Atomic detection in microwave cavity experiments: A dynamical model

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We construct a model for the atomic detection in the context of cavity quantum electrodynamics (QED) used to study coherence properties of superpositions of states of an electromagnetic mode. Analytic expressions for the atomic ionization are obtained, considering the imperfections of the measurement process due to the probabilistic nature of the interactions between the ionization field and the atoms. We provide for a dynamical content for the available expressions for the counting rates considering limited efficiency of detectors. Moreover, we include false countings. The influence of these imperfections on the information about the state of the cavity mode is obtained. In order to test the adequacy of our approach, we investigate a recent experiment reported by Maître [X. Maître *et al.*, Phys. Rev. Lett. **79**, 769 (1997)] and we obtain excellent agreement with the experimental results.

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I. INTRODUCTION

The quantum interaction between two level Rydberg atoms and one microwave mode inside a high quality factor (Q) cavity has been crucial for our understanding of dissipation and decoherence in quantum mechanics [1,2]. Usually, in cavity quantum electrodynamics (QED) experiments, Rydberg atoms cross an experimental array constituted by two Ramsey zones and a high-Q cavity. Thereafter, their final states are detected in one of two ionization zones. In the first zone is built an electrical field with amplitude sufficient to ionize the atom in the highest state. The second detection zone is designed to detect the lower atomic level.

Most of the work available in literature about the detection process is based on statistical assumptions. The pumped atoms are statistically independent, so that their arrival times are subject to a Poissonian or other statistics [3–5]. The basic idea is that atoms arrive at random and they are recorded at equally random times, so the only reproducible data are statistical. In this context one is lead to study the statistic of detector clicks. There are numerical studies [6,7] and also analytical results by Rempe and Walther [8] and by Paul and Richter [9].

Extensions of the method presented in Ref. [8] provide a deep analysis of the one-atom maser experiments. In Ref. [10], counting, waiting times and sequence statistics of detector clicks are obtained. Limits of a large number of detections have been considered and analytically soluble master equations have been derived in Ref. [11]. The results obtained within these statistical approaches lead to rather simple and physically appealing expressions for the detection probabilities in terms of the essential measurement ingredients such as the efficiency of the detectors and so on. These ingredients are treated as empirical parameters. The purpose of the present contribution is twofold: to furnish the dynamical elements leading to empirical parameters, in the context of a simple model, and to include the possibility of false detections. The idea is to unveil the physical process behind the empirical rates. A different and much more involved task would be, e.g., to obtain microscopic expressions for such quantities. We believe, however, that given the number of publications in the area and the lack of dynamical content for the derived parameters, an approach based on a simple dynamical model is an important first step in the direction of a qualitative understanding of this rather intricate process, both from the conceptual and the calculation sides. On the other hand, to our knowledge, the consideration of false counting has been given here for the first time.

In the present contribution we propose a dynamical model for the detection process. We assume that the atom undergoes the influence of a classical electrical field when it traverses the detection zones. The net effect of this field is to couple one (in the case of intrinsically inefficient detectors) or two (in the case of detectors that register false countings) discrete atomic levels to the continuum. If the atom is ionized, a transition to the continuum has occurred and a classical signal—a "click"—is generated in the correspondent detector. However, if the atom remains in one of the two discrete levels, no click is registered by the detector.

Since the atom works as a probe to the field stored in a high-Q cavity, the click or no-click registered by the detectors represents a gain of information about the state of the compound system formed by the atom and by the high-Q cavity field. Hence, the process of detection can be divided in two parts. First, the state of the compound system atom–high-Q cavity field undergoes an unitary evolution during the passage of the atom through each detection zone. Next, the resultant state is projected into a proper subspace defined as follows: if a click is registered, this subspace is formed by the set of the states that form the continuum; otherwise, this subspace corresponds to the states associated to the two discrete levels.

As an example of the applicability of the results obtained here we study a recently reported experiment [12] were the imperfections are clearly stated. Our results are in excellent agreement with the experimental findings.

This paper is organized in the following way. In Sec. II, we treat a model for intrinsically inefficient detectors. The analytical form of our results are exactly the same as those in Ref. [13], whose derivation is based on statistical and physically plausible arguments. In Sec. III, we study the possibility of the detectors perform false countings. In this case, we found that the probability of a click depends on the "nondiagonal" terms of the state of the system atom–high-Q cavity field. We calculate the fidelity of the field states in high-Q cavity after the measurement process considering the two kinds of imperfections, limited efficiency and false detections. We also apply the present model in a recently reported experiment. Section IV contains a summary of the results and conclusions.

II. MODEL FOR INEFFICIENT DETECTOR

The ionization process of an atom due to its interaction with an electromagnetic field is considered in a quantum context. Therefore a finite probability of nonexcitation will exist. That is what we call an intrinsic (i.e., quantum mechanical) inefficiency.

