Complex projection of unitary dynamics of quaternionic pure states

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Quaternionic quantum mechanics has been revealed to be a very useful framework to describe quantum phenomena. In the case of two qubit compound systems we show that the complex projection of quaternionic pure states and quaternionic unitary maps permits the description of interesting phenomena such as decoherence and optimal entanglement generation. The approach, however, presents severe limitations for the case of multipartite or higher dimensional bipartite quantum systems as we point out.

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I. INTRODUCTION

In 1936, using theoretical lattice arguments, Birkhoff and von Neumann [1] concluded that it is possible to consider the set of states of a quantum system as a vector space over real, complex, or quaternionic fields. While the real number formulation of quantum mechanics is essentially equivalent to complex quantum mechanics (CQM) [2], the research on quaternionic quantum mechanics (QQM) began much later with a series of papers by Finkelstein *et al.* [3], in the 1960's, and has been pursued since. A systematic study of QQM is given in Ref. [4], which also contains an interesting list of open problems.

At present, the most serious problem of QQM concerns the description of compound systems, since in this theory the usual definition of Kronecker product of matrices does not hold. Also, the standard definition of tensor product of Hilbert spaces cannot be used, owing to the noncommutativity of the skew field $\mathbb{Q}$. (In order to overcome this difficulty, the concept of the tensor product of quaternionic Hilbert modules has been proposed [5] which allows the description of compound systems on a mathematically well founded basis; unfortunately, the results obtained in this way do not agree in the complex limit with those of standard quantum mechanics [6].)

Experimental tests on QQM were proposed by Peres [7] and carried out by Kaiser *et al.* [8] searching for quaternionic effects manifested through noncommuting scattering phases when a particle crosses a pair of potential barriers. The null result of such tests was understood [9], since the S matrix in quaternionic scattering theory is complex, and different tests were later proposed. For a review of the experimental status of QQM we refer to Ref. [10].

Hence, the possibility of a generalization of quantum mechanics based on quaternion fields instead of complex fields is still controversial. However, the rich structures emerging from such a generalization may be very useful in the description of entanglement, dynamical maps, and decoherence phenomena in quantum physics. From this perspective we approach the analysis of some interesting applications of quaternionic quantum mechanics such as decoherence modeling and optimal entanglement generation for low dimensional systems.

The most general dynamics of the quantum state represented by a complex density matrix ρ_α can be described in terms of a dynamical map [11]

$$\rho_\alpha \rightarrow \mathcal{B}(\rho_\alpha).$$

The dynamical map represents the effect of the coupled (complex) unitary evolution of the system and its environment. In other words,

$$\rho_\alpha^A \rightarrow \rho_\alpha^A(t) = \mathrm{Tr}_B[U\rho_\alpha^A \otimes \rho_\alpha^B U^\dagger] = \mathcal{B}(\rho_\alpha^A),$$

where ρ_α^A is the state of the system and ρ_α^B is that of the environment.

The connection between complex and quaternionic maps has been recently analyzed [12–14] following a seminal idea by Kossakowski [15], and some general results were obtained, together with various hints about two-dimensional states.

In this paper we intend to apply such results to a very general framework of a two qubits dynamics, outlined in Ref. [16], to which we refer for physical motivations (and notations) (see also Ref. [17]). The plan of the paper is the following. In Sec. II we briefly introduce quaternionic quantum mechanics and, in particular, the density matrix formalism and the differential equation that rules their complex projections. In Sec. III a link is drawn between the rank of quaternionic density matrices and that one of their complex projections (proposition 1) and the limitations of our approach for multidimensional quantum systems are also discussed. Then, in Sec. IV, we describe (complex) dynamical maps for the reduced unitary evolution of two qubits in terms of the complex projection of unitary dynamics between quaternionic pure states in two interesting regimes (i.e., decoherence modelling schemes and the creation of maximally entanged Bell states). Finally, some concluding remarks are drawn in the last section.

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II. BASIC NOTATION OF QQM AND DENSITY MATRICES

We recall here some basic notations; properties of quaternionic matrices are exhaustively discussed in Ref. [18]. A (real) quaternion is usually expressed as

$$q = q_0 + q_1 i + q_2 j + q_3 k,$$

where $q_l \in \mathbb{R}$ $(l=0,1,2,3)$, $i^2=j^2=k^2=-1$, $ij=-ji=k$.

