Nonlinear management of the angular momentum of soliton clusters: Theory and experiment

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We demonstrate, both theoretically and experimentally, how to acquire nonlinear control over the angular momentum of a cluster of solitary waves. Our results, stemming from a universal theoretical model, show that the angular momentum can be adjusted by acting on the global energy input in the system. The phenomenon is experimentally ascertained in nematic liquid crystals by observing a power-dependent rotation of a two-soliton ensemble.

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I. INTRODUCTION

Angular momentum (AM) is a fundamental quantity, the importance of which has been highlighted in almost all areas of the physical sciences. Evidence of its role is found at the inception of the Universe: although the distribution of galaxies, stars, and planets is still a puzzle in astrophysics, it appears that an initial angular momentum in the early Universe prevented cluster-sized clouds from collapsing into a series of black holes—i.e., with no planets to support life [1–7]. In both classical and quantum mechanics, angular momentum is at the basis of rotational dynamics; hence, AM governs the behavior of important processes such those arising from fluid motion (e.g., initiation of cyclones [8] and fluctuations in the length of a day [9,10]), statistical complexes of rotating molecules [11], quantum optics [12], and quantized particle ensembles. In the latter area, research has been mostly conducted in two major directions of investigation: (i) the revolution of trapped particles by their interaction with classical fields carrying angular momentum, fostering nanotechnological applications such as rotors or more complex machines powered by light [13–17], and (ii) the preparation of energy packets in well-defined AM states, with implications for both fundamental physics [18,19] and quantum information systems [20,21].

However, despite the importance of AM in physics and the vast literature on the subject, neither methods to nonlinearly control AM been proposed nor thorough studies been carried out on the effect of nonlinearity on the angular momentum of a specific system. Conversely, great attention has been devoted in the past few years to solitons and solitary waves. Such waves are ubiquitous and rely on the balance between wave-packet dispersion (spreading) and nonlinearity [22]. Following the pioneering numerical experiments by Fermi, Pasta, and Ulam [23,24], the universal concept of solitons has acquired importance in several sciences, including biology [25], hydrodynamics [22], plasma physics [26], ultracold atoms [27,28], optics [29–31], gravitation [31], and beyond [32]. By virtue of their robustness, solitons have potentials in applications—from optical telecommunications to atomic interferometry in Bose-Einstein condensates (BECs)—and are the subject of vigorous theoretical studies [33–35]. Recently, the interactions between two-dimensional deterministic soliton clusters have attracted interest, encompassing a wealth of dynamics ranging from wave filamentation [36,37], to spinning [38] and spiraling [39–43]. Up to date, however, studies have focused on either specific nonlinear models or particular input wave forms [36,39,42,43], or on one-dimensional (1D) stochastic dynamics [44]; none of them discussed the role of excitation on the dynamics of a two-dimensional multisoliton ensemble.

In this paper, by employing a rather universal model for the theory and a nonlocal dielectric for the experiments, we investigate the behavior of a cluster of $(2+1)$D optical solitons, demonstrating a nonlinear approach for the control of its angular momentum. For the sake of simplicity but with no prejudice as to its general validity, we develop the analysis in the simplest case of a two-soliton cluster, starting from first principles and employing the language of symmetries [45]. The results of our approach are twofold.

(i) We demonstrate that the angular momentum of a soliton cluster exhibits a linear dependence on the nonlinear response; hence, it can be precisely managed by varying the input power.

(ii) The AM can be measured from the global revolution of the cluster which, for a fixed propagation distance, evolves linearly with excitation.

We check the theoretical results both numerically, by performing a series of $(2+1)$D simulations in nonlinear nonlocal media, and experimentally, by employing nematic liquid crystals (NLCs), nonlocal material known to support stable $(2+1)$D soliton waves [46–50]. The article is organized as follows: Section II collects our theoretical activity, including our symmetry-based analysis (Sec. II A), analytic results (Sec. II B) and numerical simulations (Sec. II C), whereas Sec. III illustrates our experiments in NLCs.

