Resonant-state expansions and the long-time behavior of quantum decay

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(Received 9 January 2007; published 3 July 2007)

It is shown that a representation of the decaying wave function as a resonant sum plus a nonexponential integral term may be written as a purely discrete resonant sum by evaluating at long times the integral term by the steepest descents method, and then expanding the resulting expression in terms of resonant states. This leads to a representation that is valid along the exponential and the inverse power in time regimes. A model calculation using the δ potential allows us to make a comparison of the expansion with numerical integrations in terms of continuum wave functions and, in the long time regime, with an exact analytic expression of the integral term obtained using the steepest descents method. The results demonstrate that resonant states give a correct description of the long-time behavior of decay.

DOI: 10.1103/PhysRevA.76.012103 PACS number(s): 03.65.Ca, 73.40.Gk

I. INTRODUCTION

Theoretical treatments of quantum decay refer to the time evolution $|\psi_t\rangle = \exp(-iHt/\hbar)|\psi_0\rangle$, of an initial state $|\psi_0\rangle$ in a system characterized by a Hamiltonian H. In some decay problems it is not suitable to separate the Hamiltonian into a part with stationary states and a part responsible for the decay, that is treated to some order of perturbation, but rather to consider the full Hamiltonian H of the system. This is usually the case when the decay originates from tunneling through a classically forbidden region. Some models corresponding to this situation consider a single particle that decays from a confining potential. As it is well known, following the work by Khalfin [1], if the energy spectra E of the system is bounded by below, i.e., $E \in (0,\infty)$, the exponential decay law cannot hold at long times. This is a common feature of a vast number of natural and artificial quantum systems. At short times there is also a departure from the exponential behavior that is related, however, to the existence of the energy moments of the Hamiltonian H [2]. The short-time behavior has been the subject of much discussion, particularly in connection with the Zeno effect [3]. The experimental verification of the departure from the exponential decay law remained elusive for decades [4,5]. A few years ago, however, it was verified in the short-time regime [6] and very recently, in the long-time regime [7]. These experimental results contradict theoretical claims of the seventies, characterized by considering the influence of the measurement apparatus on the decay process, that predicted that exponential decay should hold at all times [8–10]. Recent work on the role of the distance between the detector, modeled as an absorbing potential, and the initial decaying system shows that possible perturbing effects of measurement disappear by increasing the distance to the detector as well as by improving the detector efficiency [11]. This result favors the assumption, made in this work as in other recent one [12,13], that the decaying particle evolves with time undisturbed until it is detected. It is also worth mentioning that the $1/t^3$ long time behavior has also been derived in other fields involving quantum decay as radiative decay of atoms in photonic crystals [14] and in the decay of a local spin excitation in an inhomogeneous spin chain [15].

In addition to the decaying wave function, two other quantities of interest in studies of quantum decay are the survival and the nonescape probabilities. A common approach to calculate these quantities is to expand the decaying wave function in terms of the set of bound and continuum wave functions of the problem [12,13]. This requires numerical integration techniques, which in general are computationally time consuming and, more importantly, provide no deep physical insight on the decay process. Another approach is to consider a purely discrete analytical expansion in terms of a linear combination of resonant states and Moshinsky functions [16,17]. Using this approach, it was reported that at long times the survival probability goes as $\sim t^{-3}$, and the nonescape probability as $\sim t^{-1}$ [17]. This last result led to controversy [18–23]. A reexamination of the long time behavior of the nonescape probability using the resonant state formalism settled down the controversy showing that in fact at long times it goes as $\sim t^{-3}$ and that the previous result originated from an ambiguity in the calculation [23]. Some authors, however, have affirmed that the outgoing boundary condition character of resonant states gives an incorrect description of the long time behavior of decay [21,22]. This, in spite of the correct result obtained for the long time behavior of the survival probability. Although it is well known that resonant states form a complete representation along the internal region of the interaction [24–26], it is, however, of interest to give an answer to the above assertion in a time-dependent context.

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This work is therefore focused on the exponential and long-time regimes of quantum decay. We consider an approach where the decaying wave function is written as a sum over exponentially decaying resonant terms plus a nonexponential integral contribution [16]. The long time contribution is obtained by evaluating the integral term by the method of steepest descents [27]. This long-time contribution may be also expanded in terms of resonant states and shown to coincide exactly with the long time limit of the purely discrete expansion involving Moshinsky functions. In order to exhibit the validity of the long time expansion of the decaying wave function in terms of resonant states, we consider the exactly solvable δ potential, which allows us to make a comparison in this case with the steepest descents analytical result. We also evaluate the survival and the nonescape probabilities by numerical integration and compare it with the corresponding resonant expansions and the analytical steepest descents results. We also investigate the extent of validity of the one-term approximations of the survival and nonescape probabilities.

