lm}[\kappa_l - (\vec{\mu} \times \vec{E})_l]\partial_m(\vec{\mu} \times \vec{E}) \right) \cdot d\vec{r} \\ - i \oint \left(\vec{d} \times \vec{B} - \frac{1}{2\lambda^2}\Theta_{lm}[\kappa_l - (\vec{d} \times \vec{B})_l]\partial_m(\vec{d} \times \vec{B}) \right) \cdot d\vec{r} \\ - \frac{i}{2\lambda^2}\oint \bar{\Theta}_{ij}x_j dx_i.\end{aligned} \tag{51}$$

The previous expression (51) has a contribution due to a quantum phase in commutative space, an other contribution due to the noncommutative space, and one more contribution due to the noncommutative phase space. We can write the quantum phase (51) in the following form:

$$\phi_{PS} = \phi_{AP} + \phi_{NCS} + \phi_{NCPS}, \tag{52}$$

where ϕ_{AP} and the ϕ_{NCS} are the Anandan phase contribution and the contribution due to the space-space noncommutativity to the general dipole phase in the expression (51). The term ϕ_{NCPS} is the contribution due to noncommutativity of the momenta and is given by

$$\begin{aligned}\phi_{NCPS} = - \frac{i}{2\lambda^2}\oint \bar{\Theta}_{ij}x_j dx_i \\ + \frac{1-\lambda^2}{2\lambda^2}\oint \{\Theta_{lm}[\kappa_l - (\vec{\mu} \times \vec{E})_l]\partial_m(\vec{\mu} \times \vec{E})\} \cdot d\vec{r} \\ - \frac{1-\lambda^2}{2\lambda^2}\oint \{\Theta_{lm}[\kappa_l - (\vec{d} \times \vec{B})_l]\partial_m(\vec{d} \times \vec{B})\} \cdot d\vec{r}.\end{aligned} \tag{53}$$

This is the contribution to the noncommutative geometric phase due to noncommutativity in phase space. In this way, we can write the He-McKellar-Wilkens phase in noncommutative phase space for the particular case where $\vec{\mu} = \vec{0}$, and this expression is given by

$$\begin{aligned}\phi_{HMWPS} = - \frac{i}{2\lambda^2}\oint \bar{\Theta}_{ij}x_j dx_i \\ - \frac{1-\lambda^2}{2\lambda^2}\oint \{\Theta_{lm}[\kappa_l - (\vec{d} \times \vec{B})_l]\partial_m(\vec{d} \times \vec{B})\} \cdot d\vec{r}.\end{aligned} \tag{54}$$

Therefore, we obtain the contribution due to the NC phase space to the He-McKellar-Wilkens phase. We can see that this phase depends on the magnetic field and also on the velocity of the particle. The first term in (54) is similar in appearance to the spin factor that occurs in the partition function of a spinning particle. The connection of this spin factor with the geometric phase was investigated by Kalhede *et al.* [37] and Lévay [38]. This similarity with the spin factor and the physical implications of this term is the topic of a future presentation.

VIII. CONCLUDING REMARKS

In this paper, we study the nonrelativistic quantum dynamics of a neutral particle that possesses permanent electric and magnetic dipole moments, in the presence of electric and magnetic external fields. We use the Foldy-Wouthuysen expansion to make the transition from the classical limit of relativistic to nonrelativistic quantum mechanics. In this limit, we investigate the Aharonov-Casher and the He-McKellar-Wilkens effects in the noncommutative coordinate space. Here, we replace the star product by the Bopp shift [32] in the field terms, and then we obtain the AC and HMW quantum phases with NC corrections. We obtain the noncommutative Anandan phase and demonstrate that this is a geometric dispersive phase. Usually, a geometric phase is a local effect, while a topological phase is nonlocal. Peshkin and Lipkin [39] have shown that the Aharonov-Bohm effect is nonlocal, because its value depends upon a physical quantity in a region outside the closed path. It is a topological effect. The Aharonov-Bohm phase is proportional to the winding number of the path around the flux. It is a topological invariant, and this phase depends on topology not on distance; hence it must be nonlocal. Therefore there are no electromagnetic fields along the paths of the charged particle and there are no changes of physical quantities. They remarked that, in the case of the Aharonov-Casher effect, there are fields along the paths of the beams; then they concluded that the Aharonov-Casher effect is local due to the local interactions, and is nontopological because the phase shift depends on the local fields along the paths. In contrast with the topological phase, the geometric phase, in general, is a local effect, because it depends on the geometry and topology in the space of the parameters, but not on the topology in spacetime. Here the quantum phase depends on the fields and the velocity of the particle. This fact characterizes the noncommutative Anandan quantum phase as a geometric phase due to its dependence on the fields, and dispersive because of the dependence on the velocity [33]. The NC Aharonov-Casher effect is obtained with a limit case of (37) and agrees with the results in the literature [24,25]. The NC version of the He-McKellar-Wilkens phase is calculated for the NC quantum dynamics of electric dipoles and is a geometric dispersive phase. The noncommutative phase-space versions of Anandan's phase and the He-McKellar-Wilkens phase are obtained in this paper, and we conclude they are geometric dispersive phases.