II. MODEL

A. Symmetry-based analysis

We build a general model of nonlinear wave propagation, stemming from conservation laws and variational symmetries (as there is a one-to-one correspondence between them

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[45]) and considering a generic isolated medium with translational and rotational invariance. The medium exhibits an optical response nonlinear with the wave intensity; hence, it enforces the conservation laws of momentum (**M**), angular momentum (**A**), Hamiltonian ($\mathcal{H}$), and energy flux ($\mathcal{W}$), respectively. Such a nonlinear model is derived by defining a suitable action integral $\mathcal{I}=\int d^2\mathbf{r}dz\mathcal{L}$, the Lagrangian of which supports the variational symmetries originated by the basis of Lie generators (we adopt Einstein's summation over repeated indices):

$$\mathbf{v}_1 = \partial/\partial x,$$

$$\mathbf{v}_2 = \partial/\partial y,$$

$$\mathbf{v}_3 = \partial/\partial z,$$

$$\mathbf{v}_4 = x\partial/\partial y - y\partial/\partial x,$$

$$\mathbf{v}_5 = i\psi_\alpha \partial/\partial\psi^\alpha + \text{c.c.}, \tag{1}$$

with $\mathbf{r}=[x,y]$, z being dimensionless coordinates and ψ_α the dimensionless wave function of the αth wave packet ($\alpha \in [1,2]$ in this work). A general form for the Lagrangian $\mathcal{L}$ is as follows:

$$\mathcal{L} = \frac{1}{2}\left(i\psi_\alpha^* \frac{\partial\psi^\alpha}{\partial z} + \text{c.c.}\right) - \frac{1}{2} \times \nabla\psi_\alpha^* \nabla\psi^\alpha + \frac{1}{2}|\psi^\alpha|^2\Theta_{\alpha\beta} \otimes |\psi^\beta|^2, \tag{2}$$

$\Theta_{\alpha\beta}$ being a Hermitian tensor, $\alpha,\beta \in [1,2]$, $\nabla=[\partial/\partial x, \partial/\partial y]$, and $\otimes$ a convolution operator defined as

$$f \otimes g = \int\int d^2\mathbf{r}' f(\mathbf{r}'-\mathbf{r})g(\mathbf{r}'). \tag{3}$$

The general character of the Lagrangian (2) can be proven from the Euler-Lagrange equations of motion, which read

$$i\frac{\partial\psi_\alpha}{\partial z} + \frac{1}{2}\nabla^2\psi_\alpha + \psi_\alpha\Theta_{\alpha\beta} \otimes |\psi^\beta|^2 = 0. \tag{4}$$

The linear portion of Eq. (4) is Schrödinger like and describes the universal wave propagation in dispersive media [51]; the nonlinear portion ($\Theta_{\alpha\beta}\otimes|\psi^\beta|^2$) can model both local—i.e., for $\Theta_{\alpha\beta}(\mathbf{r})=c_{\alpha\beta}\delta(\mathbf{r})$—[29] and nonlocal [27,48,52] nonlinear responses. By construction, the Lagrangian (2) admits the symmetries generated by $\mathbf{v}_i$ ($i=1,\ldots,5$); hence, Eq. (4) possesses the following integrals of motion:

$$\mathbf{M} = \int\int d^2\mathbf{r}(i\psi_\alpha^* \nabla\psi^\alpha + \text{c.c.}), \tag{5}$$

$$\mathcal{H} = \frac{1}{2}\int\int d^2\mathbf{r}(|\psi^\alpha|^2\Theta_{\alpha\beta} \otimes |\psi^\beta|^2 - \nabla\psi_\alpha^* \nabla\psi^\alpha), \tag{6}$$

$$\mathbf{A} = \int\int d^2\mathbf{r}[\mathbf{r} \times (i\psi_\alpha^* \nabla\psi^\alpha + \text{c.c.})], \tag{7}$$

$$\mathcal{W} = \int\int d^2\mathbf{r}|\psi_\alpha|^2. \tag{8}$$

The conserved quantities (5)–(8), together with the divergence form of Eq. (8),

$$\frac{\partial}{\partial z}(|\psi_\alpha|^2) = \nabla(i\psi_\alpha^* \nabla\psi_\alpha + \text{c.c.}), \tag{9}$$

allow us to generalize the Ehrenfest theorem. With reference to Eq. (4), it can be cast in the form

$$\frac{\partial}{\partial z}\langle\psi_\alpha|\mathbf{r}|\psi_\alpha\rangle = \langle\psi_\alpha|\mathbf{p}|\psi_\alpha\rangle,$$