The Hamiltonian which describes the interaction between two level atoms and ionization field on the first detection zone, and takes into account only the intrinsic inefficiency of the process, is given by (the Hamiltonian for the second detection zone can been obtained replacing the index e by g):

$$H_{1e} = \epsilon_e|e\rangle\langle e| + \epsilon_g|g\rangle\langle g| + \int dk\epsilon_k|k\rangle\langle k| + v_e\int dk(|e\rangle\langle k| + |k\rangle\langle e|). \quad (1)$$

The first and second terms in the Hamiltonian stand for the two discrete atomic levels $|e\rangle$ and $|g\rangle$, excited and ground states, respectively, with energies ϵ_e and ϵ_g. The third term represents its continuum spectrum. The last term accounts for the coupling between the highest discrete level and the continuum. The strength of this interaction is given by the parameter v_e, assumed state independent for simplicity. This term is responsible for the ionization of the atom. Since we are dealing with a quantum mechanical process, which is intrinsically probabilistic, we will also have to consider the possibility of nonionization of the atom.

Following Cohen-Tannoudji [14], the evolution of the discrete state $|e\rangle$ according to Eq. (1) is given by

$$|\psi(t)\rangle = e^{-iH_{1e}t/\hbar}|e\rangle = \int d\mu\langle\psi_\mu^e|e\rangle e^{-i\epsilon_\mu^e t/\hbar}|\psi_\mu\rangle, \quad (2)$$

where $\{|\psi_\mu^e\rangle\}$ and $\{\epsilon_\mu^e\}$ correspond to the set of eigenvectors and eigenvalues of H_{1e}, respectively. The coefficients $\langle\psi_\mu^e|e\rangle$ and $\langle\psi_\mu^e|k\rangle$ may be written as

$$\langle\psi_\mu^e|e\rangle = \frac{1}{\left[1 + \int dk'\left(\frac{v}{\epsilon_\mu^e - \epsilon_{k'}}\right)^2\right]^{1/2}}, \quad (3a)$$

$$\langle\psi_\mu^e|k\rangle = \frac{v/(\epsilon_\mu^e - \epsilon_k)}{\left[1 + \int dk'\left(\frac{v}{\epsilon_\mu^e - \epsilon_{k'}}\right)^2\right]^{1/2}}. \quad (3b)$$

Accordingly, the ionization probability can be obtained as follows:

$$p_e = \int dk|\langle k|\psi(t)\rangle|^2 = \int dk\left|\int d\mu\langle\psi_\mu^e|e\rangle\langle k|\psi_\mu^e\rangle e^{-i\epsilon_\mu^e t/\hbar}\right|^2. \quad (4)$$

This probability defines the detector's efficiency. Therefore the nondetection probability is given by $1-p_e = |\int d\mu e^{-i\epsilon_\mu^e t/\hbar}|\langle\psi_\mu^e|e\rangle|^2|^2$. After some simplifications (see Ref. [14]) the nondetection probability can be written as

$$1 - p_e = e^{-\Gamma|t|}, \quad (5)$$

where Γ is the transition rate from discrete to the continuum level, calculated from Fermi's "golden rule." Γ is given by

$$\Gamma = \frac{2\pi\rho(E)}{\hbar}, \quad (6)$$

where $\rho(E)$ is the level density per unity energy. In the limit where the atom ionization time can be considered to be infinite (in some experimental context) we will have a perfect detector.

Following the same procedure for H_{1g} we find $p_g = \int dk|\int d\mu\langle\psi_\mu^g|g\rangle\langle k|\psi_\mu^g\rangle e^{-i\epsilon_\mu^g t/\hbar}|^2$, where $\{|\psi_\mu^g\rangle\}$ and $\{\epsilon_\mu^g\}$ correspond to the set of eigenvectors and eigenvalues of H_{1g}, respectively.

As an example of applicability of the model, let us study the interactions between two level atoms and their detection through ionization fields in cavity QED experiments. The state of the system atom–high-Q cavity field can be written as

$$\rho_{AC}(0) = \rho_{ee}|e\rangle\langle e| + \rho_{eg}|e\rangle\langle g| + \rho_{ge}|g\rangle\langle e| + \rho_{gg}|g\rangle\langle g|. \quad (7)$$

This state represents the most general state (in the system atom-cavity field) immediately before the interaction between the atom and the detectors. The symbols ρ_{ee}, ρ_{eg}, ρ_{ge}, and ρ_{gg} are operators in the cavity field subsystem.