The quaternion skew-field $\mathbb{Q}$ is an algebra of rank 4 over $\mathbb{R}$, noncommutative, and endowed with an involutive antiautomorphism (conjugation) such that

$$q \to \bar{q} = q_0 - q_1 i - q_2 j - q_3 k.$$

In a (right) n-dimensional vector space $\mathbb{Q}^n$ over $\mathbb{Q}$, every linear operator is associated in a standard way with a $n \times n$ matrix acting on the left. Moreover, in analogy with the case of vector spaces over $\mathbb{C}$, one can introduce the concepts of unitarity, Hermiticity and so on. We observe that a vector space over $\mathbb{Q}$ can be seen as a natural extension of a vector space over $\mathbb{C}$, since two further "imaginary units," j and k are used together with i.

Every linear operator A acting on $\mathbb{Q}^n$ can be written as $A=A_0+iA_1+jA_2+kA_3$, with A_r $(r=0,1,2,3)$ real matrices, or else $A=A_\alpha+jA_\beta$, where $A_\alpha=A_0+iA_1$ and $A_\beta=A_2-iA_3$.

In QQM, the Schrödinger equation becomes

$$\frac{d}{dt}|\psi\rangle = -H|\psi\rangle,$$

where H is an anti-Hermitian operator with spectral representation $H=\Sigma_a|a\rangle a\langle a|$. The energy observable defined by $|H|=\Sigma_a|a\rangle|a|\langle a|$ admits only positive eigenvalues, and is related to H by the relation $H=I|H|$, with I a phase operator [4].

The density matrix ρ_ψ associated with a pure state $|\psi\rangle$ belonging to a quaternionic n-dimensional right Hilbert space $\mathbb{Q}^n$ is defined by [4]

$$\rho_\psi = |\psi\rangle\langle\psi| \tag{1}$$

and is the same for all normalized ray representatives. By definition, density matrices ρ_ψ associated with pure states, are represented by rank one, positive definite quaternionic Hermitian operators on $\mathbb{Q}^n$ with unit trace. In analogy with CQM, quaternionic mixed states are described by positive quaternionic Hermitian operators (density matrices) ρ on $\mathbb{Q}^n$ with unit trace and rank greater than 1.

The expectation value of a quaternionic Hermitian operator A on a state $|\psi\rangle$ can be expressed in terms of ρ_ψ as [4]

$$\langle A\rangle_\psi = \langle\psi|A|\psi\rangle = \mathrm{Re}\ \mathrm{Tr}(A|\psi\rangle\langle\psi|) = \mathrm{Re}\ \mathrm{Tr}(A\rho_\psi). \tag{2}$$

Expanding $A=A_\alpha+jA_\beta$ and $\rho=\rho_\alpha+j\rho_\beta$ in terms of complex matrices A_α, A_β, ρ_α, and ρ_β, it follows that the expectation value $\langle A\rangle_\psi$ may depend on A_β or ρ_β only if both A_β and ρ_β are different from zero. Indeed,

$$\langle A\rangle_\rho = \mathrm{Re}\ \mathrm{Tr}(A\rho) = \mathrm{Re}\ \mathrm{Tr}(A_\alpha\rho_\alpha - A_\beta^*\rho_\beta), \tag{3}$$

where an asterisk denotes complex conjugation.

Thus, the expectation value of an Hermitian operator A on the state ρ depends on the quaternionic parts of A and ρ, only if both the observable and the state are represented by genuine quaternionic matrices.

However, if an observable O is described by a pure *complex* Hermitian matrix, its expectation value does not depend on the quaternionic part $j\rho_\beta$ of the state $\rho=\rho_\alpha+j\rho_\beta$. Moreover, the expectation value predicted in the standard (complex) quantum mechanics for the state ρ_α coincides with the one predicted in quaternionic quantum mechanics for the state ρ, since

$$\mathrm{Tr}(O\rho_\alpha) = \mathrm{Re}\ \mathrm{Tr}(O\rho_\alpha) = \mathrm{Re}\mathrm{Tr}(O\rho).$$

This simple observation is actually very important in our approach, in that it enables us to merge CQM in the (more general) framework of QQM, without modifying any theoretical prediction (as long as complex observables are taken into account), eluding therefore, or postponing (see Sec. V) any comparison between these theories.