The organization of the paper is as follows. In Sec. II, the long-time resonant expansion of the wave function is derived using the steepest descents method. Also, analytical expressions for the resonant expansions of the survival and nonescape probabilities are presented. Section III considers the solvable model of the δ potential: In Sec. III A, the resonant states, complex energy poles, and relevant coefficients to evaluate the resonant expansions of the previous section, are obtained. Section III B deals with the derivation of the exact analytical expressions for the steepest descents expressions for the wave function and the survival and nonescape probabilities. In Section III C the expansions of the these quantities in terms of continuum wave functions is given. Section III D provides the result of calculations among the different approaches. Finally, Sec. IV provides the concluding remarks.

II. RESONANT STATE EXPANSION USING THE STEEPEST DESCENTS METHOD

Let us consider the time evolution of decay of an initial wave function $\psi(r,0)$ confined initially, at $t=0$, along the internal region of a spherically symmetric potential of finite range, i.e., $V(r)=0$ for $r>a$, where, for the sake of simplicity, we restrict the discussion to s waves and the units employed are $\hbar=2m=1$, m being the mass of the decaying particle. As a consequence, the energy of the particle is denoted by $E=k^2$, with k the corresponding wave number. As is well known, the time evolved wave function $\psi(r,t)$ may be written in terms of the retarded Green function of the problem $g(r,r';t)$ and the initial state $\psi(r,0)$ as

$$\psi(r,t) = \int_0^a g(r,r';t)\psi(r',0)dr'. \tag{1}$$

Moreover, the retarded Green function $g(r,r';t)$ may be written, using Laplace transform techniques, as [16,17]

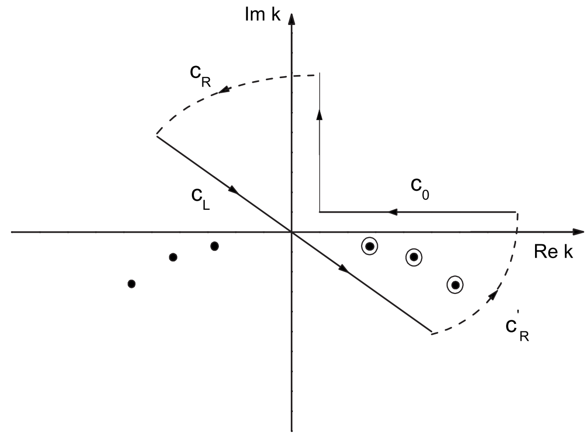

FIG. 1. Deformation of the contour C_0' in the complex k plane.

$$g(r,r';t) = \frac{i}{2\pi}\int_{C_0'} G^+(r,r';k)e^{-ik^2t}2kdk, \tag{2}$$

where $G^+(r,r';k)$ corresponds to the outgoing Green's function of the problem and the integration contour C_0' goes along the first quadrant of the complex k plane from $\gamma+i\infty$ to $\infty+i\gamma$, with γ a small constant. Since the decay refers to a process where the particle tunnels out into the continuum, here we assume, for the sake of simplicity, that the potential does not hold bound nor antibound poles. In order to evaluate Eq. (2) it is convenient to consider a closed contour formed by the sum of contours depicted in Fig. 1 and apply Cauchy's theorem. Note here, from Eq. (2), that $-1/2\pi i$ times the integral over the contour C_0' is equivalent to $1/2\pi i$ times the integral over the contour C_0 depicted in Fig. 1. We choose the path C_L as a straight line 45° off the real k axis that passes through the origin. In doing so, one picks up the residues $\rho_p(r,r')$ of the outgoing Green function at the proper complex poles k_p, i.e., such that $\mathrm{Re}[k_p]>\mathrm{Im}[k_p]$. The residue may be written in terms of the resonant states $u_p(r)$ as

$$\rho_p(r,r') = \frac{u_p(r)u_p(r')}{2k_p}, \tag{3}$$

provided the resonant states are normalized as

$$\int_0^a u_p^2(r)dr + i\frac{u_p^2(a)}{2k_p} = 1. \tag{4}$$

Taking the limit of the semicircle radii C_R and C_R' to infinity, and noting that the factor $\exp(-ik^2t)$ in the corresponding integrands guarantees that the contribution of these contours vanish in that limit, allows to rewrite Eq. (2) as a sum over exponentially decaying terms plus an integral contribution

$$g(r,r';t) = \sum_{p=1}^{\infty} u_p(r)u_p(r')e^{-ik_p^2 t} + \frac{i}{2\pi}\int_{C_L} G^+(r,r';k)e^{-ik^2 t}2kdk. \qquad (5)$$

Resonant states satisfy the Schrödinger equation of the problem with outgoing boundary conditions. This implies that the energy eigenvalues are complex, i.e., $k_p^2 = E_p = \mathcal{E}_p - i\Gamma_p/2$, where $\mathcal{E}_p$ represents the resonance position and Γ_p the corresponding decay width. The distribution of the complex poles of $G^+(r,r';k)$ along the complex k plane is well known for potentials that vanish beyond a distance: There is an infinite number of them and are seated on the lower half of the k plane [28]. At a given pole $k_p = \alpha_p - i\beta_p$, there corresponds, from time reversal considerations, a pole at $k_{-p} = -k_p^*$, which is related to the resonant state $u_{-p}(r) = u_p^*(r)$ [29]. The outgoing Green function may be expanded as an infinite sum over the full set of resonant states along the internal region of the interaction, $r < a$ and $r' < a$, and holds also if either r or r' are evaluated at $r=a$, but not both [24–26,30]. If $r_<$ and $r_>$ stand, respectively, for the smaller and larger of r and r', we denote the above conditions by the notation $[r_<, r_>) \leq a$, so the expansion reads

$$G^+(r,r';k) = \sum_{p=1}^{\infty} \left\{ \frac{u_p(r)u_p(r')}{2k_p(k-k_p)} - \frac{u_p^*(r)u_p^*(r')}{2k_p^*(k+k_p^*)} \right\}. \qquad (6)$$

