Monogamy inequality in terms of negativity for three-qubit states

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We propose an entanglement measure to quantify three-qubit entanglement in terms of negativity. A monogamy inequality analogous to the Coffman-Kundu-Wootters inequality is established. This consequently leads to a definition of residual entanglement, which is referred to as the three-π in order to distinguish it from the three-tangle. The three-π is proved to be a natural entanglement measure. By contrast to the three-tangle, it is shown that the three-π always gives greater than zero values for pure states belonging to the W and Greenberger-Horne-Zeilinger classes, implying that three-way entanglement always exists for them; the three-tangle generally underestimates the three-way entanglement of a given system. This investigation will offer an alternative tool to understand genuine multipartite entanglement.

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I. INTRODUCTION

Quantum entanglement lies at the heart of quantum-information processing and quantum computation [1]; accordingly, its quantification has drawn much attention in the last decade. In order to achieve such quantification, legitimate measures of entanglement are needed as a first step. The existing well-known bipartite measure of entanglement with an elegant formula is the concurrence derived analytically by Wootters [2], and the entanglement of formation [3,4] is a monotonically increasing function of the concurrence. They have been applied to describing quantum phase transitions in various interacting quantum many-body systems [5,6]. Another useful entanglement measure is negativity [8], regarded as a quantitative version of Peres' criterion for separability. Compared with the concurrence, the process of calculating the negativity is significantly simplified with respect to mixed states, since it does not need the convex-proof extension.

On the other hand, multipartite entanglement is a valuable physical resource in large-scale quantum-information processing [7] and plays an important role in condensed matter physics. The negativity has been used to study multipartite entanglement in a Fermi gas [9]. However, it is a formidable task to quantify multipartite entanglement since there are few well-defined multipartite entanglement measures, just as for bipartite systems. For now, the widely used basis for characterizing and quantifying tripartite entanglement is the three-tangle [10]. Very recently, a proof of the general Coffman-Kundu-Wootters (CKW) inequality for bipartite entanglement [11] and a demonstration that the CKW inequality cannot generalize to higher-dimensional systems [12] have been provided.

Recall that the concurrence of a two-qubit state ρ is defined as $\mathcal{C}(\rho)\equiv\max\{0,\sqrt{\lambda_1}-\sqrt{\lambda_2}-\sqrt{\lambda_3}-\sqrt{\lambda_4}\}$, in which $\lambda_1,\ldots,\lambda_4$ are the eigenvalues of the matrix $\rho(\sigma_y\otimes\sigma_y)\rho^*(\sigma_y\otimes\sigma_y)$ in nonincreasing order and σ_y is a Pauli spin matrix. For a pure three-qubit state ρ_{ABC}, the CKW inequality in terms of concurrence reads

$$\mathcal{C}_{AB}^2+\mathcal{C}_{AC}^2\leqslant\mathcal{C}_{A(BC)}^2, \tag{1}$$

where $\mathcal{C}_{AB}$ and $\mathcal{C}_{AC}$ are the concurrences of the mixed states $\rho_{AB}=\mathrm{Tr}_C(|\phi\rangle_{ABC}\langle\phi|)$ and $\rho_{AC}=\mathrm{Tr}_B(|\phi\rangle_{ABC}\langle\phi|)$, respectively, and $\mathcal{C}_{A(BC)}=2\sqrt{\det\rho_A}$ with $\rho_A=\mathrm{Tr}_{BC}(|\phi\rangle_{ABC}\langle\phi|)$. According to Eq. (1) the three-tangle can be defined as

$$\tau_{ABC}=C_{A(BC)}^2-C_{AB}^2-C_{AC}^2, \tag{2}$$

which is used to characterize three-way entanglement of the state [13]. For example, quantified by the three-tangle, the state $|\mathrm{GHZ}\rangle=(1/\sqrt{2})(|000\rangle+|111\rangle)$ has only three-way entanglement (the GHZ state is the Greenberger-Horne-Zeilinger state), while the state $|W\rangle=(1/\sqrt{3})(|100\rangle+|010\rangle+|001\rangle)$ has only two-way entanglement. For a general mixed three-qubit state of ρ_{ABC}, the three-tangle should be $\tau_{ABC}=\min[C_{A(BC)}^2]-C_{AB}^2-C_{AC}^2$, where $C_{A(BC)}^2$ has to be minimized for all possible decompositions of ρ_{ABC}. Now one may wonder whether there exist other entanglement measures satisfying Eq. (1), and whether the three-way entanglement of a given state provided by these entanglement measures is the same. This will help us to further understand genuine multipartite entanglement.