Let us denote by $M(\mathbb{Q})$ and $M(\mathbb{C})$ the space of $n\times m$ quaternionic and complex matrices, respectively, and let $M=M_\alpha+jM_\beta\in M(\mathbb{Q})$. We define the complex projection

$$P\colon M(\mathbb{Q}) \to M(\mathbb{C})$$

by the relation

$$P[M] = \frac{1}{2}[M - iMi] = M_\alpha. \tag{4}$$

Moreover, the probability P_c^ρ that a quaternionic state $\rho=\rho_\alpha+j\rho_\beta$ is complex can be defined as follows:

$$P_c^\rho := \mathrm{Re}\ \mathrm{Tr}(P[\rho]\rho) = \mathrm{Tr}(\rho_\alpha^2). \tag{5}$$

When we consider time-dependent quaternionic unitary dynamics

$$\rho(t) = U(t)\rho(0)U^\dagger(t), \tag{6}$$

where

$$U(t) = (U_\alpha + jU_\beta)(t) = T_o e^{-\int_0^t du H(u)} \tag{7}$$

and T_o denotes the time ordering operator, the differential equation associated with the time evolution for ρ reads

$$\frac{d}{dt}\rho(t) = -[H(t),\rho(t)], \tag{8}$$

where $H(t)=H_\alpha+jH_\beta=-\left[\frac{d}{dt}U(t)\right]U^\dagger(t)$. Finally, Eqs. (6) and (8) reduce to

$$\rho_\alpha(t) = U_\alpha\rho_\alpha(0)U_\alpha^\dagger + U_\beta^*\rho_\alpha^*(0)U_\beta^T + U_\alpha\rho_\beta^*(0)U_\beta^T - U_\beta^*\rho_\beta(0)U_\alpha^\dagger \tag{9}$$

and

$$\frac{d}{dt}\rho_\alpha = -[H_\alpha,\rho_\alpha] + H_\beta^*\rho_\beta - \rho_\beta^* H_\beta, \tag{10}$$

respectively, for the complex projection of the density matrix [13].

III. THE COMPLEX PROJECTION OF QUATERNIONIC DENSITY MATRICES

In this section, we focus our attention on the complex projection ρ_α of quaternionic density matrices $\rho=\rho_\alpha+j\rho_\beta$. First of all, from the Hermiticity of ρ and ρ_α we immediately get

$$\mathrm{Tr}\,\rho_\alpha = \mathrm{Re}\,\mathrm{Tr}\,\rho_\alpha = \mathrm{Re}\,\mathrm{Tr}\rho = \mathrm{Tr}\,\rho,$$

i.e., the complex projection of quaternionic density matrices is trace preserving (alternatively, ρ_β is traceless, being skew symmetric in order to preserve Hermiticity of $j\rho_\beta$). Moreover, we recall that [13] the complex projection ρ_α of any quaternionic density matrix $\rho=\rho_\alpha+j\rho_\beta$ is a complex density matrix. The following statement give us information about the rank of the complex projection ρ_α of any quaternionic density matrix $\rho=\rho_\alpha+j\rho_\beta$.

Proposition 1. Let $\rho=\rho_\alpha+j\rho_\beta$ be a n-dimensional quaternionic density matrix, and let $\mathrm{rank}\rho=m$. Then, $m\leq \mathrm{rank}\rho_\alpha \leq 2m$.

Proof. Let us denote by λ_p the nonzero eigenvalues of ρ [18], arranged in increasing order. By hypothesis, there is a quaternionic unitary transformation $U=U_\alpha+jU_\beta$ such that

$$\rho_m := U\rho U^\dagger = \mathrm{diag}(0,\ldots,0,\lambda_{n-m+1},\ldots,\lambda_n).$$

Then,

$$\rho = \rho_\alpha + j\rho_\beta = U^\dagger\rho_m U = U_\alpha^\dagger\rho_m U_\alpha + U_\beta^\dagger\rho_m U_\beta + j(U_\alpha^T\rho_m U_\beta - U_\beta^T\rho_m U_\alpha).$$

Now, the relaxed form of the Ostrowski theorem [19] implies (with obvious notation)

$$\lambda_p(U_{\alpha,\beta}^\dagger\rho_m U_{\alpha,\beta}) = \theta_p\lambda_p(\rho_m), \quad \theta_p \geq 0, \quad p=1,2,\ldots,n.$$

Since $\lambda_p(\rho_m)=0\ \forall p<n-m+1$ we obtain

$$\mathrm{rank}(U_{\alpha,\beta}^\dagger\rho_m U_{\alpha,\beta}) = l \leq m. \tag{11}$$

On the other hand [20],

$$\lambda_p(U_{\alpha,\beta}^\dagger\rho_m U_{\alpha,\beta}) \leq \lambda_p(\rho_\alpha), \quad p=1,2,\ldots,n. \tag{12}$$

However, according Eq. (11), $\lambda_p(U_{\alpha,\beta}^\dagger\rho_m U_{\alpha,\beta})>0$ whenever $p\geq n-l+1\geq n-m+1$ so that, from Eq. (12) we immediately get $m\leq \mathrm{rank}\,\rho_\alpha$.