The above representation for $G^+(r,r';k)$ satisfies the closure relation for resonant states [16,17]

$$\mathrm{Re}\left\{ \sum_{p=1}^{\infty} u_p(r)u_p(r') \right\} = \delta(r-r'); \quad [r_<, r_>) \leq a \qquad (7)$$

and the sum rules

$$\mathrm{Im}\left\{ \sum_{p=1}^{\infty} \frac{u_p(r)u_p(r')}{k_p} \right\} = 0; \quad [r_<, r_>) \leq a \qquad (8)$$

and

$$\mathrm{Im}\left\{ \sum_{p=1}^{\infty} u_p(r)u_p(r')k_p \right\} = 0; \quad [r_<, r_>) \leq a. \qquad (9)$$

In what follows we consider the span of values of r and r' as indicated above.

The substitution of Eq. (6) into Eq. (5) leads to the representation of the retarded Green function $g(r,r';t)$ as a linear combination of resonant states and Moshinsky functions as discussed in Refs. [16,17]. The long time $t^{-3/2}$ behavior of $g(r,r';t)$ is obtained by expanding the Moshinsky functions at long times, which go as $a/(k_p t^{1/2}) + b/(k_p t^{3/2}) + \cdots$, with a and b constants, and then using the sum rule given by Eq. (8) to eliminate the $t^{-1/2}$ contribution. This is straightforward for the decaying function and the survival amplitude, but not for the nonescape probability, which led to the controversy mentioned in the introduction. For the nonescape probability we know now that the coefficient multiplying the term that goes as t^{-1} vanishes exactly and hence the leading term of this quantity at long times goes as t^{-3} [23]. The above considerations are relevant for numerical calculations because the sums of the distinct quantities involve always a finite number of terms, and hence, the corresponding coefficients of the long-time inverse power contributions as $t^{-1/2}$ or t^{-1} are finite and remain the leading contribution unless they are subtracted explicitly from the calculation.

Here we follow an alternative approach to obtain the behavior of $g(r,r';t)$ at long times. This procedure does not rely on using the above sum rules and leads directly to the correct long-time asymptotic behavior. It exploits the fact that at long times the integrand over the k integral in Eq. (5) oscillates widely and hence it may be evaluated, to a very good approximation, by the steepest descents method. One sees that the saddle point of the exponential in Eq. (5) is at $k=0$ and hence one may perform a Taylor expansion of $G^+(r,r';k)$ around that value, namely,

$$G^+(r,r';k) = G^+(r,r';0) + k\left[\frac{\partial}{\partial k}G^+(r,r';k)\right]_{k=0} + \cdots . \qquad (10)$$

Substitution of Eq. (10) into the integral term in Eq. (5) leads to an expression where one sees that the term proportional to $G^+(r,r';0)$ vanishes because the integral over k is odd. The k integral for the next term in the Taylor expansion may be evaluated by making the change of variable $k=\sqrt{-iu}$, which gives the leading term as the inverse power in time $t^{-3/2}$. Consequently, at long times, Eq. (5) may be written approximately as

$$g(r,r';t) \approx \sum_{p=1}^{\infty} u_p(r)u_p(r')e^{-ik_p^2 t} + \eta\left\{\frac{\partial}{\partial k}G^+(r,r';k)\right\}_{k=0}\frac{1}{t^{3/2}}; \quad [r_<, r_>) \leq a, \qquad (11)$$

where $\eta = 1/(4\pi i)^{1/2}$. In general, it is difficult to obtain a closed analytical expression for the factor $[\partial G^+(r,r';k)/\partial k]_{k=0}$. An example where this is possible will be considered in the next section. One may use, however, the expansion of the outgoing Green function given by Eq. (6) to evaluate this factor and write Eq. (11) as

$$g(r,r';t) \approx \sum_{p=1}^{\infty} u_p(r)u_p(r')e^{-ik_p^2 t} - i\eta\,\mathrm{Im}\left\{\sum_{p=1}^{\infty}\frac{u_p(r)u_p(r')}{k_p^3}\right\}\frac{1}{t^{3/2}}; \quad [r_<, r_>) \leq a. \qquad (12)$$