The interaction between the atom and the first detection zone (D_e) can be separated in two steps. First, a quantum unitary evolution governed by the Hamiltonian H_{1e} given by Eq. (1) during the time interval t_1. The atom-cavity field state, after this process, is given by

$$\rho_{AC}(t_1) = e^{-iH_{1e}t_1/\hbar}\rho_{AC}(0)e^{iH_{1e}t_1/\hbar}. \quad (8)$$

Now, in the second step, at the time t_1, a classical signal is generated. If the detector clicks, we will know that the atom was ionized, so $\rho_{AC}(t_1)$ must be projected into the subspace $\{|k\rangle\}$. Although, if D_e does not click we know that the atomic

state must be projected into subspace spanned by the discrete levels $\{|e\rangle,|g\rangle\}$. The maximum value that t_1 can assume is t_1' which is the time taken by the atom to cross D_e completely. So, up to t_1', we will certainly acquire information about the system. This revealed information plays an essential role into ρ_{AC}'s evolution. So we are aware that before the interaction with D_g, the state $\rho_{AC}(t_1)$ must be projected properly into subspace.

We can calculate the probability of a click in D_e

$$p_{\text{click}D_e} = \int dk\, \text{Tr}(|k\rangle\langle k|\rho_{AC}(t_1)) = \int dk\, \text{Tr}(|k\rangle\langle k|\rho_{ee}e^{-iH_{1e}t_1/\hbar}|e\rangle\langle e|e^{iH_{1e}t_1/\hbar}). \quad (9)$$

Then, using Eq. (4), we may write

$$p_{\text{click}D_e} = p_e\, \text{Tr}_C(\rho_{ee}), \quad (10)$$

where Tr_C is the partial trace on the cavity-field subspace. This product can be interpreted as the efficiency of $D_e(p_e)$ times the probability of click on a perfect detector after the interaction with the state $\rho_{AC}(0)$ $[\text{Tr}_C(\rho_{ee})]$.

The nonclick probability is

$$p_{\text{nonclick}D_e} = \text{Tr}[(|e\rangle\langle e| + |g\rangle\langle g|)\rho_{AC}(t_1)] = \text{Tr}_C(\rho_{ee}) + \text{Tr}_C(\rho_{gg}) - p_e\, \text{Tr}_C(\rho_{ee}). \quad (11)$$

From the normalization of $\rho_{AC}(0)$, $\text{Tr}_C(\rho_{ee})+\text{Tr}_C(\rho_{gg})=1$, we can write

$$p_{\text{nonclick}D_e} = 1 - p_e\, \text{Tr}_C(\rho_{ee}). \quad (12)$$

After the nonclick stage the system is in the state

$$\rho_{\text{nonclick}}(t_1) = \frac{(|e\rangle\langle e| + |g\rangle\langle g|)\rho_{AC}(t_1)(|e\rangle\langle e| + |g\rangle\langle g|)}{N} = \frac{\rho_{gg}|g\rangle\langle g| + \rho_{ee}(1-p_e)|e\rangle\langle e| + (\rho_{eg}e^{i\epsilon_g t_1/\hbar}\int d\mu e^{-i\epsilon_\mu t_1/\hbar}|\langle\psi_\mu|e\rangle|^2|e\rangle\langle g| + \text{H.c.})}{N}, \quad (13)$$

where $N=1-p_e\,\text{Tr}_c\,\rho_{ee}$. Note that if the efficiency is equal to unity $(p_e=1)$, the reduced state operator on atomic subspace can be written as $\rho_{gg}|g\rangle\langle g|$. This result was expected, as we know, since for perfect detectors a nonclick in D_e would lead to the projection $|g\rangle\langle g|\rho_{AC}(0)|g\rangle\langle g|$.

When the atom is not ionized on D_e, it continues the journey and passes through the second detection zone (D_g). Let us set the interaction time between atom and the electromagnetic field inside D_g by t_2. The temporal evolution that models this interaction is again unitary:

$$\rho_{AC}(t_2) = e^{-iH_{1g}(t_2-t_1)\hbar}\rho_{\text{nonclick}}(t_1)e^{iH_{1g}(t_2-t_1)\hbar}. \quad (14)$$

So, the probability of click in D_g is

$$p_{\text{click }D_g} = \frac{p_g\,\text{Tr}_C(\rho_{gg})}{1-p_e\,\text{Tr}_C(\rho_{ee})}. \quad (15)$$

Note that this probability depends on the efficiency of the first detector p_e. Now let us examine some limits. For $p_e=0$, this is equivalent to the situation where the first detector is absent, so the atom interacts just with the second ionization zone D_g and the probability of click is $p_g\,\text{Tr}_C(\rho_{gg})$, as expected. If $p_e=1$, the first detector is perfect, therefore, as discussed before, the atom goes to the state $|g\rangle$ when it crosses D_e without have been detected, and the probability of click is the efficiency of D_g, p_g. If both detectors are perfect $(p_e=p_g=1)$, $p_{\text{click }D_g}=1$ because the second detector will not miss any atom prepared in $|g\rangle$.