Let us finally show that $\mathrm{rank}\,\rho_\alpha\leq 2m$. From the rank inequality,

$$\mathrm{rank}(A+B) \leq \mathrm{rank}A + \mathrm{rank}B$$

and Eq. (11) we immediately get

$$\mathrm{rank}\,\rho_\alpha \leq \mathrm{rank}\,U_\alpha^\dagger\rho_m U_\alpha + \mathrm{rank}\,U_\beta^\dagger\rho_m U_\beta \leq 2m. \quad \blacksquare$$

We recall that also a converse result holds as a consequence of the following proposition [14].

Proposition 2. Let ρ_α be a n-dimensional complex density matrix with $\mathrm{rank}\,\rho_\alpha=m>1$ and let $[x]$ denote the integer part of x. Then, for any m' with $[\frac{m+1}{2}]\leq m'\leq m$ there exists a (skew-symmetric) complex matrix ρ_β such that $\rho=\rho_\alpha+j\rho_\beta$ is a density matrix with $\mathrm{rank}\,\rho=m'$.

As a consequence of the above two propositions, we can conclude that any complex density matrix ρ_α can be obtained as the complex projection of a quaternionic pure density matrix $\rho=\rho_\alpha+j\rho_\beta$ if and only if $\mathrm{rank}\,\rho_\alpha=2$. Then, given any complex density matrix ρ_α such that $\mathrm{rank}\,\rho_\alpha=2$, we can build up a quaternionic density matrix ρ, by adding a suitable $j\rho_\beta$ such that $\mathrm{rank}\rho=1$; two dimensional examples of this fact were given in Refs. [12,13].

Now, let us consider an arbitrary pair of complex density matrices, ρ_α and ρ'_α such that $\mathrm{rank}\rho_\alpha\leq 2$ and $\mathrm{rank}\rho'_\alpha\leq 2$, and let be $\mathcal{B}$ a complex dynamical map

$$\mathcal{B}\colon\ \rho_\alpha \to \rho'_\alpha = \mathcal{B}(\rho_\alpha).$$

According to proposition 2 we can "purify" the complex states ρ_α and ρ'_α by adding suitable purely quaternionic terms $j\rho_\beta$ and $j\rho'_\beta$ respectively. Moreover, since any pair of quaternionic Hermitian matrices admitting the same eigenvalues are unitary equivalent, we immediately obtain that the map $\mathcal{B}$ can be described as the complex projection of a quaternionic unitary map between quaternionic pure states $\rho=\rho_\alpha+j\rho_\beta$ and $\rho'=\rho'_\alpha+j\rho'_\beta$:

$$U\colon\ \rho \to \rho' = U\rho U^\dagger, \quad UU^\dagger = U^\dagger U = \mathbf{1},$$

where $\rho'_\alpha=\mathcal{B}(\rho_\alpha)=P[\rho']$.

In this way, stochastic dynamics of (complex) quantum mechanical systems can be interpreted in terms of the complex projection of unitary dynamics between quaternionic pure states whenever the rank of their complex density matrices is less or equal than 2. Clearly, our approach can also be applied to higher dimensional bipartite quantum systems whenever the rank of the complex density matrices of their components, obtained via partial traces, is not higher than 2 for any time. For multipartite systems the approach applies also whenever the subdynamics obtained by partial traces of all components is described by complex density matrices with rank lower or equal to 2. The simplest example of such case will be examined in the next section.

IV. MIXED $\mathbb{C}$ QUBITS AND PURE $\mathbb{Q}$ QUBITS

Because of the relevance of two qubit quantum gates in quantum information processing, we shall now consider the dynamical maps for the reduced unitary evolution of two $\mathbb{C}$ qubits given in Ref. [16], describing them as the complex projections of unitary dynamics between pure $\mathbb{Q}$ qubits. We recall that, in virtue of Eq. (3), the expectation value of complex observables on the complex mixed state ρ_α or on the quaternionic pure state $\rho=\rho_\alpha+j\rho_\beta$ coincide.

According with propositions 1 and 2 in the preceding section, any complex (mixed) state

$$\rho_\alpha = \frac{1}{2}\begin{pmatrix} 1+a_3 & a_1 - ia_2 \\ a_1 + ia_2 & 1-a_3 \end{pmatrix} \quad (a_i \in \mathbb{R}, \quad 1-a_1^2-a_2^2-a_3^2>0) \tag{13}$$

can be purified (in its most general form), by adding the purely quaternionic Hermitian term

$$j\rho_\beta = j\frac{e^{-i\theta}\sqrt{1-a_1^2-a_2^2-a_3^2}}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \theta \in \mathbb{R}. \tag{14}$$