Then, inserting Eq. (12) into Eq. (1) yields the expression for the time dependent wave function

$$\psi(r,t) \approx \sum_{p=1}^{\infty} C_p u_p(r) e^{-i\mathcal{E}_p t} e^{-\Gamma_p t/2} - i\eta\, \mathrm{Im}\left\{\sum_{p=1}^{\infty} \frac{C_p u_p(r)}{k_p^3}\right\} \frac{1}{t^{3/2}}; \quad r \leq a, \tag{13}$$

where the exponentially decaying terms are written explicitly in terms of $\mathcal{E}_p$ and Γ_p and the coefficients C_p are defined as

$$C_p = \int_0^a \psi(r,0) u_p(r) dr. \tag{14}$$

Assuming that the initial state $\psi(r,0)$ is normalized to unity, it then follows from the closure relation given by Eq. (7) that

$$\mathrm{Re}\{C_p \bar{C}_p\} = 1, \tag{15}$$

where

$$\bar{C}_p = \int_0^a \psi^*(r,0) u_p(r) dr. \tag{16}$$

Equation (13) provides the time evolution of the decaying wave function as an expansion in terms of resonant states along the exponentially decaying and long-time regimes. It is convenient for the discussion of the next section to write separately the term corresponding to the long-time contribution of the decaying wave as

$$\psi_{\mathrm{long}}(r,t) \approx i\eta D_p \frac{1}{t^{3/2}} + \cdots; \quad r \leq a, \tag{17}$$

where the coefficient D_p is

$$D_p = -\mathrm{Im}\left\{\sum_{p=1}^{\infty} \frac{C_p u_p(r)}{k_p^3}\right\}. \tag{18}$$

The subindex of the coefficient D_p in Eqs. (17) and (18) denotes the number of poles p used in the evaluation of the corresponding summations.

Expansion of the survival and nonescape probabilities. The survival amplitude gives the probability amplitude that at time t the decaying particle remains in the initial state

$$A(t) = \int_0^a \psi^*(r,0)\psi(r,t) dr. \tag{19}$$

Substitution of Eq. (13) into Eq. (19) yields

$$A(t) \approx \sum_{p=1}^{\infty} C_p \bar{C}_p e^{-i\mathcal{E}_p t} e^{-\Gamma_p t/2} - i\eta\, \mathrm{Im}\left[\sum_{p=1}^{\infty} \frac{C_p \bar{C}_p}{k_p^3}\right] \frac{1}{t^{3/2}}. \tag{20}$$

Consequently, the survival probability $S(t)=|A(t)|^2$ may be written as

$$S(t) \approx \sum_{p,s=1}^{\infty} (C_p \bar{C}_p)(C_s^* \bar{C}_s^*) e^{-i(\mathcal{E}_p - \mathcal{E}_s)t} e^{-(\Gamma_p + \Gamma_s)t/2} - 2\,\mathrm{Im}\left[\eta^* \sum_{p=1}^{\infty} C_p \bar{C}_p e^{-i\mathcal{E}_p t} e^{-\Gamma_p t/2}\right] \mathrm{Im}\left[\sum_{p=1}^{\infty} \frac{C_p \bar{C}_p}{k_p^3}\right] \frac{1}{t^{3/2}} + |\eta|^2\, \mathrm{Im}\left[\sum_{p=1}^{\infty}\left(\frac{C_p \bar{C}_p}{k_p^3}\right)^2\right] \frac{1}{t^3}. \tag{21}$$

The nonescape probability yields the probability that at time t the particle still remains within the confining region of the potential,

$$P(t) = \int_0^a \psi^*(r,t)\psi(r,t) dr. \tag{22}$$

Hence, substitution of Eq. (13) into Eq. (22) gives

$$P(t) \approx \sum_{p,s=1}^{\infty} C_p C_s^* I_{ms} e^{-i(\mathcal{E}_p - \mathcal{E}_s)t} e^{-(\Gamma_p+\Gamma_s)t/2} - \mathrm{Re}\left\{\eta \sum_{p,s=1}^{\infty}\left[\frac{C_p C_s^*}{k_p^3} I_{ps} - \frac{C_p^* C_s^*}{(k_p^3)^*} \bar{I}_{ps}\right] e^{i\mathcal{E}_s t} e^{-\Gamma_s t/2}\right\} \frac{1}{t^{3/2}} + \frac{1}{2}|\eta|^2\, \mathrm{Re}\left\{\sum_{p,s=1}^{\infty}\left[\frac{C_p C_s^*}{k_p^3 (k_s^3)^*} I_{ps} - \frac{C_p^* C_s^*}{(k_p^3)^*(k_s^3)^*} \bar{I}_{ps}\right]\right\} \frac{1}{t^3}, \tag{23}$$

where the coefficients I_{ps} and $\bar{I}_{ps}$ are given, respectively, by

$$I_{ps} = \int_0^a u_p(r) u_s(r)^* dr = \frac{u_p(a) u_s^*(a)}{i(k_p - k_s^*)} \tag{24}$$

and

$$\bar{I}_{ps} = \int_0^a u_p(r)^* u_s(r)^* dr = \delta_{ps} - \frac{u_p^*(a) u_s^*(a)}{i(k_p^* + k_s^*)}. \tag{25}$$

Equations (21) and (23) provide, respectively, resonant expansions that are valid along the exponential and long-times regimes for decay.