A more complete analysis of $p_{\text{click }D_g}$ for different values of $\text{Tr}\rho_{gg}(0)$ is shown in Fig. 1. The behavior of the curves associate to $\text{Tr}_C(\rho_{gg})=0.5$ and $\text{Tr}_C(\rho_{gg})=0.01$ reflects the fact that a nonclick on a very efficient D_e $(p_e\approx 1)$ raises the probability $p_{\text{click }D_g}$, even if the atom is practically prepared in the state $|e\rangle\langle e|$ $[\text{Tr}_C(\rho_{gg})=0.01]$. On the other hand, if the atom is practically prepared in the state $|g\rangle\langle g|$ $[\text{Tr}_C(\rho_{gg})=0.99]$, p_e does not affect $p_{\text{click}D_g}$.

The probability of nonclick in D_g is

$$1-p_{\text{click }D_g} = \frac{1-p_e\,\text{Tr}_C\,\rho_{ee}-p_g\,\text{Tr}_C\,\rho_{gg}}{1-p_e\,\text{Tr}_C\,\rho_{ee}}. \quad (16)$$

When the atom crosses both detectors without being detected, it reduces the field state inside the cavity to

$$\rho_C' = \frac{\text{Tr}_A[(|e\rangle\langle e| + |g\rangle\langle g|)\rho_{AC}(t_2-t_1)(|e\rangle\langle e| + |g\rangle\langle g|)]}{\text{Tr}[(|e\rangle\langle e| + |g\rangle\langle g|)\rho_{AC}(t_2-t_1)(|e\rangle\langle e| + |g\rangle\langle g|)]}. \quad (17)$$

Here, Tr_A stands for the trace in the atomic variables. Now, using the definition (4) we may write

$$\rho_C' = \frac{(1-p_e)\rho_{ee}+(1-p_g)\rho_{gg}}{1-p_e\,\text{Tr}_C(\rho_{ee})-p_g\,\text{Tr}_C(\rho_{gg})}. \quad (18)$$

The form of this result is in complete agreement with the one in [13], where the authors used statistics arguments to derive the expression (18).

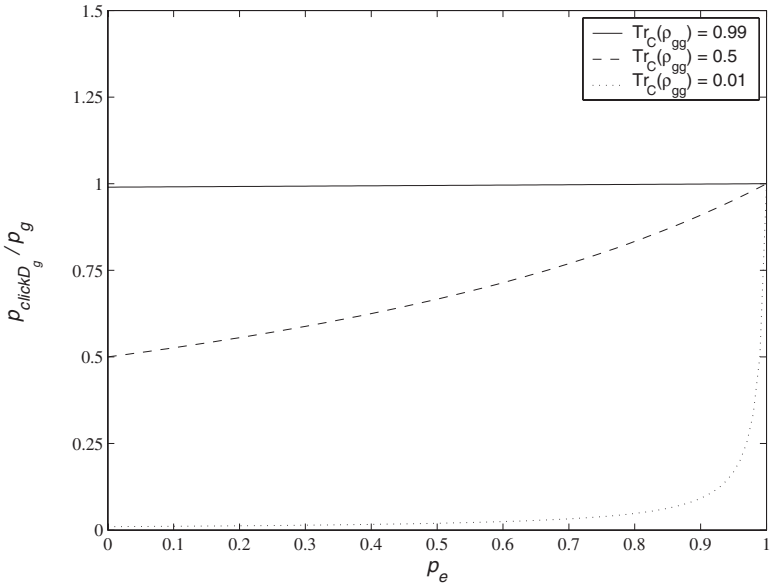

FIG. 1. Influence of the efficiency of the detector D_e (p_e) on the "normalized" probability of click in the detector D_g ($p_{\text{click}D_g}/p_g$), for different values of $\text{Tr}_C(\rho_{gg})$. The efficiency of the detector D_g (p_g) just limits the maximum value reached by $p_{\text{click}D_g}$ and does not modify its qualitative behavior as function of p_e.

III. MODEL FOR FALSE DETECTIONS

In addition to the previous intrinsically inefficient detector we extend the model to include false detections. The Hamiltonian for the first detection zone D_e is given by (the Hamiltonian for the second detection zone can be obtained replacing the index e by g)

$$H_{2e} = \epsilon_e|e\rangle\langle e| + \epsilon_g|g\rangle\langle g| + \int dk\epsilon_k|k\rangle\langle k| + w_e\int dk(|e\rangle\langle k| + |k\rangle\langle e|) + w_g\int dk(|g\rangle\langle k| + |k\rangle\langle g|), \tag{19}$$

where w_e and w_g are real coupling constants. The second interaction term (the last one in the equation above) represents the coupling between $|g\rangle$ and the continuum, so it is responsible for wrong detections. On the other hand, the previous one is responsible for the correct ones.