$$\frac{\partial}{\partial z}\langle\psi_\alpha|\mathbf{p}|\psi_\alpha\rangle = \langle\psi_\alpha|\nabla\Theta_\alpha|\psi_\alpha\rangle, \tag{10}$$

with $\mathbf{p}=-i\nabla$ being the momentum operator, $\Theta_\alpha=\Theta_{\alpha\beta}\otimes|\psi^\beta|^2$ and $\langle f_\alpha|a_\alpha\rangle=\int\int d^2\mathbf{r} f_\alpha^* g_\alpha$ the inner product defining the space metric. Equations (10) are a generalization of models either obtained in local media [39,43,53] or derived in one dimension [44]. Out of the whole spectrum of solutions to Eq. (4), we are interested in the evolution of a nonlinear cluster, the solitary "particles" (solitons) of which are found as invariant solutions of Eq. (4) (written for $\alpha=\beta$) with respect to the global symmetry group generated by

$$\mathbf{v} = \mathbf{v}_3 + \mathbf{v}_4 - x_\alpha\mathbf{v}_1 + y_\alpha\mathbf{v}_2 + \mu\mathbf{v}_5. \tag{11}$$

Then the method of characteristics [45] yields the functional form of each solitary wave:

$$\psi_\alpha(\mathbf{r};z) = \phi(|\mathbf{r}-\mathbf{r}_\alpha|)\exp(i\mu z), \tag{12}$$

with $\phi(|\mathbf{r}-\mathbf{r}_\alpha|)$ being a complex envelope and $[x_\alpha,y_\alpha]=\mathbf{r}_\alpha=\langle\psi_\alpha|\mathbf{r}|\psi_\alpha\rangle/\int\int d^2\mathbf{r}|\psi_\alpha|^2$ the soliton "center of mass." It is worth remarking that the term "particle" is correct as long as the solitons do not overlap: the intrinsic nonintegrable nature of Eq. (4), in fact, does not assure the isospectrality [54] of the cluster evolution—i.e., the fact that after collision(s) the soliton ensemble evolves with a constant number of wave packets. However, if solitons do not overlap, it is possible to assume that the effects of the interaction terms ($\Theta_{\alpha\beta}\otimes|\psi_\beta|^2$ for $\alpha\neq\beta$) do not affect their functional form but only their phase μ or their center of mass $\mathbf{r}_\alpha$. Otherwise stated, we assume an "adiabatic" regime of interaction, using the term adiabatic as in the context of soliton perturbation theory: when the perturbation terms are small and out of resonance; hence, the deformation of the soliton may appear only through the soliton parameters such as the amplitude and velocity, and the balance between nonlinearity and dispersion is given by an adiabatic relation (see [55] Chap. 5, p. 75). Such a hypothesis allows us to cast Eq. (10) in a potential form. By substituting Eq. (12) into Eq. (10), after some straightforward algebra, we obtain