III. MODEL

A convenient model to study the time evolution of quantum decay is the δ potential. This model was considered many years ago by Winter [33] and since then by many authors. The reason being that its mathematical simplicity does not prevent that it describes correctly the main physical features of the time evolution of decay along the exponential and nonexponential long-time regimes.

We consider a δ-potential of radius a and intensity λ, namely,

$$V(r) = \lambda\,\delta(r-a), \tag{26}$$

and as initial state, the simple analytical expression provided by the infinite box state

$$\psi(r,0) = \left(\frac{2}{a}\right)^{1/2} \sin\left(\frac{q\pi r}{a}\right), \tag{27}$$

where $q=1,2,\ldots$.

In spite of its mathematical simplicity the infinite box initial state possesses the main physical features of presumably more realistic initial states and it has the advantage, as shown explicitly below, that it provides simple analytic expressions for the expansion coefficients defined by Eqs. (14) and (16).

A. Complex poles and resonant states

The resonant states of the problem obey the Schrödinger equation of the problem with complex energy eigenvalues. They read

$$u_p(r) = \begin{cases} A_p \sin(k_p r), & r \leqslant a, \\ B_p e^{ik_p r}, & r \geqslant a, \end{cases} \tag{28}$$

From the continuity of the above solutions and the discontinuity of its derivatives with respect to r (due to the δ-function interaction) at the boundary value $r=a$, it follows that the k_p's satisfy the equation

$$2ik_p + \lambda(e^{2ik_p a} - 1) = 0. \tag{29}$$

For $\lambda > 1$ one may write the approximate analytical solutions to Eq. (29) as

$$k_p \approx \frac{p\pi}{a}\left(1 - \frac{1}{\lambda a}\right) - i\frac{1}{a}\left(\frac{p\pi}{\lambda a}\right)^2. \tag{30}$$

Using the above expression for k_p as the initial value in the Newton-Rapshon method, i.e., $k_p^{r+1}=k_p^r - F(k_p^r)/\dot{F}(k_p^r)$, with $\dot{F}=[dF/dk]_{k=k_p}$ yield the solutions k_p with the desired degree of approximation according to the number of iterations.

The normalization coefficient of resonant states may be evaluated by substitution of Eq. (28), for $r \leqslant a$, into Eq. (4), to obtain the analytical expression

$$A_p = \left[\frac{2\lambda}{\lambda a + e^{-2ik_p a}}\right]^{1/2}. \tag{31}$$

Similarly, using Eqs. (28), (31), and (27) into Eq. (14) allows one to write the expansion coefficient C_p as the analytic expression

$$C_p = \left[\frac{\lambda a}{\lambda a + e^{-2ik_p a}}\right]^{1/2}\left[\frac{2q\pi \sin(k_p a)(-1)^q}{k_p^2 a^2 - q^2\pi^2}\right]. \tag{32}$$

It is worth noticing that as the intensity of the potential $\lambda \to \infty$, the complex poles $k_p=\alpha_p - i\beta_p$ tend to the real infinite box eigenvalues, and similarly, the resonant eigenfunctions $u_p(r)$ tend to the infinite box model eigenfunctions. This means that for a finite value of the intensity λ, an initial state $\psi(r,0)$ with $q=p$, is closer to the resonant state $u_p(r)$ than to any other resonant state.

For a given finite value of the intensity λ and the radius a of the δ potential one may then evaluate the set of complex poles $\{k_p\}$ and the expansion coefficients $\{C_p\}$, note that in this case $C_p=\bar{C}_p$, which are the required input to calculate the different quantities of interest for the time evolution of decay.

B. Exact analytic expression for the steepest descents contribution

As is well known, the outgoing Green function $G^+(r,r';k)$ may be expressed in terms of the regular function $\phi(k,r)$, the Jost function $f_+(k,r)$, and the Jost solutions as [28]

$$G^+(r,r';k) = -\frac{\phi(k,r_<)f_+(k,r_>)}{J_+(k)}, \tag{33}$$

where, as indicated above, $r_<$ and $r_>$ stand, respectively, for the smaller and larger of r and r'. For the δ potential one may obtain a simple exact analytical expression for $G^+(r,r';k)$. Choosing, $r<r'$ and $r' \leqslant a$, we find [28]

$$\phi(k,r) = \frac{\sin(kr)}{k}, \tag{34}$$

$$f_+(k,r') = e^{ikr'} - \frac{\lambda}{k}\sin[k(r'-a)]e^{ika}, \tag{35}$$

$$J_+(k) = 1 + \frac{\lambda}{k}\sin(ka)e^{ika}, \tag{36}$$

and hence $G^+(r,r';k)$ may be written as

$$G^+(r,r';k) = -\frac{\sin(kr)}{k} \times \left[\frac{\exp(ikr') - (\lambda/k)\sin[k(r'-a)]\exp(ika)}{1 + (\lambda/k)\sin(ka)\exp(ika)}\right]. \tag{37}$$

Note that the poles of $G^+(r,r';k)$, which follow from the vanishing of $J_+(k)$ given by Eq. (36), correspond precisely to the condition given by Eq. (29).