For simplicity we are going to define the complex coefficient

$$q^e_{a,b} = \int d\eta e^{-i\epsilon^e_\eta t/\hbar}\langle \phi^e_\eta|a\rangle\langle b|\phi^e_\eta\rangle, \tag{20}$$

where $\{|\phi^e_\eta\rangle\}$ and $\{\epsilon^e_\eta\}$ are the set of eigenvectors and eigenvalues of H_{2e}, respectively. a and b are indexes that may represent continuum and discrete eigenvectors. The explicit form of the coefficients inside the integral and relative discussions are in the Appendix. The notation is as follows: the upper index indicates which detection zone the atom is passing through. The two lower indexes, a and b, represent its initial incoming state and its final state after traversing the detector, respectively. One can notice that $\int dk|q^e_{e,k}|^2$ is the probability of an atom prepared in $|e\rangle$ to be ionized inside D_e, this can be understood as the efficiency of D_e. $\int dk|q^e_{g,k}|^2$ is the probability of an atom prepared in $|g\rangle$ to be ionized inside D_e, i.e., the probability of a wrong detection.

We can see, from Eq. (19), that the unitary evolution of this system allows for an indirect coupling between the two discrete levels. So we can take into account $|q^e_{e,g}|^2$, which is the probability of a transition between the two discrete levels. $|q^e_{e,e}|^2$ ($|q^e_{g,g}|^2$) is the probability of an atom prepared in $|e\rangle$ ($|g\rangle$) to interact with the electromagnetic field inside D_e and does not change level. We can also notice that $\int dk|q^g_{e,k}|^2$ ($\int dk|q^g_{g,k}|^2$) is the probability that an atom prepared in $|e\rangle$ ($|g\rangle$) to be ionized inside D_g.

A. An example: cavity QED

As we did for the intrinsically inefficient detectors, the interaction between atoms and false counting detectors can be separated in two processes: first, an unitary evolution of the initial state operator, generated by H_{2e} (H_{2g}) where H_{2e} (H_{2g}) have the form shown in Eq. (19), and then a projection in a proper subspace, which represents the classical information, click or nonclick, of the detector.

Starting from the initial state given by Eq. (7), and using the definitions in Eq. (20), the probability of click in D_e can be written as

$$p_{\text{click }D_e} = \int dk|q^e_{e,k}|^2\,\text{Tr}_C(\rho_{ee}) + \int dk|q^e_{g,k}|^2\,\text{Tr}_C(\rho_{gg}) + \left(\int dk q^{e*}_{e,k}q^e_{g,k}\text{Tr}_C(\rho_{eg}) + \text{H.c.}\right). \tag{21}$$

This expression shows us that $p_{\text{click }D_e}$ is sensitive to interference terms ρ_{eg} and ρ_{ge}. If we calculate the value of $p_{\text{click }D_e}$ for the initial state $\rho_{AC}(0)=\rho_{ee}|e\rangle\langle e| + \rho_{gg}|g\rangle\langle g|$, the answer would be different from Eq. (21). However, if we do the same, but replace the false counting detectors by inefficient or perfect detectors, the calculated probability would be the same for the two different initial states. That is due to the fact that this case is insensitive to interference terms.

In order to compare the modifications on the cavity field due to atomic interaction with inefficient detectors and false

counting detectors, we calculate the fidelity of the different state operators. Fidelity between the states ρ_A and ρ_B measures the overlap between them and is given by

$$F(\rho_A,\rho_B)=\left(\mathrm{Tr}\sqrt{\rho_A^{1/2}\rho_B\rho_A^{1/2}}\right)^2. \tag{22}$$

First, let us calculate the fidelity between state operators ρ_A^e (which describe the system after the atomic ionization inside the first detection zone of inefficient detectors), and ρ_B^e (which describe the system after the atomic ionization inside the first detection zone of false counting detectors). For simplicity, assume that the system atom–high-Q cavity field is found in the following entangled state just before the atom reaches the detection zones:

$$\rho_{AC}(0)=\frac{1}{2}(|e,0\rangle\langle e,0|+|e,0\rangle\langle g,1|+|g,1\rangle\langle e,0|+|g,1\rangle\langle g,1|). \tag{23}$$

After an unitary evolution and the projection on the continuum subspace, ρ_A^e and ρ_B^e can be written as

$$\rho_A^e=|0\rangle\langle 0|, \tag{24a}$$

$$\rho_B^e=\frac{\int dk(|q_{e,k}^e|^2|0\rangle\langle 0|+|q_{g,k}^e|^2|1\rangle\langle 1|+q_{g,k}^{e*}q_{e,k}^e|1\rangle\langle 0|+q_{e,k}^{e*}q_{g,k}^e|0\rangle\langle 1|)}{\int dk(|q_{e,k}^e|^2+|q_{g,k}^e|^2)}, \tag{24b}$$

and the fidelity

$$F(\rho_A^e,\rho_B^e)=\frac{\int dk|q_{e,k}^e|^2}{\int dk(|q_{e,k}^e|^2+|q_{g,k}^e|^2)}. \tag{25}$$

Notice that if the wrong detection probability goes to zero ($\int dk|q_{g,k}^e|^2\to 0$), the fidelity goes to one, $F(\rho_A^e,\rho_B^e)\to 1$, so ρ_A^e and ρ_B^e are identical.