We outline now a procedure which allows to obtain in a simple way quaternionic unitary transformation connecting two pure $\mathbb{Q}$ qubits. Let $\rho(0)$ and $\rho(t)$ denote the initial and final pure $\mathbb{Q}$ qubits, respectively,

$$\rho(t) = U(t)\rho(0)U^\dagger(t). \tag{15}$$

Then, with reference to the notation in Eq. (13), there exists [4] a suitable (time-dependent) unimodular quaternion $p(t)$ such that

$$p(t)[a_1(t) - ia_2(t) - je^{-i\theta}\sqrt{1-a_1^2(t)-a_2^2(t)-a_3^2(t)}]\bar{p}(t) = a_1(t) + i\sqrt{1-a_1^2(t)-a_3^2(t)}.$$

As a consequence,

$$\rho_c(t) = p(t)\rho(t)\bar{p}(t) = \frac{1}{2}\begin{pmatrix} 1+a_3(t) & a_1(t)+i\sqrt{1-a_1^2(t)-a_3^2(t)} \\ a_1(t)-i\sqrt{1-a_1^2(t)-a_3^2(t)} & 1-a_3(t) \end{pmatrix}$$

is a pure complex state.

Then, we obtain from Eq. (15)

$$\bar{p}(t)\rho_c(t)p(t) = U(t)\bar{p}(0)\rho_c(0)p(0)U^\dagger(t)$$

and finally

$$\rho_c(t) = [p(t)U(t)\bar{p}(0)]\rho_c(0)[p(0)U^\dagger(t)\bar{p}(t)] = U_c(t)\rho_c(0)U_c^\dagger(t), \tag{16}$$

where $U_c(t)=p(t)U(t)\bar{p}(0)$ obviously denotes a complex unitary matrix which satisfy Eq. (16). By solving Eq. (16) and coming back, the quaternionic unitary matrix immediately follows:

$$U(t) = \bar{p}(t)U_c(t)p(0). \tag{17}$$

This computation procedure developed will allows us, in the next subsections, to describe the complex dynamical maps introduced in Ref. [16], in terms of the complex projection of quaternionic unitary dynamics of quaternionic pure states. As we mentioned in the Introduction, we refer to the cited paper for an exhaustive, physical discussion of this problem. The system we will study is composed of two $\mathbb{C}$ qubits A and B, parametrized by the Bloch vectors $\vec{a} \equiv (a_1,a_2,a_3)$ and $\vec{b}\equiv(b_1,b_2,b_3)$, respectively, according to Eq. (13). The most general Hamiltonian for two qubits (in the interaction picture) can be written as

$$H = \sum_i \gamma_i \sigma_i^A \otimes \sigma_i^B,$$

where the parameters γ_1, γ_2, and γ_3 are constant if one assumes that there is no free evolution for individual qubits. The evolution operator U of the overall state ρ^{AB} assumes in this case the simple form

$$U = \prod_{j=1}^{3}[\cos(\gamma_j t)\mathbf{1}^A \otimes \mathbf{1}^B - i\sin(\gamma_j t)\sigma_j^A \otimes \sigma_j^B]. \tag{18}$$

Some particular forms of operator (18) will be looked into in the next subsections.

A. Decoherence modeling

In this subsection, we shall study a kind of evolution whose dynamical map is used in some decoherence modeling schemes. Assume some interaction characterized by $\gamma_1=0$, $\gamma_2=\gamma_3=1$, and consider two initially simply separable states where, in particular, $\rho_\alpha^A(0)$ is (complex) pure while $\rho_\alpha^B(0)$ is fully mixed. The parameters a_i and b_i characterizing $\rho_\alpha^A(0)$ and $\rho_\alpha^B(0)$, respectively, according with Eq. (13), are then

$$a_1 = 1, \quad a_2 = a_3 = b_1 = b_2 = b_3 = 0.$$

More explicitly, the initial complex $\mathbb{C}$ qubits are

$$\rho_\alpha^A(0) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \rho_\alpha^B(0) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the evolution operator of the overall system ρ^{AB} is given by

$$U = [\cos t\mathbf{1}^A \otimes \mathbf{1}^B - i\sin t\sigma_2^A \otimes \sigma_2^B][\cos t\mathbf{1}^A \otimes \mathbf{1}^B - i\sin t\sigma_3^A \otimes \sigma_3^B] = \begin{pmatrix} e^{-it}\cos t & 0 & 0 & i(\cos t - \sin t)\sin t \\ 0 & e^{it}\cos t & -i(\cos t + \sin t)\sin t & 0 \\ 0 & -i(\cos t + \sin t)\sin t & e^{it}\cos t & 0 \\ i(\cos t - \sin t)\sin t & 0 & 0 & e^{-it}\cos t \end{pmatrix}. \quad (19)$$