From the Eq. (37) one may obtain after some simple algebra

$$\left\{\frac{\partial}{\partial k}G^+(r,r';k)\right\}_{k=0} = -i\frac{rr'}{(1+\lambda a)^2}. \tag{38}$$

Substitution of Eq. (38) into Eq. (11) allows to write, using Eq. (1), $\psi(r,t)$ at asymptotically long times as

$$\psi_{\rm long}(r,t) \approx i\eta D_e(r)\frac{1}{t^{3/2}}, \tag{39}$$

where $D_e(r)$ stands for

$$D_e(r) = -\frac{r}{(1+\lambda a)^2}\int_0^a r\psi(r,0)dr = -\frac{r}{(1+\lambda a)^2}\mathcal{C}(q), \tag{40}$$

with

$$\mathcal{C}(q) = -\frac{a\sqrt{2a}}{q\pi}(-1)^q. \tag{41}$$

Using Eqs. (39)–(41) and the definition of the survival amplitude, given by Eq. (19), gives the following expression for the survival probability at asymptotically long times:

$$S_{\text{long}}(t) \approx \frac{|\eta|^2 \mathcal{C}^4(q)}{(1+\lambda a)^4}\frac{1}{t^3} \tag{42}$$

and analogously, using instead of Eq. (19), Eq. (22), the expression for nonescape probability at asymptotically long times

$$P_{\text{long}}(t) \approx \frac{|\eta|^2 a^3 \mathcal{C}^2(q)}{3(1+\lambda a)^4}\frac{1}{t^3}. \tag{43}$$

C. Expansions in terms of continuum wave functions

As is well known, the time evolution of the decaying wave solution given by Eq. (1) may also be calculated by expanding the retarded Green function in terms of the complete set of continuum wave functions of the problem, the so called *physical* wave solutions $\psi^+(k,r)$ to obtain

$$\psi(r,t) = \int_0^\infty C(k)\psi^+(k,r)e^{-ik^2t}dk, \tag{44}$$

where

$$C(k) = \int_0^a \psi(r,0)^* \psi^+(r,k)dr \tag{45}$$

and

$$\psi^+(k,r) = \sqrt{\frac{2}{\pi}}\begin{cases}\sin(kr)/J_+(k), & r \leqslant a,\\ (i/2)[e^{-ikr} - \zeta(k)e^{ikr}], & r \geqslant a,\end{cases} \tag{46}$$

where the S matrix $\zeta(k)=J_-(k)/J_+(k)$, with $J_-(k)=J_+^*(k)$. One may evaluate analytically Eq. (45) using Eqs. (27) and (46). Substitution of Eq. (44) into Eq. (19) allows one to write an expression for the survival probability as an expansion in terms of continuum wave functions, namely,

$$S(t) = \int_0^\infty |C(k)|^2 e^{-ik^2t}dk. \tag{47}$$

Similarly, using Eq. (44) into Eq. (22) allows also to write the nonescape probability as an expansion in terms of continuum wave functions

$$P(t) = \int_0^\infty dk' \int_0^\infty dk C^*(k')C(k) \times \int_0^a dr[\psi^+(k',r)]^* \psi^+(k,r)e^{-i(k^2-k'^2)t}. \tag{48}$$

The above expressions for the survival and nonescape probabilities may be evaluated by numerical integration in a similar fashion as discussed elsewhere [12,13]. Evidently these expressions do not provide insight into the physics of the time evolution of decay.

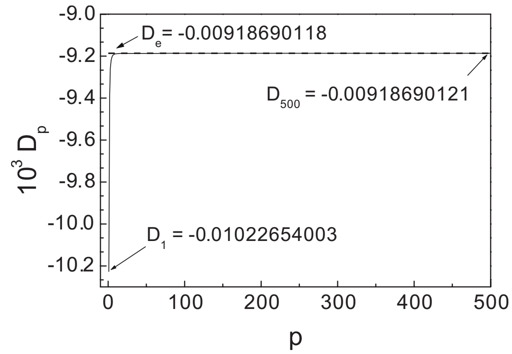

FIG. 2. Coefficients D_p vs p and D_e given, respectively, by Eqs. (18) and (40).

D. Results

It is of interest to make a comparison between Eqs. (17) and (39) for the asymptotic long-time expression of the decaying wave function $\psi_{\text{long}}(r,t)$. As discussed above, they differ by the values of the coefficients D_p and D_e, given respectively, by Eqs. (18) and (40). The former coefficient being obtained using the resonant expansion, and the latter by using the steepest descents method. The parameters of the δ potential are $\lambda=6$ and $a=1$, and the initial state $\psi(r,0)$ is chosen with $q=1$. Figure 2 provides a plot of D_p (full line) as a function of the number of resonant poles, from $p=1$ up to $p=500$ evaluated at the boundary value $r=a$. The value of D_e is indicated by the dashed line. We see that there is an excellent agreement between both calculations. Note that already the value $p=1$ provides a good approximation for D_e.

