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Disentanglement in a two-qubit system subjected to dissipation environments

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We investigate the time evolution of entanglement of various entangled states of a two-qubit system exposed to either thermal or squeezed reservoirs. We show that, except for the vacuum reservoir, the sudden-death of entanglement always exists in the thermal and squeezed reservoirs. We present explicit expression for the sudden-death time of entanglement for various entangled states. We find that the sudden-death of entanglement results from the portion of the double excitation component in the initial entangled state. In this sense, the maximally entangled states of a two-qubit system that do not have the double excitation component is more robust against the quantum fluctuations of the vacuum reservoir.

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I. INTRODUCTION

The state superposition principle of quantum theory allows a quantum system to be in a linear and coherent superposition of all possible states. This results in the coherent correlations of different states in a superposition leading to quantum states that are fundamentally different from classical correlations. In a one-party quantum system, quantum coherence leads to many novel phenomena such as lasing without inversion [1], enhancement of refractive index [2], electromagnetically induced transparency [3], correlated emission laser [4]. A multiparty quantum system, in addition to local quantum coherence that exists within each of subsystems, may have nonlocal or distributed quantum coherence that exists among several distinct subsystems. This nonlocal quantum correlation or quantum entanglement plays a crucial role in quantum information processing such as quantum teleportation [5], quantum dense coding [6], quantum cryptography [7], and quantum computing [8].

Quantum entanglement is, however, too fragile to play a real role in the real world because the inevitable interaction of the quantum systems with their surrounding environments leads to decoherence effects. On the other hand, it is essential to maintain quantum entanglement for a longer time for many applications of interest. In order to achieve this objective, a deep understanding of the decoherence mechanism is desirable. In recent years, several investigations have focused on this subject [9–14]. In particular, Yu and Eberly [11,12] showed that the dynamics of the quantum entanglement between two qubits interacting independently with either quantum noise or classical noise displays a completely different behavior from the dynamics of the local coherence. Instead of the exponential decay in time of the local coherence, quantum entanglement may disappear within a finite time in the dynamical evolution. This phenomenon is called "entanglement sudden death" [11]. The sudden death of entanglement of a two-qubit system under the influence of independent environments has been experimentally demonstrated in an all-optical setup [14].

In this paper, we consider a two-qubit system interacting with two independent thermal or squeezed environments. For a general class of two-qubit entangled states, we obtain an explicit expression of the sudden death time. We find that the appearance of entanglement sudden death strongly depends on the initial entangled state and the environment.

II. MODEL

We consider two two-level atoms 1 and 2 that present a two-qubit system and interact independently with their local environments, as schematically shown in Fig. 1. There is no direct interaction between the atoms. The correlation between the atoms results only from an initial quantum entanglement between them.

In the interaction picture, the Hamiltonian of the atom-field coupled system has the form ($\hbar=1$)

$$H=\sum_{\mathbf{k}}[g_{\mathbf{k}}^{(1)}e^{i(\omega-\nu_k)t}|a_1\rangle\langle b_1|a_{\mathbf{k}}+\text{H.c.}]+\sum_{\mathbf{k}}[g_{\mathbf{k}}^{(2)}e^{i(\omega-\nu_k)t}|a_2\rangle\langle b_2|b_{\mathbf{k}}+\text{H.c.}], \tag{1}$$

where $|b_i\rangle$ and $|a_i\rangle$ are the ground and excited states of the atom i, ω is the frequency separation between the atomic states, $a_{\mathbf{k}}(b_{\mathbf{k}})$ is the annihilation operator for the photons of the reservoir surrounding atom 1(2) in mode $\mathbf{k}$, ν_k is the frequency of the mode $\mathbf{k}$, and $g_{\mathbf{k}}^{(i)}$ is the coupling constant of the interaction between atom i and the local reservoir. When writing the Hamiltonian (1), we assume that the rotating-wave approximation is valid. The same model has been employed by Yu and Eberly [11,12]. In their investigation, the local reservoirs are in the vacuum state. In the present inves-

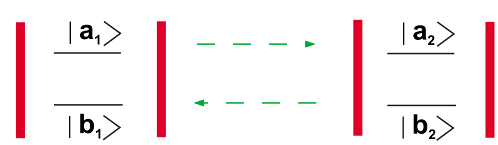

FIG. 1. (Color online) Two two-level atoms, initially prepared in an entangled state, have no directional interaction with each other but independently interact with their local reservoirs.

tigation, we consider two cases where the local reservoirs are in thermal and squeezed vacuum states. We find that the thermal and the squeezed reservoir always lead to the sudden-death of entanglement irrespective of the initial entangled state of the atoms.

III. SUDDEN-DEATH OF ENTANGLEMENT IN THERMAL RESERVOIR

In this section, we consider the dynamics of entanglement of the atoms in a thermal reservoir. According to the general quantum reservoir theory [15], with the Hamiltonian (1), we can derive the following equation of motion for the reduced density matrix of the atoms interacting with their local thermal reservoirs of mean thermal photon numbers m and n:

$$\begin{aligned}\frac{d\rho}{dt} &= -\frac{1}{2}\gamma_1(m+1)[\sigma_+^1\sigma_-^1\rho - 2\sigma_-^1\rho\sigma_+^1 + \rho\sigma_+^1\sigma_-^1] \\ &\quad - \frac{1}{2}\gamma_1 m[\sigma_-^1\sigma_+^1\rho - 2\sigma_+^1\rho\sigma_-^1 + \rho\sigma_-^1\sigma_+^1] \\ &\quad - \frac{1}{2}\gamma_2(n+1)[\sigma_+^2\sigma_-^2\rho - 2\sigma_-^2\rho\sigma_+^2 + \rho\sigma_+^2\sigma_-^2] \\ &\quad - \frac{1}{2}\gamma_2 n[\sigma_-^2\sigma_+^2\rho - 2\sigma_+^2\rho\sigma_-^2 + \rho\sigma_-^2\sigma_+^2], \end{aligned} \tag{2}$$

where γ_i is the spontaneous emission rate of atom i, and $\sigma_\pm^i$ are the raising (+) and lowering (−) operators of atom i, defined as $\sigma_+^i = |a_i\rangle\langle b_i|$ and $\sigma_-^i = |b_i\rangle\langle a_i|$. When deriving Eq. (2), we assume that the interaction between the atoms and the reservoirs is weak and there is no back reaction effect of the atoms on the reservoirs. It means that the reservoirs are at all times in the initial uncorrelated thermal equilibrium mixture of photon number states. Here, we also assume that the correlation time between the atoms and the reservoirs is much shorter than the characteristic time of the dynamic evolution of the atoms such as spontaneous emission life and entanglement sudden-death time so that the Markov approximation is valid.