Hence, the final states (obtained by partial traces from the time-evolved density matrix of the whole system) are [16]

$$\rho_\alpha^A(t) = \frac{1}{2}\begin{pmatrix} 1 & (\cos 2t)^2 \\ (\cos 2t)^2 & 1 \end{pmatrix},$$

$$\rho_\alpha^B(t) = \frac{1}{2}\begin{pmatrix} 1 & (\sin 2t)^2 \\ (\sin 2t)^2 & 1 \end{pmatrix}.$$

As it is evident from the formulas above, subsystems A and B alternatively oscillate between pure and fully mixed states, swapping purity each other. Hence a process of decoherence modelling can be obtained considering a $\mathbb{C}$ qubit A in a pure initial state, which interacts successively with a chain of identical samples B_i of the $\mathbb{C}$ qubit B. If the time interaction τ is short enough, in each step the state of A loses so to say a lot of purity, so that we can obtain as output of these repeated interactions, a fully mixed state. Just this physical situation has been fully exploited in a recent paper [21], where the authors shown rigorously (under some natural assumptions) that the system A approaches an asymptotic state which does not depend on the initial state and satisfies an average second law of thermodynamics.

Let us now describe this dynamics as the complex projection of a quaternionic unitary dynamics between quaternionic pure $\mathbb{Q}$ qubits. By purification, the initial and final pure $\mathbb{Q}$ qubits, respectively, can be easily obtained [see Eqs. (13), (14)]

$$\rho^A(0) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \rho^B(0) = \frac{1}{2}\begin{pmatrix} 1 & -je^{-i\theta} \\ je^{-i\theta} & 1 \end{pmatrix}, \quad \theta \in \mathbb{R}$$

and

$$\rho^A(t) = \frac{1}{2}\begin{pmatrix} 1 & (\cos 2t)^2 \\ (\cos 2t)^2 & 1 \end{pmatrix} + j\frac{e^{-i\varphi}\sqrt{1-(\cos 2t)^4}}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varphi \in \mathbb{R}, \quad (20)$$

$$\rho^B(t) = \frac{1}{2}\begin{pmatrix} 1 & (\sin 2t)^2 \\ (\sin 2t)^2 & 1 \end{pmatrix} + j\frac{e^{-i\theta}\sqrt{1-(\sin 2t)^4}}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (21)$$

The quaternionic unitary evolution operators can be immediately computed by using the computation procedure outlined at the beginning of this section

$$U^A(t) = U_\alpha^A + jU_\beta^A = \begin{pmatrix} (\cos 2t)^2 - je^{-i\varphi}\sqrt{1-(\cos 2t)^4} & 0 \\ 0 & 1 \end{pmatrix}, \quad (22)$$

$$U^B(t) = U_\alpha^B + jU_\beta^B = \begin{pmatrix} \sqrt{1-(\sin 2t)^4} + je^{-i\theta}(\sin 2t)^2 & 0 \\ 0 & 1 \end{pmatrix}. \quad (23)$$

The corresponding anti-Hermitian quaternionic Hamiltonians [4] are, respectively, given by

$$H^A(t) = H_\alpha^A + jH_\beta^A = -\left(\frac{d}{dt}U^A(t)\right)U^{A\dagger}(t) = \begin{pmatrix} j\dfrac{4e^{-i\varphi}\sin 2t\cos 2t}{|\sin 2t|\sqrt{1+(\cos 2t)^2}} & 0 \\ 0 & 0 \end{pmatrix} \quad (24)$$

and

$$H^B(t) = H_\alpha^B + jH_\beta^B = -\left(\frac{d}{dt}U^B(t)\right)U^{B\dagger}(t) = \begin{pmatrix} j\dfrac{-4e^{-i\theta}\sin 2t\cos 2t}{|\cos 2t|\sqrt{1+(\sin 2t)^2}} & 0 \\ 0 & 0 \end{pmatrix}. \quad (25)$$

The expectation value of the (Hermitian) energy observables $|H^A(t)|$ and $|H^B(t)|$ [4] on the states $\rho^A(t)$ and $\rho^B(t)$, respectively, reads

$$\langle |H^A(t)| \rangle_{\rho^A(t)} = \mathrm{Re}\,\mathrm{Tr}(|H^A(t)|\rho^A(t)) = \frac{2|\cos 2t|}{\sqrt{1+(\cos 2t)^2}},$$

$$\langle |H^B(t)| \rangle_{\rho^B(t)} = \mathrm{Re}\,\mathrm{Tr}(|H^B(t)|\rho^B(t)) = \frac{2|\sin 2t|}{\sqrt{1+(\sin 2t)^2}}.$$

From Eqs. (15) and (20)–(23), the complex dynamical maps for the complex projection states ρ_α^A and ρ_α^B can be given in terms of U_α^A, U_α^B, U_β^A, U_β^B and ρ_β^A, ρ_β^B, according with the general formula (9) in Sec. II.