Figure 3 exhibits a plot of the natural logarithm of the survival probability $S(t)$ as a function of time, using the resonant expansion given by Eq. (21) (solid line) in the one-term approximation $p=1$. The plotted curve is already undistinguishable from the result of a purely brute force numerical integration using Eq. (47) (dashed-dotted line). The long-time asymptotic steepest descents result given by Eq. (42) is also plotted (dotted-line). One sees that it approaches the

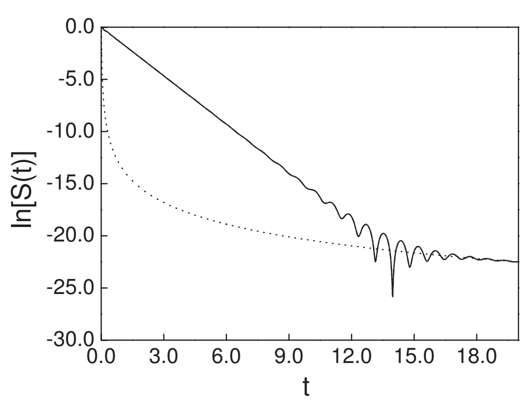

FIG. 3. Survival probability $S(t)$ as a function of time. The resonant expansion (solid line) and numerical integration (dashed-dotted line) calculations are undistinguishable. Also shown is the long-time steepest descents calculation (dotted-line). The parameters of the δ potential are $\lambda=6$ and $a=1$. See text.

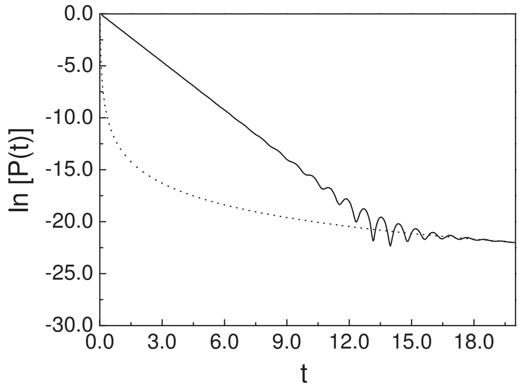

FIG. 4. Nonescape probability $P(t)$ as a function of time. The resonant expansion (solid line) and numerical integration (dashed-dotted line) calculations are undistinguishable. Also shown is the long-time steepest descents calculation (dotted-line). The parameters of the δ potential are $\lambda=6$ and $a=1$. See text.

other two curves at long times. The parameters of the δ potential employed in the calculation are $\lambda=6$ and $a=1$.

In an analogous way, using the same potential parameters, Fig. 4, exhibits a plot of the natural logarithm for the nonescape probability $P(t)$ in terms of the resonant expansion given Eq. (23) (solid line), also in the one-term $p=1$ approximation, as in the previous case. Again this curve is undistinguishable from the curve that results by numerical integration of Eq. (48) (dashed-dotted line). It is worth mentioning that in this case the numerical integration was performed by using a FORTRAN IMSL integration routine (DTWODQ) for a two dimensional integral based on the Gauss-Konrod quadrature method [31]. In general, however, the possibility of having analytical solutions of known functions not only provides more insight into the physics of the problem, but also, more versatile and faster computational tools. The long-time steepest descents calculation, using Eq. (43), is also plotted (dotted line). It approaches as in the previous case, the other curves at long times.

The excellent agreement obtained using the one-term approximation for the survival and nonescape probabilities in the above calculations, is due to the fact that in each case there is a large overlap between the initial state, with $q=1$, and the lowest energy resonant state with $p=1$, i.e., $\mathrm{Re}(C_1\bar{C}_1)=0.99068$, which in view of the closure relation given by Eq. (15), implies that the rest of the other coefficients contributes a small amount. The reason for the large overlap mentioned above is, as discussed at the end of Sec. III. A, that the momentum of the initial state wave is closer to the resonant state $u_1(r)$, than to any of the others resonant states. To exemplify the above considerations, Figs. 5 and 6 exhibit, respectively, for the survival and the nonescape probabilities, the contributions with $p=1$ and $p=1-10$, which are undistinguishable, $p=2$ and $p=3$. One sees clearly the predominance of the contributions with $p=1$. We have found that the single-term approximation remains an excellent approximation for intensities of the δ potential $\lambda \geqslant 2$. It is worth mentioning that in the long time limit, the one-term approximation for the survival probability is comparable to the exact single resonance formula discussed in Ref. [32],

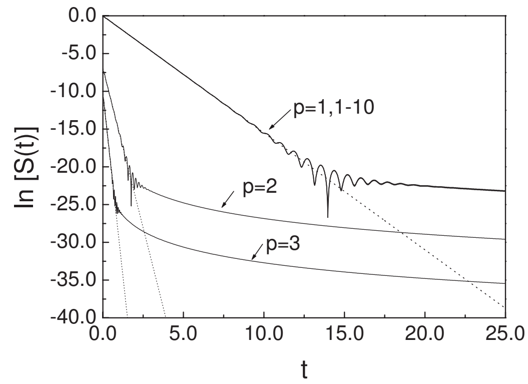

FIG. 5. Resonant contributions of the survival probability $S(t)$ as a function of time, with the value of p as indicated, to show the validity of the corresponding one-term approximation. The parameters of the δ potential are $\lambda=6$ and $a=1$. See text.