The solution of Eq. (2) depends on the initial state of the atoms. We note that, for a class of the initial states that will be considered below, the solution of Eq. (2) has the matrix form in the representation spanned by two-qubit product states $|1\rangle = |a_1, a_2\rangle, |2\rangle = |a_1, b_2\rangle, |3\rangle = |b_1, a_2\rangle, |4\rangle = |b_1, b_2\rangle$

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{3}$$

A measure of entanglement shared by both the atoms is given by concurrence [16]. In order to calculate the concurrence, we first consider the matrix

$$M = \rho(\sigma_Y^1 \otimes \sigma_Y^2)\rho^*(\sigma_Y^1 \otimes \sigma_Y^2), \tag{4}$$

where

$$\sigma_Y^1 \otimes \sigma_Y^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

In the representation under consideration [Eq. (3)], the matrix M has the explicit form

$$M = \begin{pmatrix} \rho_{14}\rho_{41} + \rho_{11}\rho_{44} & 0 & 0 & 2\rho_{11}\rho_{14} \\ 0 & \rho_{23}\rho_{32} + \rho_{22}\rho_{33} & 2\rho_{22}\rho_{23} & 0 \\ 0 & 2\rho_{32}\rho_{33} & \rho_{23}\rho_{32} + \rho_{22}\rho_{33} & 0 \\ 2\rho_{44}\rho_{41} & 0 & 0 & \rho_{14}\rho_{41} + \rho_{11}\rho_{44} \end{pmatrix}. \tag{6}$$

Eigenvalues of the matrix (6) are easily found to be

$$\begin{aligned} \lambda_1 &= (\sqrt{\rho_{22}\rho_{33}} + \sqrt{\rho_{23}\rho_{32}})^2, \\ \lambda_2 &= (\sqrt{\rho_{22}\rho_{33}} - \sqrt{\rho_{23}\rho_{32}})^2, \\ \lambda_3 &= (\sqrt{\rho_{11}\rho_{44}} + \sqrt{\rho_{14}\rho_{41}})^2, \\ \lambda_4 &= (\sqrt{\rho_{11}\rho_{44}} - \sqrt{\rho_{14}\rho_{41}})^2. \end{aligned} \tag{7}$$

In terms of these eigenvalues, the concurrence can be expressed as

$$C = \mathrm{Max}\{0, 2(\sqrt{\rho_{23}\rho_{32}} - \sqrt{\rho_{11}\rho_{44}}), 2(\sqrt{\rho_{14}\rho_{41}} - \sqrt{\rho_{22}\rho_{33}})\}. \tag{8}$$

In the following, we use this formalism to investigate the dynamics of entanglement for several different initial cases.

(1) Consider the initial state $\rho(0) = (1-a)(|a_1, b_2\rangle + |b_1, a_2\rangle)(\langle a_1, b_2| + \langle b_1, a_2|)/2 + a|a_1, a_2\rangle\langle a_1, a_2|$ $(0 \leqslant a < 1)$. In this state, the maximally entangled state $(|a_1, b_2\rangle + |b_1, a_2\rangle)/\sqrt{2}$ is mixed with the excited state $|a_1, a_2\rangle$. For this initial state, the solution of Eq. (2) is given by Eq. (A2).

First let us consider the simple case of the standard vacuum reservoir, i.e., $n = m = 0$. For simplicity sake, we assume $\gamma_1 = \gamma_2 = \gamma$. From Eq. (A2), we obtain the eigenvalues of the resulting matrix M

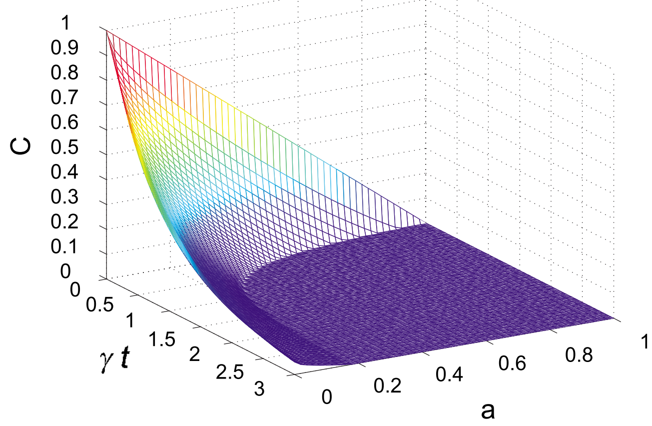

FIG. 2. (Color online) The time evolution of the concurrence in the vacuum reservoir when the atom initially in the state $(1-a)(|a_1,b_2\rangle+|b_1,a_2\rangle)(\langle a_1,b_2|+\langle b_1,a_2|)/2+a|a_1,a_2\rangle\langle a_1,a_2|$.

$$\lambda_1=[(1-ae^{-\gamma t})e^{-\gamma t}]^2,$$

$$\lambda_2=[a(1-e^{-\gamma t})e^{-\gamma t}]^2,$$

$$\lambda_3=\lambda_4=a[1-(a+1)e^{-\gamma t}+ae^{-2\gamma t}]e^{-2\gamma t}. \tag{9}$$

According to the concurrence formulation

$$C=\mathrm{Max}\{0,\sqrt{\lambda_1}-\sqrt{\lambda_2}-2\sqrt{\lambda_3}\}, \tag{10}$$

we find that the disentanglement process lasts for an infinite time period when $0\leq a\leq 3-2\sqrt{2}$. However, there exists the sudden-death phenomenon when $3-2\sqrt{2}<a\leq 1$. The sudden-death time is given by

$$t_d=\frac{1}{\gamma}\ln\frac{2a}{(1+a)-\sqrt{2}(1-a)}. \tag{11}$$

In Fig. 2, the time evolution of the concurrence for various values of the parameter a is shown.

Instead of mixing the excited state $|e_1,e_2\rangle$, we may mix the ground state $|g_1,g_2\rangle$ with the maximally entangled state. In that case, we find that the sudden-death of entanglement never happens. Thus the spontaneous emission of the initial portion of the double excitation is responsible for the sudden-death entanglement. From Eq. (11), we also see that the larger the portion of the double excitation in the initial state, the shorter the death time is.