A deeper and more evident physical insight is obtained if one takes into account the differential evolution equation associated with the complex projections of the quaternionic unitary dynamics for the density matrices $\rho_\alpha^A(t)$ and $\rho_\alpha^B(t)$ [see Eq. (10)], which assume the form

$$\frac{d}{dt}\rho_\alpha^A(t) = -[H_\alpha^A,\rho_\alpha^A] + H_\beta^{A*}\rho_\beta^A - \rho_\beta^{A*}H_\beta^A = -\sin 4t\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{26}$$

$$\frac{d}{dt}\rho_\alpha^B(t) = -[H_\alpha^B,\rho_\alpha^B] + H_\beta^{B*}\rho_\beta^B - \rho_\beta^{B*}H_\beta^B = \sin 4t\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{27}$$

Equations (26) and (27) clearly show that the evolutions of subsystems A and B are strictly correlated, as we expects from a physical point of view.

Finally, the probabilities $P_c^{\rho^A}(t)$ and $P_c^{\rho^B}(t)$ that the quaternionic states $\rho^A(t)$ and $\rho^B(t)$ are complex are respectively given by [see Eq. (5)]

$$P_c^{\rho^A}(t) = \text{Re Tr}\{P[\rho^A(t)]\rho^A(t)\} = \frac{1}{2}[1 + (\cos 2t)^4], \tag{28}$$

$$P_c^{\rho^B}(t) = \text{Re Tr}\{P[\rho^B(t)]\rho^B(t)\} = \frac{1}{2}[1 + (\sin 2t)^4] \tag{29}$$

and coincide with the probabilities that the reduced complex density matrices $\rho_\alpha^A(t)$ and $\rho_\alpha^B(t)$ become pure [16], since at the same time the quaternionic terms $\rho_\beta^{A,B}$ vanish [see Eqs. (20) and (21)].

B. Optimal entanglement generation

Another interesting regime to study is related to the creation of maximally entangled Bell states. Assume that we have initially two complex pure states, with Bloch vectors $\vec{a}\equiv(1,0,0)$ and $\vec{b}\equiv(0,1,0)$, respectively. Assume some interaction associated with the following parameters:

$$\gamma_3 = 1, \quad \gamma_1 = \gamma_2 = 0.$$

The initial complex $\mathbb{C}$ qubits are given, respectively, by [16]

$$\rho_\alpha^A(0) = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \quad \rho_\alpha^B(0) = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}$$

and the evolution operator of the overall system assumes in this case the form

$$U = \cos t\mathbf{1}^A \otimes \mathbf{1}^B - i\sin t\sigma_3^A \otimes \sigma_3^B = \begin{pmatrix}e^{-it} & 0 & 0 & 0\\ 0 & e^{it} & 0 & 0\\ 0 & 0 & e^{it} & 0\\ 0 & 0 & 0 & e^{-it}\end{pmatrix}. \tag{30}$$

Then, we obtain by partial traces the final states

$$\rho_\alpha^A(t) = \frac{1}{2}\begin{pmatrix}1 & \cos 2t\\ \cos 2t & 1\end{pmatrix},$$

$$\rho_\alpha^B(t) = \frac{1}{2}\begin{pmatrix}1 & -i\cos 2t\\ i\cos 2t & 1\end{pmatrix}.$$

At the time $t_{\text{Bell}}=\pi/4$, the purity of each $\mathbb{C}$ qubit goes to a minimum and (if we choose a suitable basis for ρ^B) the overall state $\rho^{AB}(t=t_{\text{Bell}})$ is equivalent to a Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. We recall that an entanglement optimization procedure can be then implemented as described in Ref. [16].