To this end, the main result of this paper is to provide a monogamy inequality in terms of negativity. In Sec. II, we recall some basic concepts of negativity. In Sec. III, we deduce the monogamy inequality in terms of negativity. In Secs. IV and V, the three-π measure, analogous to the three-tangle is defined, which is shown to be a natural entanglement measure. By calculation on the $|W\rangle$ state, the $|\mathrm{GHZ}\rangle$ state, and the superposed states of the two states, the three-π is shown to be greater than zero, i.e., for such states there always exists three-way entanglement. It is also shown that the three-π is always not less than the three-tangle for any tripartite pure states and can be extended to mixed three-qubit states. In Sec. VI, the monogamy inequality is extended to pure multiqubit states. The conclusions are in Sec. VII.

II. BASIC CONCEPTS OF NEGATIVITY

For either a pure or mixed state ρ, in the tensor product $\mathcal{H}_A\otimes\mathcal{H}_B$ of two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ for two subsystems A and B, the partial transpose with respect to the A subsystem is $(\rho^{T_A})_{ij,kl}=(\rho)_{kj,il}$ and the negativity is defined by $\mathcal{N}=(\|\rho^{T_A}\|-1)/2$ where the trace norm $\|R\|$ is given by $\|R\|$

$=\mathrm{Tr}\sqrt{RR^{\dagger}}$. $\mathcal{N}>0$ is the necessary and sufficient inseparable condition for the $2\otimes 2$ and $2\otimes 3$ bipartite quantum systems [14]. In order for any maximally entangled state in $2\otimes 2$ systems to have the negativity 1, it can be reexpressed as

$$\mathcal{N}=\|\rho^{T_A}\|-1, \tag{3}$$

with only a change of the constant factor 2. Therefore $\mathcal{N}=1$ for Bell states like $(1/\sqrt{2})(|01\rangle+|01\rangle)$ and vanishes for factorized states. For pure two-qubit systems in terms of the coefficients $\{\phi_{00},\phi_{01},\phi_{10},\phi_{11}\}$ of $|\phi_{AB}\rangle$ with respect to an orthonormal basis, the concurrence is defined as $\mathcal{C}_{AB}=2|\phi_{00}\phi_{11}-\phi_{01}\phi_{10}|$. From Eq. (3) it is easy to check that $\mathcal{N}_{AB}=\mathcal{C}_{AB}$ for such systems. Now let us consider pure three-qubit systems A, B, and C in the standard basis $\{|ijk\rangle\}$, where each index takes the values 0 and 1: $|\phi\rangle_{ABC}=\Sigma_{ijk}\phi_{ijk}|ijk\rangle$. For our goal it is necessary to show $\mathcal{N}_{A(BC)}=\mathcal{C}_{A(BC)}$. The density matrix of $|\phi\rangle_{ABC}$ is $\rho=|\phi\rangle_{ABC}\langle\phi|$ and $\rho^{T_A}=\Sigma_{ijk,i'j'k'}\phi_{ijk}\phi^{*}_{i'j'k'}|i'jk\rangle\langle ij'k'|$. Following from Eq. (3) we arrive at

$$\begin{aligned}\mathcal{N}_{A(BC)}&=\Big\|\sum_{ijk,i'j'k'}\phi_{ijk}\phi^{*}_{i'j'k'}|i'jk\rangle\langle ij'k'|\Big\|-1\\&=\Big\|\sum_{ijk}\phi_{ijk}|jk\rangle\langle i|\otimes\sum_{i'j'k'}\phi^{*}_{i'j'k'}|i'\rangle\langle j'k'|\Big\|-1\\&=\|R\otimes R^{\dagger}\|-1=\|R\|^2-1=2\sqrt{\lambda_0\lambda_1}=\mathcal{C}_{A(BC)},\end{aligned} \tag{4}$$