(2) Consider the initial state $\rho(0)=a(|a_1,a_2\rangle+|b_1,b_2\rangle)(\langle a_1,a_2|+\langle b_1,b_2|)/4+(2-a)|a_1,a_2\rangle\langle a_1,a_2|/2$ $(0<a\leq 2)$. In this state, the maximally entangled state $(|a_1,a_2\rangle+|b_1,b_2\rangle)/\sqrt{2}$ is mixed with the double excited state $|a_1,a_2\rangle$. Here, unlike the state considered in the previous example, the maximally entangled state itself contains the double excitation component. For this initial state, the solution of Eq. (2) is given by Eq. (A3).

For the case $n=m=0$ and $\gamma_1=\gamma_2=\gamma$, we obtain the eigenvalues of the matrix M

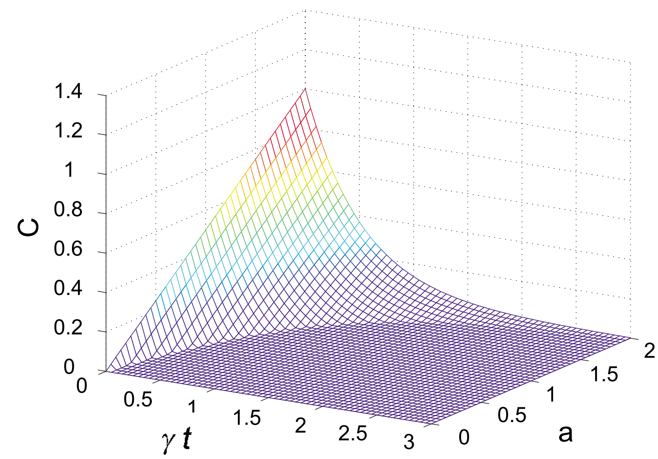

FIG. 3. (Color online) The time evolution of the concurrence for various values of the parameter a when the atoms initially in the state $a(|a_1,a_2\rangle+|b_1,b_2\rangle)(\langle a_1,a_2|+\langle b_1,b_2|)/4+(2-a)|a_1,a_2\rangle\langle a_1,a_2|/2$.

$$\lambda_1=\lambda_2=\left[\left(1-\frac{a}{4}\right)(1-e^{-\gamma t})e^{-\gamma t}\right]^2,$$

$$\lambda_3=\left\{\sqrt{\left(1-\frac{a}{4}\right)\left[1-2\left(1-\frac{a}{4}\right)e^{-\gamma t}+\left(1-\frac{a}{4}\right)e^{-2\gamma t}\right]}+\frac{a}{4}e^{-\gamma t}\right\}^2,$$

$$\lambda_4=\left\{\sqrt{\left(1-\frac{a}{4}\right)\left[1-2\left(1-\frac{a}{4}\right)e^{-\gamma t}+\left(1-\frac{a}{4}\right)e^{-2\gamma t}\right]}-\frac{a}{4}e^{-\gamma t}\right\}^2, \tag{12}$$

and the concurrence

$$C=\left[a-2+\left(2-\frac{a}{2}\right)e^{-\gamma t}\right]e^{-\gamma t}. \tag{13}$$

From Eq. (13), we see that the entanglement can survive for an infinite time period only when $a=2$. The sudden-death always happens when $0\leq a<2$. The sudden death time is given by

$$t_d=\frac{1}{\gamma}\ln\frac{a-4}{2(a-2)}. \tag{14}$$

The time evolution of the concurrence is shown in Fig. 3.

This example has two interesting features. One is that when $a=2$ the entanglement can last for an infinite period in spite of the existence of the double excitation component in the ideal entangled state. Another is that the asymptotical behavior of entanglement in time is immediately destroyed no matter how small amount of additional double excitation component is mixed in the ideal entangled state. In this sense, the entangled state $(|a_1,a_2\rangle+|b_1,b_2\rangle)/\sqrt{2}$ is more frag-

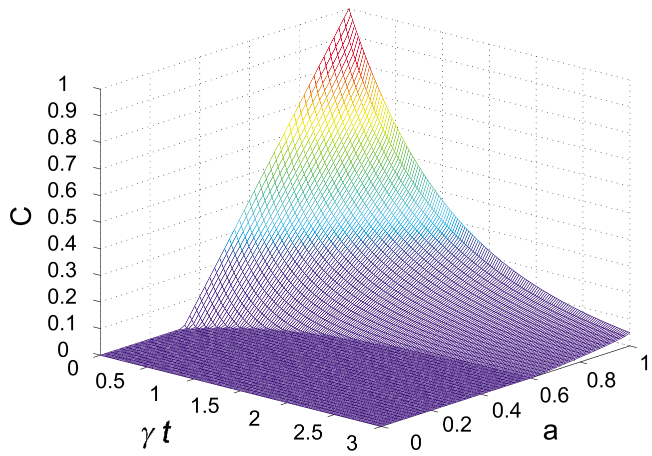

FIG. 4. (Color online) The time evolution of the concurrence when the atoms are initially in the Werner state.

ile than the entangled state $(|a_1,b_2\rangle+|b_1,a_2\rangle)/\sqrt{2}$ against the quantum fluctuation of the vacuum.

(3) Consider the Werner state [17]

$$\rho(0)=\frac{a}{2}(|a_1,b_2\rangle-|b_1,a_2\rangle)(\langle a_1,b_2|-\langle b_1,a_2|) + \frac{1-a}{4}(|a_1,a_2\rangle\langle a_1,a_2|+|b_1,b_2\rangle\langle b_1,b_2|+|a_1,b_2\rangle\langle a_1,b_2| + |b_1,a_2\rangle\langle b_1,a_2|). \quad (15)$$

In this state, the maximally entangled state $(|a_1,b_2\rangle-|b_1,a_2\rangle)/\sqrt{2}$ is mixed with the equally-weighted four possible states. For this initial state, the solution of Eq. (2) is given by Eq. (A4). Under the conditions $n=m=0$ and $\gamma_1=\gamma_2=\gamma$, we find the eigenvalues of the matrix M

$$\lambda_1=\left\{\frac{1}{4}[2(1+a)-(1-a)e^{-\gamma t}]e^{-\gamma t}\right\}^2,$$

$$\lambda_2=\left[\frac{1-a}{4}(2-e^{-\gamma t})e^{-\gamma t}\right]^2,$$

$$\lambda_3=\lambda_4=\frac{1-a}{16}[4-4e^{-\gamma t}+(1-a)e^{-2\gamma t}]e^{-2\gamma t}, \quad (16)$$

and the concurrence

$$C=\text{Max}\left\{0,\left[a-\sqrt{(1-a)\left(1-e^{-\gamma t}+\frac{1-a}{4}e^{-2\gamma t}\right)}\right]e^{-\gamma t}\right\}. \quad (17)$$