As above, let us describe this dynamics as the complex projection of a quaternionic unitary dynamics between quaternionic pure $\mathbb{Q}$ qubits. By purification the initial and final pure $\mathbb{Q}$ qubits, respectively, can be easily obtained:

$$\rho^A(0) = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \quad \rho^B(0) = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}$$

and

$$\rho^A(t) = \frac{1}{2}\begin{pmatrix}1 & \cos 2t\\ \cos 2t & 1\end{pmatrix} + j\frac{e^{-i\varphi}}{2}\begin{pmatrix}0 & -\sin 2t\\ \sin 2t & 0\end{pmatrix},$$

$$\varphi \in \mathbb{R}, \tag{31}$$

$$\rho^B(t) = \frac{1}{2}\begin{pmatrix}1 & -i\cos 2t\\ i\cos 2t & 1\end{pmatrix} + j\frac{e^{-i\theta}}{2}\begin{pmatrix}0 & -\sin 2t\\ \sin 2t & 0\end{pmatrix},$$

$$\theta \in \mathbb{R}. \tag{32}$$

The quaternionic unitary evolution operators read

$$U^A(t) = \begin{pmatrix}\cos 2t + je^{-i\varphi}\sin 2t & 0\\ 0 & 1\end{pmatrix}, \tag{33}$$

$$U^B(t) = \begin{pmatrix}\cos 2t + ke^{-i\theta}\sin 2t & 0\\ 0 & 1\end{pmatrix} \tag{34}$$

and the corresponding anti-Hermitian quaternionic Hamiltonians turn out to be constant:

$$H^A(t) = -\left(\frac{d}{dt}U^A(t)\right)U^{A\dagger}(t) = \begin{pmatrix}-2je^{-i\varphi} & 0\\ 0 & 0\end{pmatrix} \tag{35}$$

and

$$H^B(t) = -\left(\frac{d}{dt}U^B(t)\right)U^{B\dagger}(t) = \begin{pmatrix}-2ke^{-i\theta} & 0\\ 0 & 0\end{pmatrix}. \tag{36}$$

The expectation value of the (Hermitian) energy observables $|H^{A,B}(t)|$ on the states $\rho^{A,B}(t)$ reads

$$\langle|H^{A,B}(t)|\rangle_{\rho^{A,B}(t)} = \text{Re Tr}[|H^{A,B}(t)|\rho^{A,B}(t)] = 1.$$

We remark that, in this case, the quaternionic unitary operators $U^A(t)$ and $U^B(t)$ satisfy a one-parameter semigroup composition law $U(t)U(t')=U(t+t')$ for all t, t'. The time evolution of the density matrices $\rho_\alpha^{A,B}(t)$ is ruled by the differential equations

$$\frac{d}{dt}\rho_\alpha^A(t) = -[H_\alpha^A,\rho_\alpha^A] + H_\beta^{A*}\rho_\beta^A - \rho_\beta^{A*}H_\beta^A = \sin 2t\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{37}$$

$$\frac{d}{dt}\rho_{\alpha}^{B}(t) = -[H_{\alpha}^{B},\rho_{\alpha}^{B}] + H_{\beta}^{B*}\rho_{\beta}^{B} - \rho_{\beta}^{B*}H_{\beta}^{B} = \sin 2t\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{38}$$

The probabilities $P_c^{\rho^A}(t)$ and $P_c^{\rho^B}(t)$ that the quaternionic states $\rho^A(t)$ and $\rho^B(t)$ are complex are, respectively, given by [see Eq. (5)]

$$P_c^{\rho^A}(t) = \text{Re Tr}\{P[\rho^A(t)]\rho^A(t)\} = \frac{1}{2}[1 + (\cos 2t)^2], \tag{39}$$

$$P_c^{\rho^B}(t) = \text{Re Tr}\{P[\rho^B(t)]\rho^B(t)\} = \frac{1}{2}[1 + (\cos 2t)^2] \tag{40}$$

and coincide with the probabilities that the reduced complex density matrices $\rho_{\alpha}^{A}(t)$ and $\rho_{\alpha}^{B}(t)$ become pure [16], since at the same time the quaternionic terms $\rho_{\beta}^{A,B}$ vanish [see Eqs. (31) and (32)].

V. CONCLUDING REMARKS

The main implication of the above results is very surprising. For any compound system made of two $\mathbb{C}$ qubits, each subsystem can be described by a pure $\mathbb{Q}$ qubit, which undergoes a unitary quaternionic time evolution. Hence, one can attribute to each subsystem "individual" properties, in contrast to what happens in the realm of CQM where reduced density matrices do not allow a similar interpretation. Nevertheless, the correlations between subsystems do not disappear at all, but are implicitly taken into account in such individual evolutions, as the examples analyzed above point out. These results point to an apparently puzzling situation, in which the same state of a physical system is entangled in CQM, while it seems to be "separable" in QQM. Moreover, the physical interpretation of the interaction discussed in Sec. IV A that can be given in the realm of CQM ("decoherence modeling") is untenable in QQM, where the two $\mathbb{Q}$ qubits are described by pure states all the time even under the effect of interaction. The apparent purity loss can be simply attributed to experimental inadequacies (we recall once again that the influence of quaternionic terms ρ_{β} can only be revealed by measuring genuinely quaternionic observables, see Sec. II).

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