where $R=\Sigma_{i'j'k'}\phi^{*}_{i'j'k'}|i'\rangle\langle j'k'|$, and λ_0 and λ_1 are eigenvalues of $RR^{\dagger}$. The third formula obtained is based on the property of the trace norm $\|G\otimes Q\|=\|G\|\cdot\|Q\|$, the observation that $RR^{\dagger}=\Sigma_{i'j'k',ijk}\phi_{ijk}\phi^{*}_{i'j'k'}|i'\rangle\langle j'k'|\cdot|jk\rangle\langle i|$, and that $\|R\|$ is equal to the sum of the square root of the eigenvalues λ_i of $RR^{\dagger}$ with $\lambda_0+\lambda_1=1$. From the further observation that λ_0 and λ_1 are also the eigenvalues of the reduced density matrix $\rho_A=\mathrm{Tr}_{BC}(|\phi\rangle_{ABC}\langle\phi|)$, whose matrix elements are $\mu_{00}=\Sigma_{jk}\phi_{0jk}\phi^{*}_{0jk}$, $\mu_{01}=\Sigma_{jk}\phi_{0jk}\phi^{*}_{1jk}$, $\mu_{10}=\Sigma_{jk}\phi_{1jk}\phi^{*}_{0jk}$, and $\mu_{11}=\Sigma_{jk}\phi_{1jk}\phi^{*}_{1jk}$, and that the concurrence between A and BC is defined as $\mathcal{C}_{A(BC)}=\sqrt{2(1-\mathrm{Tr}\,\rho_A^2)}=2\sqrt{\lambda_0\lambda_1}$, the last formula is obtained. The next sections are devoted to one of the main results of this paper.

III. MONOGAMY INEQUALITY IN TERMS OF NEGATIVITY

For any pure $2\otimes 2\otimes 2$ states $|\phi\rangle_{ABC}$, the entanglement quantified by the negativity between A and B, between A and C, and between A and the single object BC satisfies the following CKW-inequality-like monogamy inequality:

$$\mathcal{N}^2_{AB}+\mathcal{N}^2_{AC}\leqslant\mathcal{N}^2_{A(BC)}, \tag{5}$$

where $\mathcal{N}_{AB}$ and $\mathcal{N}_{AC}$ are the negativities of the mixed states $\rho_{AB}=\mathrm{Tr}_C(|\phi\rangle_{ABC}\langle\phi|)$ and $\rho_{AC}=\mathrm{Tr}_B(|\phi\rangle_{ABC}\langle\phi|)$, respectively.

In order to prove Eq. (5) it is helpful to recall the theorem appearing in [15], which states that for any $m\otimes n$ $(m\leqslant n)$ mixed state ρ, the concurrence $\mathcal{C}(\rho)$ satisfies

$$\sqrt{\frac{2}{m(m-1)}}(\|\rho^{T_A}\|-1)\leqslant\mathcal{C}(\rho). \tag{6}$$

In our considered qubit system, $m=n=2$. Therefore it follows from Eqs. (3) and (6) that $\mathcal{N}\leqslant\mathcal{C}$, implying that the negativity is never greater than the concurrence in this case. Thus for the state $|\phi\rangle_{ABC}$ we have

$$\mathcal{N}_{AB}\leqslant\mathcal{C}_{AB},\quad\mathcal{N}_{AC}\leqslant\mathcal{C}_{AC}. \tag{7}$$

Observing Eqs. (1), (4), and (7), the conclusion in Eq. (5) can be proved.

In a similar way, if one takes the different focus B and C, the following monogamy inequalities:

$$\mathcal{N}^2_{BA}+\mathcal{N}^2_{BC}\leqslant\mathcal{N}^2_{B(AC)} \tag{8}$$

and

$$\mathcal{N}^2_{CA}+\mathcal{N}^2_{CB}\leqslant\mathcal{N}^2_{C(AB)} \tag{9}$$

hold also.

Now one is naturally concerned about the tightness of the monogamy inequality in Eq. (5). All pure three-qubit states can be sorted into six classes through stochastic local operation and classical communication (SLOCC) [13]. (1) The A-B-C class including product states; (2) A-BC, (3) B-BC, and (4) C-AB classes including bipartite entanglement states; (5) W and (6) GHZ classes including genuine tripartite entanglement states. For the first four classes it is easy to verify that Eqs. (5), (8), and (9) turn out to be an equality, being the same as the CKW inequality. However, it is different for the W class. For the pure state of ABC