From Eq. (17), we see that the entanglement lasts for an infinite period when $\frac{\sqrt{5}-1}{2}<a\leq 1$. The sudden death of entanglement happens after a time

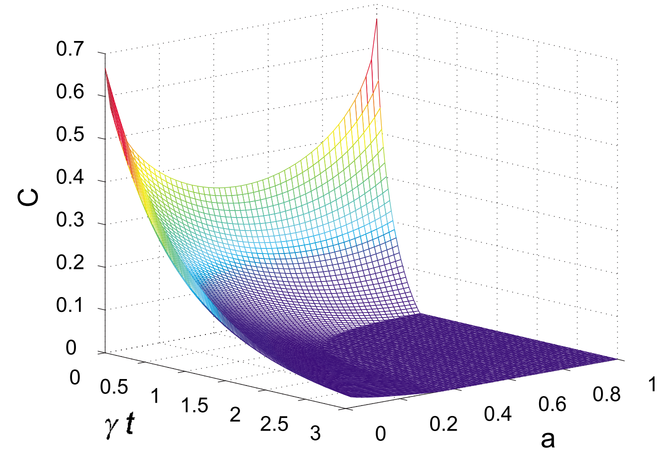

FIG. 5. (Color online) The time evolution of the concurrence when the atom initially in the state $\frac{a}{3}|a_1a_2\rangle\langle a_1a_2| + \frac{1-a}{3}|b_1b_2\rangle\langle b_1b_2| + \frac{1}{3}(|a_1b_2\rangle+|b_1a_2\rangle)(\langle a_1b_2|+\langle b_1a_2|)$.

$$t_d=\frac{1}{\gamma}\ln\frac{(a-1)}{2[\sqrt{a(a+1)}-1]} \quad (18)$$

when $0\leq a<\frac{\sqrt{5}-1}{2}$. The time evolution of the concurrence is shown in Fig. 4.

(4) Finally we consider the initial state $\rho(0)=\frac{a}{3}|a_1a_2\rangle\langle a_1a_2| + \frac{1-a}{3}|b_1b_2\rangle\langle b_1b_2| + \frac{1}{3}(|a_1b_2\rangle+|b_1a_2\rangle) \times(\langle a_1b_2|+\langle b_1a_2|)$. Unlike the initial state in example (1), this state contains both the double excitation and the ground state components at the same time. This state was considered by Yu and Eberly and obtained the sudden-death time with $a=1$ [11]. We reconsider this example for the sake of completeness and find the sudden-death time for any value of the parameter a. For this initial state, the solution of Eq. (2) is given by Eq. (A5).

For the standard vacuum $n=m=0$, and assuming $\gamma_1=\gamma_2=\gamma$, we find the eigenvalues of the matrix M

$$\lambda_1=\left\{\frac{1}{3}[(2+a)-ae^{-\gamma t}]e^{-\gamma t}\right\}^2,$$

$$\lambda_2=\left[\frac{a}{3}(1-e^{-\gamma t})e^{-\gamma t}\right]^2,$$

$$\lambda_3=\lambda_4=\frac{a}{9}[3-2(a+1)e^{-\gamma t}+ae^{-2\gamma t}]e^{-2\gamma t}, \quad (19)$$

and the concurrence

$$C=\text{Max}\left\{0,\frac{2}{3}[1-\sqrt{3a-2a(a+1)e^{-\gamma t}+a^2e^{-2\gamma t}}]e^{-\gamma t}\right\}. \quad (20)$$

From Eq. (20), we find that when $1/3<a\leq 1$ the sudden death of entanglement always happens and the sudden-death time is given by

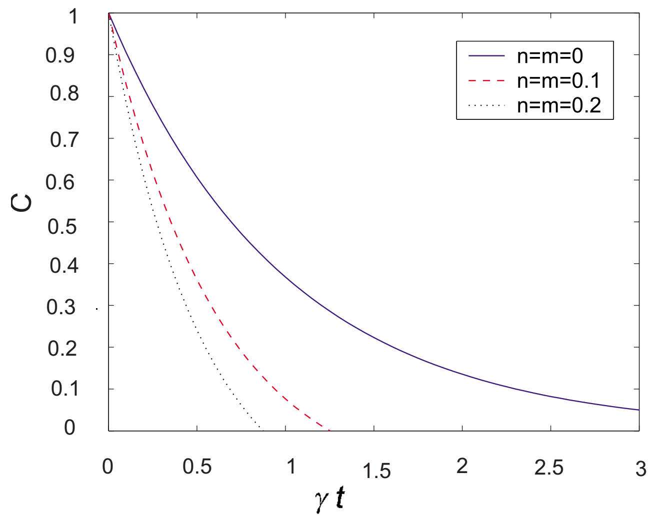

FIG. 6. (Color online) The time evolution of the concurrence in a thermal reservior when the atoms are initially in the entangled state state $(|a_1,b_2\rangle+|b_1,a_2\rangle)/\sqrt{2}$. The solid line is for the vacuum case ($n=m=0$), the dashed line for the thermal reservior with $n=m=0.1$, and the dotted line for the thermal reservior with $n=m=0.2$.

$$t_d=\frac{1}{\gamma}\ln\frac{a}{(a+1)-\sqrt{a^2-a+2}}. \tag{21}$$

The results obtained by Yu and Eberly [11] are recovered for the case $a=1$. From Eq. (21), t_d becomes infinite when $a\leqslant 1/3$, which means there is no sudden-death of entanglement. The time evolution of the concurrence for various values of the parameter a is shown in Fig. 5.

In the above examples, we see that the quantum fluctuation of the vacuum reservoir is not sufficient to destroy the entanglement in a finite time in some situations. The sudden-death of entanglement results from the decay of the mixed double excitation state component.

When the mean thermal photon number is not zero, we find that in the thermal reservoir the entanglement sudden-death always happens no matter which entangled state the atoms are initially in and no matter how small the nonzero mean thermal photon number is. In Fig. 6, the time evolution of the concurrence is plotted for the thermal reservoir with the nonzero mean photon number when the atoms are initially in the entangled state $(|a_1,b_2\rangle+|b_1,a_2\rangle)/\sqrt{2}$. In the first example above, we have shown that the entanglement can last for an infinite period in the vacuum reservoir for this initial entangled state. However, as shown in Fig. 6, the sudden-death of entanglement always happens in a thermal reservoir of nonzero mean photon number. It is also observed that the death time decreases as the mean thermal photon number becomes large.