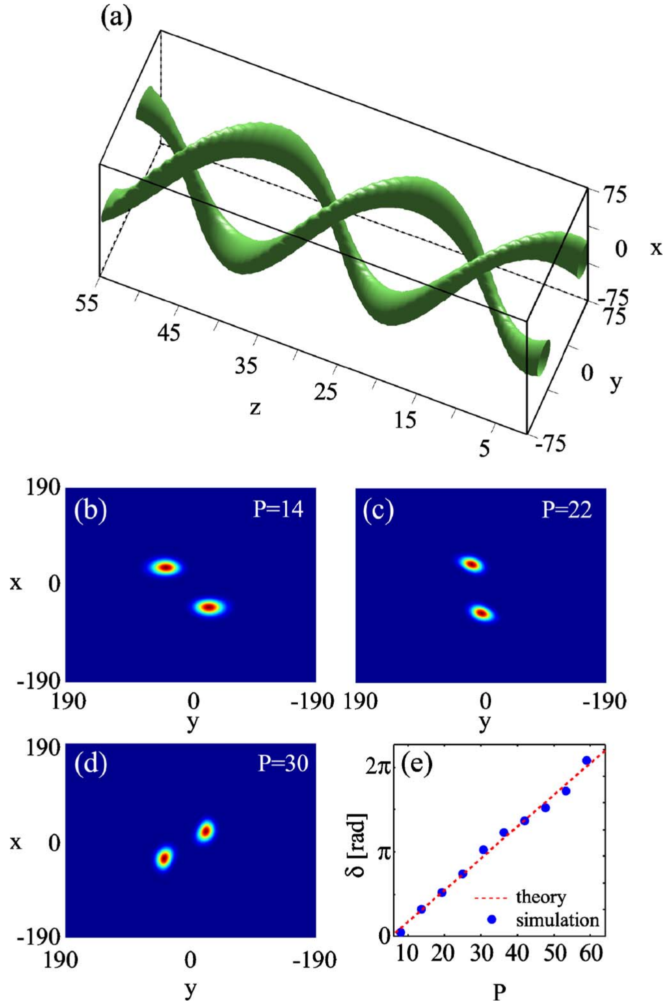

FIG. 1. (Color online) Summary of numerical simulations: (a) isosurface plot showing soliton spiraling for $P=8$, (b)–(d) soliton output positions at different powers, and (e) output revolution angle δ versus power P. All simulations are performed for $k_0=2.75$, $y_0=4$, and $\sigma=14$.

$$m\frac{\partial^2\mathbf{q}}{\partial z^2}+\boldsymbol{\nabla}_q U(\mathbf{q})=\mathbf{0}, \tag{13}$$

with $\mathbf{q}=\mathbf{r}_1-\mathbf{r}_2=[q_x,q_y]$, $m=\int\int d^2\mathbf{r}|\psi_\alpha|^2$ the soliton "mass," $\nabla_q=[\partial/\partial q_x,\partial/\partial q_y]$, $U(\mathbf{q})=-[U_{12}(\mathbf{q})+U_{21}(\mathbf{q})]$, and the pair-interaction potential

$$U_{\alpha\beta}=\int\int d^2\mathbf{r}'|\phi(|\mathbf{r}'+\mathbf{q}|)|^2\Theta_\alpha(|\mathbf{r}'|),\quad \alpha\neq\beta. \tag{14}$$

It is worth remarking that the only hypotheses in deriving Eqs. (13) and (14) is an adiabatic regime of soliton interactions. Equations (5)–(12), in fact, are rigorous.

B. Theoretical results and discussion

Equation (13) defines the motion of a classical mass point (of mass m) subject to the nonlinear pairwise potential (14). The Lagrangian density of Eq. (13) is

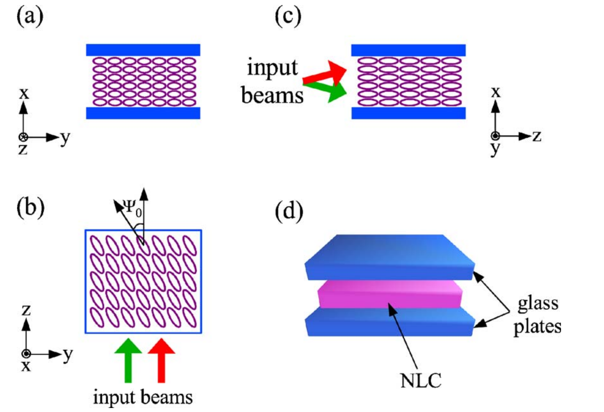

FIG. 2. (Color online) Sketch of the planar cell with nematic liquid crystals: (a) front view, (b) top view, (c) side view, and (d) perspective view. The ellipses indicate the molecular orientation (optic axis) in the plane (y,z).

$$\mathcal{L}=\dot{\mathbf{q}}\dot{\mathbf{q}}/2-U(\mathbf{q}) \tag{15}$$

and admits variational symmetries generated by the basis

$$\mathbf{v}_1=\partial/\partial z,$$

$$\mathbf{v}_2=q_x\partial/\partial q_y-q_y\partial/\partial q_x, \tag{16}$$

the latter originating — through Noether's theorem — the conservation laws of energy $\mathcal{E}$ and angular momentum $\mathbf{L}$, respectively:

$$\mathcal{E}=m\frac{\dot{\mathbf{q}}\dot{\mathbf{q}}}{2}+U(\mathbf{q}), \tag{17}$$

$$\mathbf{L}=m\mathbf{q}\times\dot{\mathbf{q}}. \tag{18}$$

Noticeably, Eq. (18) states that (i) the angular momentum is conserved, and hence it is controlled by the input conditions, and (ii) the angular momentum depends linearly on the soliton "mass" m.