where $C_r\bar{C}_r=1.0-i\beta_r/\alpha_r$. In that formulation the dynamics depends only on the input of the complex pole $k_r=\alpha_r-i\beta_r$. In our example of the δ potential, where the intensity $\lambda=6$, $C_1\bar{C}_1=0.990677-i0.0680$ and $\beta_1/\alpha_1=0.0509$. For larger values of λ, the resonance becomes much sharper and $C_1\bar{C}_1$ gets closer to the value of $C_r\bar{C}_r$. For example, for $\lambda=500$, $C_1\bar{C}_1=0.999986-i1.2689\times 10^{-5}$ and $\beta_1/\alpha_1=1.2502\times 10^{-5}$. The one-term approximation is not longer valid when $q>1$. Indeed if $q=j$, with $j=2,3,\ldots$, then resonant terms up to $p=j+1$ are required to calculate the survival probability, and correspondingly, up to $p=j$, for the nonescape probability. As an example of this, Fig. 7 exhibits plots of $S(t)$ and $P(t)$ as a function of time for an initial state with $q=3$. The corresponding long-time steepest descents curves are plotted also (dotted-lines).

IV. CONCLUDING REMARKS

The main results of this work are represented by Eqs. (13), (21), and (23), which provide, respectively, analytical expressions for the resonant expansions of the decaying

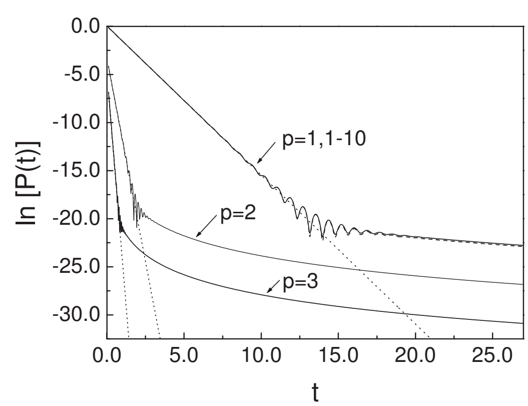

FIG. 6. Resonant contributions to the nonescape probability $P(t)$ as a function of time, with the value of p as indicated, to show the validity of the corresponding one-term approximation. The parameters of the δ potential are $\lambda=6$ and $a=1$. See text.

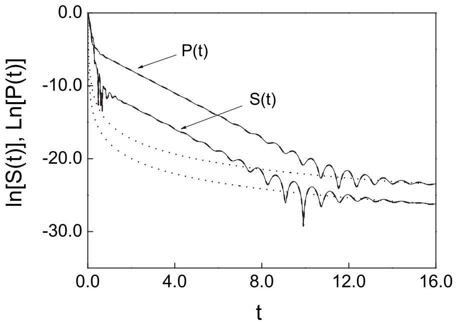

FIG. 7. Resonant contributions to the survival $S(t)$ and nonescape $P(t)$ probabilities as a function of time for an initial state with $q=3$. The resonant sums run up to $p=4$ and $p=10$, for $S(t)$, and $p=3$ and $p=10$ terms, for $P(t)$ (full lines). The corresponding steepest descents calculations of the long-time behavior are also plotted (dotted-line). The parameters of the δ potential are $\lambda=6$ and $a=1$. See text.

wave function and the survival and nonescape probabilities along the exponential and the long-time regimes. It is worth emphasizing that the purely resonant expansions that we have derived do not require to fulfil the sum rule given by Eq. (8) to obtain the respective long-time behaviors, as occurs in a representation involving Moshinsky functions [16,17]. As a consequence, the present approach allows to investigate the validity of one-term and few-term approximations with the consequent simplicity in the calculations. Here, the closure relationship given by Eq. (15), which involves the overlap between the initial state and the distinct resonant states, plays a relevant role. It is also worth stressing out that the exact analytical expressions derived from the steepest descents method [Eqs. (39), (42), and (43)], validate the long time behavior of decay obtained in terms of resonant states. In particular, this also demonstrates that the criticisms raised by some authors were unfounded. The present formulation may be particularly suitable for a systematic study of the effect of different types of initial states in the time evolution of decay, as for example, initial states that do not belong to the domain of the Hamiltonian, which have led to some intriguing results as a fractal-like behavior of the time evolution of the nonescape probability as a function of time [34], and also to study in more detail the transient behavior that appears in the exponential-nonexponential transition at long times. These intriguing peculiar beats have also been noted in other areas of quantum decay [14,15]. We believe that artificial quantum structures, where there is more freedom and flexibility in the characterization of the parameters of the system may be appropriate to study these type of effects.

ACKNOWLEDGMENTS

J. V. acknowledges the support from 10ma. Convocatoria Interna-UABC under Grant No. 184, and Programa de Intercambio Académico 2006-1, UABC.

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