IV. ENTANGLEMENT SUDDEN-DEATH IN SQUEEZED RESERVOIR

In this section, we consider the case in which atoms 1 and 2 are exposed in broadband squeezed vacuum reservoirs. According to the general quantum reservoir theory [15], we derive the equation of motion for the reduced density matrix of the atoms independently interacting with their local squeezed reservoirs

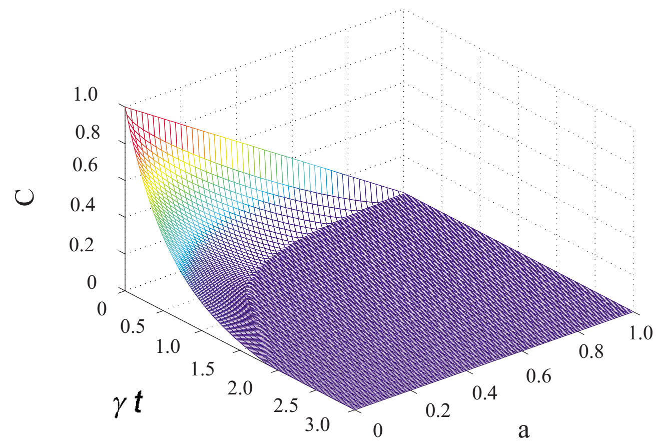

FIG. 7. (Color online) The time evolution of the concurrence when the atoms initially in the state $\frac{1-a}{2}(|a_1,b_2\rangle+|b_1,a_2\rangle)(\langle a_1,b_2|+\langle b_1,a_2|)+a|a_1,a_2\rangle\langle a_1,a_2|$ in the squeezed reservior with $r=0.3$.

$$\dot{\rho}=\sum_{\alpha=1,2}\Bigg[-\frac{\gamma_\alpha}{2}\cosh^2(r_\alpha)(\sigma_+^\alpha\sigma_-^\alpha\rho-2\sigma_-^\alpha\rho\sigma_+^\alpha+\rho\sigma_+^\alpha\sigma_-^\alpha) - \gamma_\alpha e^{-i\theta_\alpha}\sinh(r_\alpha)\cosh(r_\alpha)\sigma_-^\alpha\rho\sigma_-^\alpha-\frac{\gamma_\alpha}{2}\sinh^2(r_\alpha)(\sigma_-^\alpha\sigma_+^\alpha\rho - 2\sigma_+^\alpha\rho\sigma_-^\alpha+\rho\sigma_-^\alpha\sigma_+^\alpha)-\gamma_\alpha e^{i\theta_\alpha}\sinh(r_\alpha)\cosh(r_\alpha)\sigma_+^\alpha\rho\sigma_+^\alpha\Bigg], \tag{22}$$

where r_1 and r_2 are the squeezing parameters of the reservoirs, and θ_1 and θ_2 are the squeezing angles. When deriving the master equation (22), besides using the same assumptions as when deriving the master equation (2) but assuming the reservoirs to be in the squeezed vacuum instead of the uncorrelated thermal equilibrium mixture of photon number states, we also assume that squeezing bandwidths of the squeezed reservoirs are much larger than the atomic linewidths.

For the initial state

$$\rho(0)=\frac{1-a}{2}(|a_1,b_2\rangle+|b_1,a_2\rangle)(\langle a_1,b_2|+\langle b_1,a_2|) + a|a_1,a_2\rangle\langle a_1,a_2|, \tag{23}$$

the solution of the master equation (22) is given by Eq. (B2). With the solution, we can calculate the concurrence by using Eqs. (7) and (8). In the preceding section, we have found that for the initial state (23) the entanglement lasts for an infinite time in a vacuum reservoir when $a<1$. However, in the squeezed reservoir, we find that the entanglement sudden-death always happens no matter how small portion of the double excitation is in the initial state. This is shown explicitly in the numerical results plotted in Figs. 7 and 8. In Figs. 9 and 10, the time evolution of the concurrence is shown for

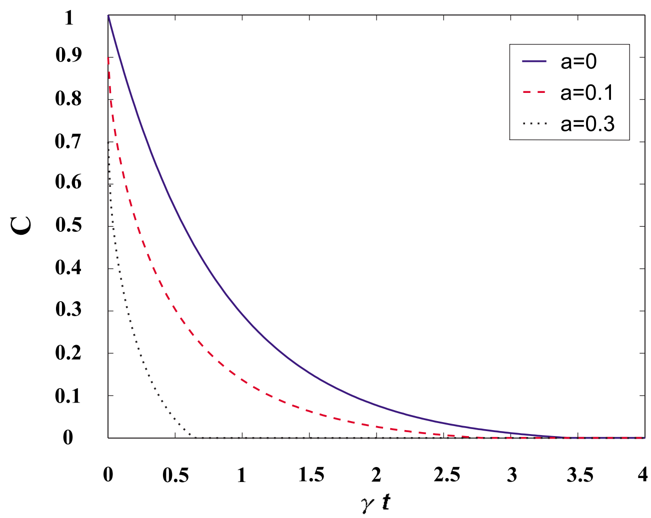

FIG. 8. (Color online) The time evolution of the concurrence when the atoms initially in the state $\frac{1-a}{2}(|a_1,b_2\rangle+|b_1,a_2\rangle)(\langle a_1,b_2| +\langle b_1,a_2|)+a|a_1,a_2\rangle\langle a_1,a_2|$ in the squeezed reservior with $r=0.1$.

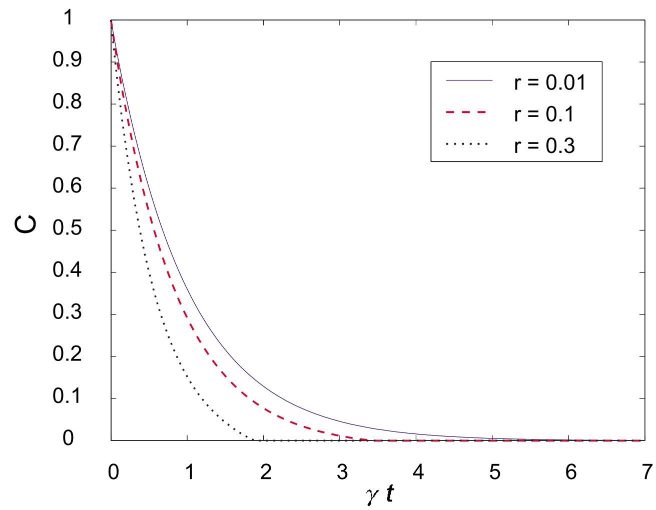

FIG. 10. (Color online) The time evolution of the concurrence when the atoms initially in the state $\frac{1}{2}(|a_1,b_2\rangle+|b_1,a_2\rangle)(\langle a_1,b_2| +\langle b_1,a_2|)$ in the squeezed rerservior with the squeezing parameter $r=0.01,0.1,0.3$.

different values of the degree of squeezing. We see that the sudden-death time of entanglement becomes shorter as r increases, i.e., the entanglement survives for a shorter time with increasing the degree of squeezing.