As a result, a change in the soliton *global* power $\int\int d^2\mathbf{r}|\psi|^2=\int\int d^2\mathbf{r}(|\psi_1|^2+|\psi_2|^2)$ leads to a linear variation of AM, provided the adiabatic condition [i.e. Eqs. (12)–(17)] is fulfilled. To measure the AM observable, hereby we develop an original method by exploiting soliton *spiraling*. Such dynamics occurs when the initial momentum $\dot{\mathbf{q}}(0)=\dot{\mathbf{r}}_1(0)-\dot{\mathbf{r}}_2(0)$ balances the attractive force provided by the bound potential $U(\mathbf{q})$ [39], resulting in a rigid rotation of the (two-) soliton ensemble with constant separation d and invariant angular velocity $\omega=\mathcal{L}/\mathcal{M}=\partial\delta/\partial z$, δ being the angle spanned by $\mathbf{q}$ and $\mathcal{M}$ the "momentum of inertia" of the classical system (13) [39,43]. Following a straightforward integration of $\mathcal{L}/\mathcal{M}=\partial\delta/\partial z$, the soliton cluster revolves with an angle δ which, for a given z, evolves linearly with angular momentum $\mathcal{L}$. As a result, provided the adiabatic condition remains valid, the revolution angle δ of the soliton ensemble is expected to increase linearly with soliton input power $P_{\rm in}=\int\int d^2\mathbf{r}|\psi|^2$.

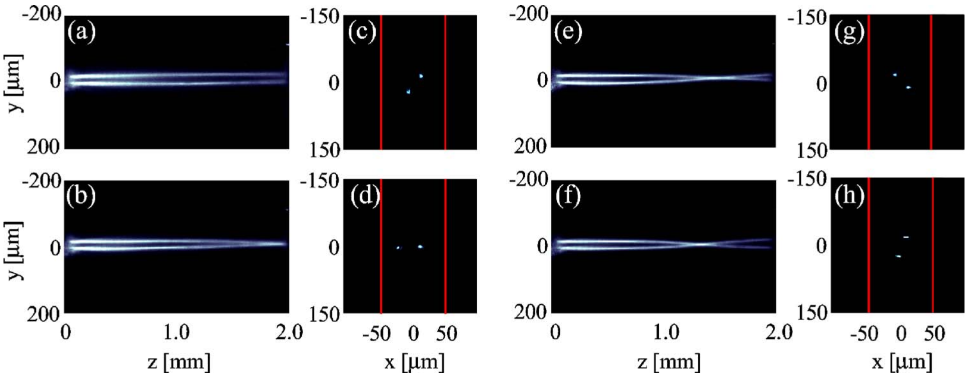

FIG. 3. (Color online) Experimental results: (a),(b),(e),(f) evolution in the plane (y,z) and (c),(d),(g),(h) output intensity profiles of the two-soliton cluster for increasing input powers $P_{\mathrm{in}}=2.1$, 2.7, 3.3, and 3.9 mW. The borders of the NLC cell are indicated with by solid lines in (c),(d),(g),(h).

C. Numerical simulations in nonlocal media

We numerically verify our analytical results by performing a series of (2+1)D nonlinear simulations from Eqs. (4) in the general case of a material with a nonlocal response. The latter is defined by employing a Gaussian kernel for the Hermitian tensor $\Theta_{\alpha\beta}$:

$$\Theta_{\alpha\beta} = \frac{1}{\pi\sigma^2}\exp(-r^2/\sigma^2), \tag{19}$$

with σ being the degree of nonlocality in the medium. Figure 1 summarizes our numerical results, obtained for $\sigma=14$ and an input consisting of two Gaussian beams $\psi_\pm$ carrying opposite momenta along x:

$$\psi_\pm = \frac{\sqrt{P}}{\sqrt{\pi\omega}}\exp\left(-\frac{x^2+|y\pm y_0|^2}{\omega^2}\pm ik_0x\right), \tag{20}$$

with global power $P=\int\int d^2\mathbf{r}|\psi_+|^2+|\psi_-|^2\in[8,60]$, $\omega=2$, $y_0=4$, and initial momentum $k_0=2.75$. The latter value was chosen in order to observe soliton spiraling [Fig. 1(a)]. As expected, when the power P increases, the soliton particles rotate [Figs. 1(b)–1(d)] with a revolution angle δ linearly depending on the excitation [Fig. 1(e)], in perfect agreement with theory (Secs. II A and II B).