Now, we are going to calculate the fidelity between state operators ρ_A^g, which describes the system after the atomic ionization inside the second detection zone of inefficient detectors, and ρ_B^g, which describes the system the atomic ionization inside the second detection zone of false counting detectors. The calculation is as follows: interaction of an atom with the first detection zone D_e, modeled by unitary evolution of the state given by Eq. (23) and projection on the discrete subspace. Then, the interaction with the second detection zone D_g, modeled again by unitary evolution of the resultant state and projection, but now on the continuum subspace. After this, we can write the fidelity as

$$F(\rho_A^g,\rho_B^g)=\frac{1}{A}\int dk(|q_{g,g}^{e*}|^2|q_{g,k}^{g*}|^2+q_{g,g}^{e*}q_{g,e}^eq_{g,k}^{g*}q_{e,k}^g+q_{g,g}^eq_{g,e}^{e*}q_{g,k}^gq_{e,k}^{g*}+|q_{g,e}^e|^2|q_{e,k}^g|^2), \tag{26}$$

where

$$A=\int dk(|q_{g,e}^e|^2|q_{g,k}^g|^2+q_{g,e}^{e*}q_{e,e}^eq_{g,k}^{g*}q_{e,k}^g+q_{g,e}^eq_{e,e}^{e*}q_{g,k}^gq_{e,k}^{g*}+|q_{e,e}^e|^2|q_{e,k}^g|^2)+\int dk(|q_{g,g}^e|^2|q_{g,k}^g|^2+q_{g,g}^{e*}q_{g,e}^eq_{g,k}^{g*}q_{e,k}^g+q_{g,g}^eq_{g,e}^{e*}q_{g,k}^gq_{e,k}^{g*}+|q_{g,e}^e|^2|q_{e,k}^g|^2). \tag{27}$$

As we are considering that any transition from a discrete state to the continuum generates a classical signal, we must not admit the possibility of indirect coupling between $|e\rangle$ and $|g\rangle$ mediated by the continuum. Therefore, we must assume that $|q_{g,e}^e|^2=0$ and we may write

$$F(\rho_A^g,\rho_B^g)=\frac{\int dk|q_{g,g}^e|^2|q_{g,k}^g|^2}{\int dk(|q_{g,g}^e|^2|q_{g,k}^g|^2+|q_{e,e}^e|^2|q_{e,k}^g|^2)}. \tag{28}$$

If the wrong detections probability in D_g goes to zero ($\int dk|q_{e,k}^g|^2\to 0$) the fidelity goes to one [$F(\rho_A^g,\rho_B^g)\to 1$].

B. An application to a recent experiment

As an example of applicability of this model, let us study an experiment reported in [12], where the time decay of a single photon Fock state prepared in a microwave cavity was measured. In this experiment, a pair of atoms interacts, separately, with an electromagnetic field mode stored in the high-Q cavity. The first atom is prepared in e state and undergoes a π pulse transferring the excitation to the field mode. After some time delay T, the second atom prepared in g state undergoes a π pulse and absorbs the photon left inside the

cavity by the first atom. Both atoms have their final state detected in ionization zones, the results are recorded for several realizations of this procedure in order to measure, as a function of the delay T, the conditional probability $\Pi_{ge}=P_{ge}/(P_{ge}+P_{gg})$ of the second atom to be in the e state provided the first one was detected in the g state. Here, P_{ge} is the probability to detect the first atom in D_g and the second in D_e and P_{gg} is the probability to detect both atoms in D_g. Ideally, the conditional probability Π_{ge} should be equal to unity at $T=0$. Although, due to some experimental imperfections pointed out by the authors, the value of the conditional probability extrapolated to zero delay is 70%–80%.

Let us apply the model for false counting detectors to calculate the conditional probability Π_{ge}. According to [12], due to coupling dispersions, the state of the system atom-cavity mode after the passage of the first atom through the cavity can be written as

$$\rho_1 = |\alpha|^2|g,1\rangle\langle g,1| + |\beta|^2|e,0\rangle\langle e,0| + (\gamma|g,1\rangle\langle e,0| + \text{H.c.}), \tag{29}$$

where $|\alpha|^2=0.94$ and $|\beta|^2=0.06$.