V. SUMMARY

We considered a two-qubit system consisting of two two-level atoms that are spatially separated from each other and independently coupled to local reservoirs that may be in either a thermal state or a vacuum squeezed state. We investigated the dynamical evolution of entanglement between the atoms coupled to the reservoirs. We show that, for a certain class of two-qubit entangled states, the entanglement measured by concurrence can suddenly disappear during the dynamic evolution in the vacuum reservoir. We find explicit expressions for the entanglement sudden death time for various entangled states. In contrast with the vacuum reservoir, we find that sudden death of entanglement always happens in the thermal reservoir with nonzero mean photon number and the squeezed reservoir. The exponential decay of entanglement is a very special result to the vacuum reservoir. The above results are not just for discrete-variable quantum systems. In fact, for continuous-variable two-party quantum systems, it has been shown that entanglement initially in a two-mode squeezed state disappears in a finite time period in the thermal environment but can last for an infinite time in the vacuum environment [18,19].

Note added in proof. Entanglement sudden death has been observed recently by Almeida *et al.* [20]. See also the paper by Eberly and Yu [21].

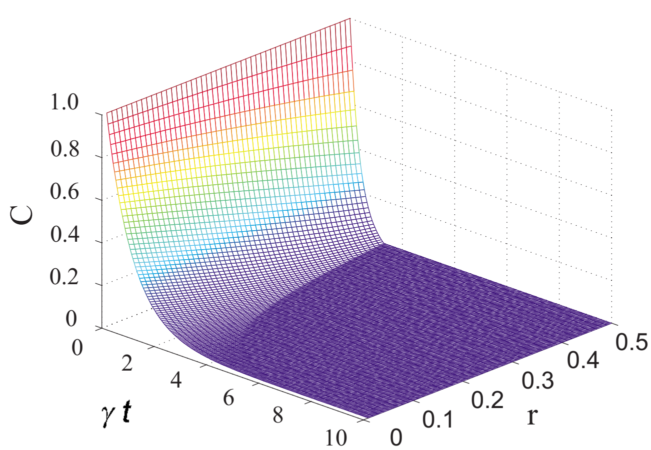

FIG. 9. (Color online) The time evolution of the concurrence when the atoms initially in the state $\frac{1}{2}(|a_1,b_2\rangle+|b_1,a_2\rangle)(\langle a_1,b_2| +\langle b_1,a_2|)$ in the squeezed rerservior.

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APPENDIX A

In the representation spanned by two-qubit product states $|1\rangle=|a_1,a_2\rangle, |2\rangle=|a_1,b_2\rangle, |3\rangle=|b_1,a_2\rangle, |4\rangle=|b_1,b_2\rangle$, Eq. (2) can be written in the matrix form

$$\dot{\rho}_{11}=-[(m+1)\gamma_1+(n+1)\gamma_2]\rho_{11}+n\gamma_2\rho_{22}+m\gamma_1\rho_{33},$$

$$\dot{\rho}_{22}=-[(m+1)\gamma_1+n\gamma_2]\rho_{22}+(n+1)\gamma_2\rho_{11}+m\gamma_1\rho_{44},$$

$$\dot{\rho}_{33} = -[(n+1)\gamma_2 + m\gamma_1]\rho_{33} + (m+1)\gamma_1\rho_{11} + n\gamma_2\rho_{44},$$

$$\dot{\rho}_{44} = -[m\gamma_1 + n\gamma_2]\rho_{44} + (m+1)\gamma_1\rho_{22} + (n+1)\gamma_2\rho_{33},$$

$$\dot{\rho}_{23} = -\left[\left(m+\frac{1}{2}\right)\gamma_1 + \left(n+\frac{1}{2}\right)\gamma_2\right]\rho_{23},$$

$$\dot{\rho}_{32} = -\left[\left(m+\frac{1}{2}\right)\gamma_1 + \left(n+\frac{1}{2}\right)\gamma_2\right]\rho_{32},$$

$$\dot{\rho}_{14} = -\left[\left(m+\frac{1}{2}\right)\gamma_1 + \left(n+\frac{1}{2}\right)\gamma_2\right]\rho_{14},$$

$$\dot{\rho}_{41} = -\left[\left(m+\frac{1}{2}\right)\gamma_1 + \left(n+\frac{1}{2}\right)\gamma_2\right]\rho_{41}. \tag{A1}$$

The solution of this equation depends on the initial state. In the following, we list solutions of Eq. (A1) for various initial states.

(1) For the initial state

$$\rho(0) = (1-a)(|a_1,b_2\rangle + |b_1,a_2\rangle)(\langle a_1,b_2| + \langle b_1,a_2|)/2 + a|a_1,a_2\rangle\langle a_1,a_2| (0 \leq a < 1),$$

the solution of Eq. (A1) is given by

$$\rho_{11} = \frac{1}{2(2m+1)(2n+1)}\{2mn + m[a(2n+1)+1] \times e^{-(2n+1)\gamma_2 t} + n[a(2m+1)+1]e^{-(2m+1)\gamma_1 t} + (4amn + 3am + 3an - 2mn - m - n + 2a) \times e^{-[(2m+1)\gamma_1 + (2n+1)\gamma_2]t}\},$$

$$\rho_{22} = \frac{1}{2(2m+1)(2n+1)}\{2m(n+1) - m[a(2n+1)+1] \times e^{-(2n+1)\gamma_2 t} + (n+1)[a(2m+1)+1]e^{-(2m+1)\gamma_1 t} - (4amn + 3am + 3an - 2mn - m - n + 2a)e^{-[(2m+1)\gamma_1 + (2n+1)\gamma_2]t}\},$$

$$\rho_{33} = \frac{1}{2(2m+1)(2n+1)}\{2(m+1)n + (m+1)[a(2n+1)+1] \times e^{-(2n+1)\gamma_2 t} - n[a(2m+1)+1]e^{-(2m+1)\gamma_1 t} - (4amn + 3am + 3an - 2mn - m - n + 2a)e^{-[(2m+1)\gamma_1 + (2n+1)\gamma_2]t}\},$$

$$\rho_{44} = \frac{1}{2(2m+1)(2n+1)}\{2(m+1)(n+1) - (m+1)[a(2n+1)+1]e^{-(2n+1)\gamma_2 t} - (n+1) \times [a(2m+1)+1]e^{-(2m+1)\gamma_1 t} + (4amn + 3am + 3an - 2mn - m - n + 2a)e^{-[(2m+1)\gamma_1 + (2n+1)\gamma_2]t}\},$$

$$\rho_{23} = \rho_{32} = \frac{1-a}{2}\exp\left[-\left(m+\frac{1}{2}\right)\gamma_1 t - \left(n+\frac{1}{2}\right)\gamma_2 t\right]. \tag{A2}$$