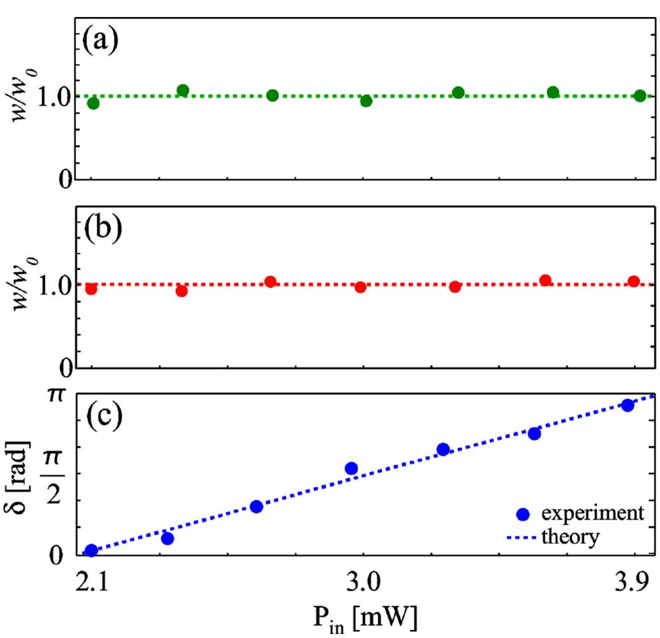

FIG. 4. (Color online) (a),(b) Output spot size w of the two solitons normalized to $w_0=w(2.1\text{ mW})$ and (c) revolution angle δ versus input power P_{in}.

III. EXPERIMENTS IN NEMATIC LIQUID CRYSTALS

We carry out experiments in a planar glass cell with a 100-μm-thick layer of planarly anchored liquid crystals $E7$ (Fig. 2). The sample is similar to those previously employed for the investigation of accessible solitons in highly nonlocal media [48], but with a preorientation $\Psi_0\approx\pi/6$ with respect to z in the (y,z) plane to make the voltage bias unnecessary [56]. We perform the experiments with a near-infrared ($\lambda=1.064$ μm) cw source and high-resolution silicon charge-coupled-device cameras for imaging both the soliton propagation in (y,z) and their output transverse positions or profiles. We launch two extraordinary-wave Gaussian beams to generate solitons with opposite momenta along x, compensating for the walk-off in (y,z) by input wave-front tilts. The input momenta are chosen in order to achieve soliton spiraling within the bulk of the sample.

Our experimental results are illustrated in Fig. 3, showing images of light propagating in (y,z) [Figs. 3(a)–3(d)] and output intensity profiles in (x,y) [Figs. 3(a), 3(b), 3(e), and 3(f)] as the excitation P_{in} increases. In agreement with both analytical and numerical predictions, as the power changes from 2.1 [Figs. 3(a) and 3(e)] to 3.9 mW [Figs. 3(c), 3(d), 3(g), and 3(h)], the AM of the soliton cluster changes as well, as demonstrated by the rotation of $\approx 180°$ in the output plane [Figs. 3(e) and 3(h)]. Remarkably enough, each soliton profile evolves nearly unmodified [Figs. 4(a)4(b)] and the rotation angle δ is linear with power [Fig. 4(c)], demonstrating the nonlinear control over the overall angular momentum. Owing to the giant nonlinear response of NLC, the resulting sensitivity $\Delta\delta/\Delta P_{\mathrm{in}}=0.5\pi$ (rad)/mW is quite large.

IV. CONCLUSIONS

Stemming from first principles and with no specific assumptions on the dielectric response, we theoretically disclosed an original method to gain nonlinear control over the angular momentum of a cluster of (two) spatial solitons. Such nonlinear management of the soliton-interaction potential is a remarkable example of all-optical control over lightinduced guided-wave interconnect. The theoretical findings were verified against both numerical and actual experiments in a nonlocal dielectric—namely, nematic liquid crystals—using a cluster of two nonlocal spatial solitons with unprecedented power-dependent revolutions as large as $500\pi/\mathrm{W}$. Due to the general character of the model we derived, our findings might allow some progress in other areas where solitons are actively investigated, such as plasma physics and Bose-Einstein condensates. We also envision the possibility of novel applications in soliton-based nonlinear optics.

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