Following the model for false counting detectors, the probability to the first atom to be detected in D_g is given by

$$p_{\text{click } D_g}^{(1^{\text{st}}\text{ atom})} = \int dk\ \text{Tr}(|k\rangle\langle k|e^{-iH_{2e}t_2/\hbar}\rho_1' e^{iH_{2e}t_2/\hbar}), \tag{30}$$

where

$$\rho_1' = \frac{1}{N}[(|e\rangle\langle e| + |g\rangle\langle g|)e^{-iH_{2e}t_1/\hbar}\rho_1 e^{iH_{2e}t_1/\hbar}(|e\rangle\langle e| + |g\rangle\langle g|)]. \tag{31}$$

N is the normalization constant, t_1 is the time that the atom takes to cross D_e, and t_2 is the time that the atom takes to be ionized inside D_g.

Developing the calculation from Eq. (30), using the definition (20), and assuming that $|q_{e,g}^e|^2=|q_{g,e}^e|^2=0$, we can write

$$p_{\text{click } D_g}^{(1^{\text{st}}\text{ atom})} = \frac{|\alpha|^2(1-\int dk|q_{g,k}^e|^2)\int dk|q_{g,k}^g|^2 + |\beta|^2(1-\int dk|q_{e,k}^e|^2)\int dk|q_{e,k}^g|^2}{|\alpha|^2(1-\int dk|q_{g,k}^e|^2) + |\beta|^2(1-\int dk|q_{e,k}^e|^2)}. \tag{32}$$

The detection of the first atom in D_g reduces the state of the electromagnetic field inside the cavity to $|1\rangle$. After the passage of the second atom, the state of the system atom-cavity mode can be written as

$$\rho_2 = |\beta|^2|g,1\rangle\langle g,1| + |\alpha|^2|e,0\rangle\langle e,0| + (\delta|g,1\rangle\langle e,0| + \text{H.c.}). \tag{33}$$

Through a similar procedure used to calculate Eq. (32), we can calculate the probability of the second atom to be detected in D_g ($p_{\text{click } D_g}^{(2^{\text{nd}}\text{ atom})}$) and in D_e ($p_{\text{click } D_e}^{(2^{\text{nd}}\text{ atom})}$):

$$p_{\text{click } D_g}^{(2^{\text{nd}}\text{ atom})} = \frac{|\beta|^2(1-\int dk|q_{g,k}^e|^2)\int dk|q_{g,k}^g|^2 + |\alpha|^2(1-\int dk|q_{e,k}^e|^2)\int dk|q_{e,k}^g|^2}{|\beta|^2(1-\int dk|q_{g,k}^e|^2) + |\alpha|^2(1-\int dk|q_{e,k}^e|^2)}, \tag{34}$$

and

$$p_{\text{click } D_e}^{(2^{\text{nd}}\text{ atom})} = |\beta|^2\int dk|q_{g,k}^e|^2 + |\alpha|^2\int dk|q_{e,k}^e|^2. \tag{35}$$

In Ref. [12], the authors inform the efficiency of the detectors ($\int dk|q_{e,k}^e|^2=\int dk|q_{g,k}^g|^2=0.35$), the false counting rate on D_e ($\int dk|q_{g,k}^e|^2=0.13$) and the false counting rate on D_g ($\int dk|q_{e,k}^g|^2=0.1$). Therefore, the conditional probability at $T=0$ yields

$$\Pi_{ge}(0) = \frac{P_{ge}}{P_{ge}+P_{gg}} = \frac{p_{\text{click } D_g}^{(1^{\text{st}}\text{ atom})}p_{\text{click } D_e}^{(2^{\text{nd}}\text{ atom})}}{p_{\text{click } D_g}^{(1^{\text{st}}\text{ atom})}p_{\text{click } D_e}^{(2^{\text{nd}}\text{ atom})} + p_{\text{click } D_g}^{(1^{\text{st}}\text{ atom})}p_{\text{click } D_g}^{(2^{\text{nd}}\text{ atom})}} = 0.738, \tag{36}$$

a value close to the one obtained by the extrapolation to zero time delay (between the atoms) of the exponencial fit shown in Fig. 2 of Ref. [12].

IV. CONCLUSIONS

We have presented a dynamical model for the detection process of atomic levels on field ionization detectors. On the context of cavity QED, the model allows us to calculate the reduced state operator, for the field inside the cavity, after the classical signal generated by the detectors.

The detailed analysis of the detection process also let us introduce naturally the effects of realistic features of the detectors (e.g., efficiency and false counting rates) on the study of microwave cavity experiments. For intrinsically inefficient detectors, we found that the probability of a click in the second detection zone is sensitive to the efficiency of the first one. Besides, our results are in complete agreement with those obtained in Ref. [13] by different methods.

If one allows the detectors to register false countings, the probability of a click is sensitive to the "nondiagonal" or coherence terms of the state of the system atom–high-Q cavity field. In fact, false countings are a consequence of the coupling between the two discrete atomic levels to the continuum in each detection zone. As a result of this coupling, a click registered in any detector does not provide an *unequivocal* information about the atomic state. The detectors acts as a "beam splitter," mixing the two "paths" e and g, and, to some degree, these "paths" become undistinguishable.