(2) For the initial state $\rho(0)=a(|a_1,a_2\rangle+|b_1,b_2\rangle)(\langle a_1,a_2| + \langle b_1,b_2|)/4+(2-a)|a_1,a_2\rangle\langle a_1,a_2|/2 (0 < a \leq 2)$, the solution of Eq. (A1) is given by

$$\rho_{11} = \frac{1}{4(2m+1)(2n+1)}\{4mn - m[2an - 4n + a - 4] \times e^{-(2n+1)\gamma_2 t} - n[2am - 4m + a - 4] \times e^{-(2m+1)\gamma_1 t} - [am + an - 4mn - 4m - 4n + a - 4] \times e^{-[(2m+1)\gamma_1 + (2n+1)\gamma_2]t}\},$$

$$\rho_{22} = \frac{1}{4(2m+1)(2n+1)}\{4m(n+1) + m[2an - 4n + a - 4] \times e^{-(2n+1)\gamma_2 t} - (n+1)[2am - 4m + a - 4] \times e^{-(2m+1)\gamma_1 t} + [am + an - 4mn - 4m - 4n + a - 4] \times e^{-[(2m+1)\gamma_1 + (2n+1)\gamma_2]t}\},$$

$$\rho_{33} = \frac{1}{4(2m+1)(2n+1)}\{4(m+1)n - (m+1) \times [2an - 4n + a - 4]e^{-(2n+1)\gamma_2 t} + n[2am - 4m + a - 4] \times e^{-(2m+1)\gamma_1 t} + [am + an - 4mn - 4m - 4n + a - 4] \times e^{-[(2m+1)\gamma_1 + (2n+1)\gamma_2]t}\},$$

$$\rho_{44} = \frac{1}{4(2m+1)(2n+1)}\{4(m+1)(n+1) + (m+1) \times [2an - 4n + a - 4]e^{-(2n+1)\gamma_2 t} + (n+1) \times [2am - 4m + a - 4]e^{-(2m+1)\gamma_1 t} - [am + an - 4mn - 4m - 4n + a - 4]e^{-[(2m+1)\gamma_1 + (2n+1)\gamma_2]t}\},$$

$$\rho_{14} = \rho_{41} = \frac{a}{4}\exp\left[-\left(m+\frac{1}{2}\right)\gamma_1 t - \left(n+\frac{1}{2}\right)\gamma_2 t\right]. \tag{A3}$$

(3) For the Werner state (15), the solution of Eq. (A1) is given by

$$\rho_{11} = \frac{1}{4(2m+1)(2n+1)}\{4mn + 2me^{-(2n+1)\gamma_2 t} + 2ne^{-(2m+1)\gamma_1 t} + [1 - a(2m+1)(2n+1)]e^{-[(2m+1)\gamma_1 + (2n+1)\gamma_2]t}\},$$

$$\rho_{22} = \frac{1}{4(2m+1)(2n+1)}\{4m(n+1) - 2me^{-(2n+1)\gamma_2 t} + 2(n+1)e^{-(2m+1)\gamma_1 t} - [1 - a(2m+1)(2n+1)] \times e^{-[(2m+1)\gamma_1 + (2n+1)\gamma_2]t}\},$$

$$\rho_{33} = \frac{1}{4(2m+1)(2n+1)}\{4(m+1)n + 2(m+1)e^{-(2n+1)\gamma_2 t} - 2ne^{-(2m+1)\gamma_1 t} - [1 - a(2m+1)(2n+1)] \times e^{-[(2m+1)\gamma_1 + (2n+1)\gamma_2]t}\},$$

$$\rho_{44}=\frac{1}{4(2m+1)(2n+1)}\{4(m+1)(n+1)-2(m+1)\times e^{-(2n+1)\gamma_2 t}-2(n+1)e^{-(2m+1)\gamma_1 t}+[1-a(2m+1)(2n+1)]e^{-[(2m+1)\gamma_1+(2n+1)\gamma_2]t}\},$$

$$\rho_{23}=\rho_{32}=-\frac{a}{2}\exp\left[-\left(m+\frac{1}{2}\right)\gamma_1 t-\left(n+\frac{1}{2}\right)\gamma_2 t\right]. \quad (A4)$$

(4) For the initial state $\rho(0)=\frac{a}{3}|a_1a_2\rangle\langle a_1a_2|+\frac{1-a}{3}|b_1b_2\rangle\times\langle b_1b_2|+\frac{1}{3}(|a_1b_2\rangle+|b_1a_2\rangle)(\langle a_1b_2|+\langle b_1a_2|)$, the solution of Eq. (A1) is given by

$$\rho_{11}=\frac{1}{3(2m+1)(2n+1)}\{3mn+m(2an+a-n+1)e^{-(2n+1)\gamma_2 t}+n(2am+a-m+1)e^{-(2m+1)\gamma_1 t}+(am+an+a-mn-m-n)e^{-[(2m+1)\gamma_1+(2n+1)\gamma_2]t}\},$$

$$\rho_{22}=\frac{1}{3(2m+1)(2n+1)}\{3m(n+1)-m(2an+a-n+1)\times e^{-(2n+1)\gamma_2 t}+(n+1)(2am+a-m+1)e^{-(2m+1)\gamma_1 t}-(am+an+a-mn-m-n)e^{-[(2m+1)\gamma_1+(2n+1)\gamma_2]t}\},$$

$$\rho_{33}=\frac{1}{3(2m+1)(2n+1)}\{3(m+1)n+(m+1)(2an+a-n+1)e^{-(2n+1)\gamma_2 t}-n(2am+a-m+1)e^{-(2m+1)\gamma_1 t}-(am+an+a-mn-m-n)e^{-[(2m+1)\gamma_1+(2n+1)\gamma_2]t}\},$$

$$\rho_{44}=\frac{1}{3(2m+1)(2n+1)}\{3(m+1)(n+1)-(m+1)(2an+a-n+1)e^{-(2n+1)\gamma_2 t}-(n+1)(2am+a-m+1)e^{-(2m+1)\gamma_1 t}+(am+an+a-mn-m-n)e^{-[(2m+1)\gamma_1+(2n+1)\gamma_2]t}\},$$

$$\rho_{23}=\rho_{32}=\frac{1}{3}\exp\left[-\left(m+\frac{1}{2}\right)\gamma_1 t-\left(n+\frac{1}{2}\right)\gamma_2 t\right]. \quad (A5)$$