ACKNOWLEDGMENTS

This work was partially supported by CNPq, CAPES/PROCAD, CNPQ/FINEP/PADCT, and PRONEX/CNPQ/FAPESQ. We thank A. Yu. Petrov for the critical reading of the manuscript. J.R.N. was partially supported by FAPESP.

[1] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).
[2] R. G. Chambers, Phys. Rev. Lett. **5**, 3 (1960).
[3] M. Peshkin and A. Tonomura, *The Aharonov-Bohm Effect* (Springer-Verlag, Berlin, 1989).
[4] Y. Aharonov and A. Casher, Phys. Rev. Lett. **53**, 319 (1984).
[5] A. Cimmino, G. I. Opat, A. G. Klein, H. Kaiser, S. A. Werner, M. Arif, and R. Clothier, Phys. Rev. Lett. **63**, 380 (1989).
[6] K. Sangster, E. A. Hinds, S. M. Barnett, and E. Riis, Phys. Rev. Lett. **71**, 3641 (1993).
[7] X.-G. He and B. H. J. McKellar, Phys. Rev. A **47**, 3424 (1993).
[8] M. Wilkens, Phys. Rev. Lett. **72**, 5 (1994).
[9] H. Wei, R. Han, and X. Wei, Phys. Rev. Lett. **75**, 2071 (1995).
[10] J. Anandan, Phys. Rev. Lett. **85**, 1354 (2000).
[11] J. Anandan, Phys. Lett. A **138**, 347 (1989).
[12] C. Furtado and C. A. de Lima Ribeiro, Phys. Rev. A **69**, 064104 (2004).
[13] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001).
[14] A. Connes, M. R. Douglas, and A. Schwarz, J. High Energy Phys. 9802, 003 (1998).
[15] N. Seiberg and E. Witten, J. High Energy Phys. 9909, 032 (1999).
[16] L. Susskind, e-print arXiv:hep-th/0101029.
[17] O. F. Dayi and A. Jellal, J. Math. Phys. **43**, 4592 (2002).
[18] B. Basu and Subir Ghosh, Phys. Lett. A **346**, 133 (2005).
[19] M. Chaichian, M. M. Sheikh-Jabbari, and A. Tureanu, Phys. Rev. Lett. **86**, 2716 (2001).
[20] M. Chaichian, A. Demichev, P. Presnajder, M. M. Sheikh-Jbbari, and A. Tureanu, Nucl. Phys. B **611**, 383 (2001).
[21] M. Chaichian, P. Presnajder, M. M Sheikh-Jabbari, and A. Tureanu, Phys. Lett. B **527**, 149 (2002).
[22] H. Falomir, J. Gamboa, M. Loewe, F. Méndez, and J. C. Rojas, Phys. Rev. D **66**, 045018 (2002).
[23] K. Li and S. Dulat, Eur. Phys. J. C **46**, 825 (2006).
[24] B. Mirza and M. Zarei, Eur. Phys. J. C **32**, 583 (2004).
[25] K. Li and J. Wang, e-print arXiv:hep-th/0608100.
[26] S. A. Alavi, Phys. Scr., T **7**, 366 (2003).
[27] B. Basu, Subir Ghosh, and S. Dhar, Europhys. Lett. **76**, 395 (2006).
[28] X. Calmet and M. Selvaggi, Phys. Rev. D **74**, 037901 (2006).
[29] Franz Gross, *Relativistic Quantum Mechanics and Field Theory* (Wiley Classics, New York, 1999).
[30] A. J. Silenko, Russ. Phys. J. **48**, 788 (2005).
[31] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).
[32] T. Curtright, D. Fairlie, and C. Zachos, Phys. Rev. D **58**, 025002 (1998).
[33] G. Badurek, H. Weinfurter, R. Gähler, A. Kollmar, S. Wehinger, and A. Zeilinger, Phys. Rev. Lett. **71**, 307 (1993).
[34] K. Li, J. Wang, and C. Chen, Mod. Phys. Lett. A **20**, 2165 (2005).
[35] V. P. Nair and A. P. Polychronakos, Phys. Lett. B **505**, 267 (2001).
[36] J.-Z. Zhang, Phys. Lett. B **584**, 204 (2004).
[37] J. Grundberg, T. H. Hansson, and A. Karlhede, Phys. Rev. D **41**, 2642 (1990).
[38] P. Lévay, J. Math. Phys. **32**, 2347 (1991).
[39] M. Peshkin and H. J. Lipkin, Phys. Rev. Lett. **74**, 2847 (1995).