We have also shown, by applying the present model in the experiment reported in [12], that it is possible to calculate, from a more realistic way, the empirical data produced in cavity QED experiments.

APPENDIX: EVALUATION OF THE COEFFICIENTS $\langle a|\phi_\eta^e\rangle$

In order to calculate the coefficients inside the integral in Eq. (20), let us write the eigenvalue equation for H_{2e},

$$H_{2e}|\phi_\eta^e\rangle = [H_{e(0)} + H_{2e(I)} + H_{2e(II)}]|\phi_\eta^e\rangle = \epsilon_\eta^e|\phi_\eta^e\rangle, \quad \text{(A1)}$$

where $|\phi_\eta^e\rangle$ and ϵ_η^e are eigenvectors and eigenvalues of the Hamiltonian H_{2e}, $H_{e(0)}=\epsilon_e|e\rangle\langle e| + \epsilon_g|g\rangle\langle g| + \int dk\epsilon_k|k\rangle\langle k|$, $H_{2e(I)}=w_e\int dk(|e\rangle\langle k| + |k\rangle\langle e|)$, and $H_{2e(II)}=w_g\int dk(|g\rangle\langle k| + |k\rangle\times\langle g|)$.

Let us project equation (A1) onto $|k\rangle$,

$$\langle k|H_{2e}|\phi_\eta^e\rangle = \langle k|H_{e(0)}|\phi_\eta^e\rangle + \langle k|H_{2e(I)}|\phi_\eta^e\rangle + \langle k|H_{2e(II)}|\phi_\eta^e\rangle$$
$$= \epsilon_k\langle k|\phi_\eta^e\rangle + w_e\langle e|\phi_\eta^e\rangle + w_g\langle g|\phi_\eta^e\rangle = \epsilon_\eta^e\langle k|\phi_\eta^e\rangle. \quad \text{(A2)}$$

We can do the same with the discrete states $|g\rangle$ and $|e\rangle$,

$$\langle g|H_{2e}|\phi_\eta^e\rangle = \langle g|H_{e(0)}|\phi_\eta^e\rangle + \langle g|H_{2e(I)}|\phi_\eta^e\rangle + \langle g|H_{2e(II)}|\phi_\eta^e\rangle$$
$$= \epsilon_g\langle g|\phi_\eta^e\rangle + w_g\int dk\langle k|\phi_\eta^e\rangle = \epsilon_\eta^e\langle g|\phi_\eta^e\rangle, \quad \text{(A3a)}$$

$$\langle e|H_{2e}|\phi_\eta^e\rangle = \langle e|H_{e(0)}|\phi_\eta^e\rangle + \langle e|H_{2e(I)}|\phi_\eta^e\rangle + \langle e|H_{2e(II)}|\phi_\eta^e\rangle$$
$$= \epsilon_e\langle e|\phi_\eta^e\rangle + w_e\int dk\langle k|\phi_\eta^e\rangle = \epsilon_\eta^e\langle e|\phi_\eta^e\rangle. \quad \text{(A3b)}$$

From the eigenvector's normalization, we can also obtain the following expression

$$|\langle g|\phi_\eta^e\rangle|^2 + |\langle e|\phi_\eta^e\rangle|^2 + \int dk|\langle k|\phi_\eta^e\rangle|^2 = 1. \quad \text{(A4)}$$

Defining the fundamental energy level as $\epsilon_g=0$, and using Eq. (A3a) and (A3b) we can write

$$\langle e|\phi_\eta^e\rangle = \frac{\epsilon_\eta^e w_e}{w_g(\epsilon_\eta^e - \epsilon_e)}\langle g|\phi_\eta^e\rangle. \quad \text{(A5)}$$

From Eq. (A2), we have

$$\langle k|\phi_\eta^e\rangle = \frac{1}{\epsilon_\eta^e - \epsilon_k}\left(w_g + \frac{\epsilon_\eta^e w_e^2}{w_g(\epsilon_\eta^e - \epsilon_e)}\right)\langle g|\phi_\eta^e\rangle, \quad \text{(A6)}$$

and from the normalization condition, we obtain

$$\langle g|\phi_\eta^e\rangle = \left\{1 + \left[\frac{\epsilon_\eta^e w_e}{w_g(\epsilon_\eta^e - \epsilon_e)}\right]^2 + \left[w_g + \frac{\epsilon_\eta^e w_e^2}{w_g(\epsilon_\eta^e - \epsilon_e)}\right]^2 \int dk\left(\frac{1}{\epsilon_\eta^e - \epsilon_k}\right)^2\right\}^{-1/2}. \quad \text{(A7)}$$

Therefore, Eqs. (A5)–(A7) give us the explicit form of the three coefficients.

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