APPENDIX B

In the representation employed in writing Eq. (A1), the master equation (22) yields the following equation for the various matrix elements:

$$\dot{\rho}_{11}=-[\gamma_1\cosh^2(r_1)+\gamma_2\cosh^2(r_2)]\rho_{11}+\gamma_2\sinh^2(r_2)\rho_{22}+\gamma_1\sinh^2(r_1)\rho_{33},$$

$$\dot{\rho}_{22}=-[\gamma_1\cosh^2(r_1)+\gamma_2\sinh^2(r_2)]\rho_{22}+\gamma_2\cosh^2(r_2)\rho_{11}+\gamma_1\sinh^2(r_1)\rho_{44},$$

$$\dot{\rho}_{33}=-[\gamma_1\sinh^2(r_1)+\gamma_2\cosh^2(r_2)]\rho_{33}+\gamma_1\cosh^2(r_1)\rho_{11}+\gamma_2\sinh^2(r_2)\rho_{44},$$

$$\dot{\rho}_{44}=-[\gamma_1\sinh^2(r_1)+\gamma_2\sinh^2(r_2)]\rho_{44}+\gamma_1\cosh^2(r_1)\rho_{22}+\gamma_2\cosh^2(r_2)\rho_{33},$$

$$\dot{\rho}_{23}=-\frac{1}{2}\left\{[\gamma_1\cosh(2r_1)+\gamma_2\cosh(2r_2)]\rho_{23}+\frac{1}{2}[\gamma_2 e^{i\theta_2}\sinh(2r_2)\rho_{14}+\gamma_1 e^{i\theta_1}\sinh(2r_1)\rho_{41}]\right\},$$

$$\dot{\rho}_{32}=-\frac{1}{2}\left\{[\gamma_1\cosh(2r_1)+\gamma_2\cosh(2r_2)]\rho_{32}+\frac{1}{2}[\gamma_1 e^{i\theta_1}\sinh(2r_1)\rho_{14}+\gamma_2 e^{i\theta_2}\sinh(2r_2)\rho_{41}]\right\},$$

$$\dot{\rho}_{14}=-\frac{1}{2}\left\{[\gamma_1\cosh(2r_1)+\gamma_2\cosh(2r_2)]\rho_{14}+\frac{1}{2}[\gamma_2 e^{i\theta_2}\sinh(2r_2)\rho_{23}+\gamma_1 e^{i\theta_1}\sinh(2r_1)\rho_{32}]\right\},$$

$$\dot{\rho}_{41}=-\frac{1}{2}\left\{[\gamma_1\cosh(2r_1)+\gamma_2\cosh(2r_2)]\rho_{41}+\frac{1}{2}[\gamma_1 e^{i\theta_1}\sinh(2r_1)\rho_{23}+\gamma_2 e^{i\theta_2}\sinh(2r_2)\rho_{32}]\right\}. \quad (B1)$$

For the initial state (23), the solution of Eq. (B1) is given by

$$\rho_{11}=\frac{1}{2(C_2+S_2)(C_1+S_1)}[2S_1S_2+S_2(aS_1+aC_1+C_1-S_1)\times e^{-(C_1+S_1)t}+S_1(aS_2+aC_2+C_2-S_2)e^{-(C_2+S_2)t}+(aS_1C_2+aC_1S_2+2aC_2C_1-C_2S_1-C_1S_2)\times e^{-(C_1+S_1+C_2+S_2)t}],$$

$$\rho_{22}=\frac{1}{2(C_2+S_2)(C_1+S_1)}[2S_1C_2+C_2(aS_1+aC_1+C_1-S_1)\times e^{-(C_1+S_1)t}-S_1(aS_2+aC_2+C_2-S_2)e^{-(C_2+S_2)t}-(aS_1C_2+aC_1S_2+2aC_2C_1-C_2S_1-C_1S_2)e^{-(C_1+S_1+C_2+S_2)t}],$$

$$\rho_{33} = \frac{1}{2(C_2+S_2)(C_1+S_1)}[2C_1S_2 - S_2(aS_1 + aC_1 + C_1 - S_1) \times e^{-(C_1+S_1)t} + C_1(aS_2 + aC_2 + C_2 - S_2)e^{-(C_2+S_2)t} - (aS_1C_2 + aC_1S_2 + 2aC_2C_1 - C_2S_1 - C_1S_2)e^{-(C_1+S_1+C_2+S_2)t}],$$

$$\rho_{44} = \frac{1}{2(C_2+S_2)(C_1+S_1)}[2C_1C_2 - C_2(aS_1 + aC_1 + C_1 - S_1) \times e^{-(C_1+S_1)t} - C_1(aS_2 + aC_2 + C_2 - S_2)e^{-(C_2+S_2)t} + (aS_1C_2 + aC_1S_2 + 2aC_2C_1 - C_2S_1 - C_1S_2)e^{-(C_1+S_1+C_2+S_2)t}],$$

$$\rho_{23} = \rho_{32} = \frac{1-a}{2}e^{-1/2(C_3+C_4)t}\cosh\left[\frac{1}{2}(S_3+S_4)t\right],$$

$$\rho_{14} = \rho_{41} = -\frac{1-a}{2}e^{-1/2(C_3+C_4)t}\sinh\left[\frac{1}{2}(S_3+S_4)t\right], \tag{B2}$$

where

$$C_\alpha = \gamma_\alpha \cosh^2(r_\alpha), \quad S_\alpha = \gamma_\alpha \sinh^2(r_\alpha), \quad (\alpha = 1,2),$$

$$C_\beta = \gamma_\beta \cosh(2r_\beta), \quad S_\beta = \gamma_\beta \sinh(2r_\beta), \quad (\beta = 3,4). \tag{B3}$$

In the above, for simplicity, we assume $\theta_1 = \theta_2 = 0$.

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