$$|\phi\rangle=\alpha|100\rangle+\beta|010\rangle+\gamma|001\rangle, \tag{10}$$

which belongs to the W class. Substituting $\mathcal{N}^2_{AB}=4\alpha^2\beta^2+2\gamma^4-2\gamma^2\sqrt{\gamma^4+4\alpha^2\beta^2}$, $\mathcal{N}^2_{AC}=4\alpha^2\gamma^2+2\beta^4-2\beta^2\sqrt{\beta^4+4\alpha^2\gamma^2}$, and $\mathcal{N}^2_{A(BC)}=4\alpha^2(\beta^2+\gamma^2)$ into Eq. (5) we have $\gamma^4+\beta^4<\gamma^2\sqrt{\gamma^4+4\alpha^2\beta^2}+\beta^2\sqrt{\beta^4+4\alpha^2\gamma^2}$, with the result that the inequality in Eq. (5) is strict because $\alpha\neq 0$, $\beta\neq 0$, and $\gamma\neq 0$, while the CKW inequality can only be an equality for the W class [13].

Having seen that both the equality and inequality in Eq. (5) can be satisfied by some three-qubit states, we can define the residual entanglement, which is referred to as the three-π in order to distinguish it from the three-tangle in the following main results of this paper.

IV. THE THREE-π ENTANGLEMENT MEASURE

The difference between the two sides of Eq. (5) can be interpreted as the residual entanglement

$$\pi_A=\mathcal{N}^2_{A(BC)}-\mathcal{N}^2_{AB}-\mathcal{N}^2_{AC}. \tag{11}$$

Likewise, Eqs. (8) and (9) create the corresponding residual entanglement as

$$\pi_B=\mathcal{N}^2_{B(AC)}-\mathcal{N}^2_{BA}-\mathcal{N}^2_{BC} \tag{12}$$

and

$$\pi_C = \mathcal{N}^2_{C(AB)} - \mathcal{N}^2_{CA} - \mathcal{N}^2_{CB}, \tag{13}$$

respectively. The subscripts A, B, and C in π_A, π_B, and π_C mean that qubit A, qubit B, and qubit C are taken as the focus, respectively. Unlike the three-tangle, in general $\pi_A \neq \pi_B \neq \pi_C$, which can be easily confirmed after calculating them for the state in Eq. (10). This indicates that the residual entanglement corresponding to the different focus varies under permutations of the qubits. We take π_{ABC} (referred to as the three-π) as the average of π_A, π_B, and π_C, i.e.,

$$\pi_{ABC} = \frac{1}{3}(\pi_A + \pi_B + \pi_C), \tag{14}$$

which thus becomes invariant under permutations of the qubits since, for example, permutation of qubit A and qubit B accordingly only leads to exchanging π_A and π_B with each other in π_{ABC}.

As we will prove here, the three-π in Eq. (14) is a natural entanglement measure satisfying three necessary conditions [16]. The first condition is that the three-π should be local unitary (LU) invariant. After the LU transformations U_A, U_B, and U_C have acted separately on a pure three-qubit state ρ_{ABC}, the state can read $\rho'_{ABC}=U_A\otimes U_B\otimes U_C\rho_{ABC}U_A^\dagger\otimes U_B^\dagger\otimes U_C^\dagger$. It is necessary to prove that the six squared negativities in Eq. (14) are invariant under the three simultaneous LU transformations. Since $\rho'_A=\mathrm{Tr}_{BC}\,\rho'_{ABC}=U_A\rho_A U_A^\dagger$ and $\mathcal{N}'_{A(BC)}=\mathcal{C}'_{A(BC)}=\sqrt{2(1-\mathrm{Tr}\,\rho_A'^2)}=\mathcal{N}_{A(BC)}$, $\mathcal{N}_{A(BC)}$ is LU invariant. Similarly, $\mathcal{N}_{B(AC)}$ and $\mathcal{N}_{C(AB)}$ are also LU invariant; while $\rho'_{AB}=\mathrm{Tr}_C\,\rho'_{ABC}=U_A\otimes U_B\rho_{AB}U_A^\dagger\otimes U_B^\dagger$, together with the property that the negativity itself is LU invariant [17,18], leads to $\mathcal{N}(\rho'_{AB})=\mathcal{N}(\rho_{AB})$. Thus $\mathcal{N}(\rho_{AB})$ is LU invariant, and so are $\mathcal{N}(\rho_{BC})$ and $\mathcal{N}(\rho_{AC})$. Now we finish proving the first condition.

Observation of Eqs. (5), (8), and (9), shows that $\pi_{ABC}\geqslant 0$; thus the second condition is satisfied. Moreover, it is easy to verify that $\pi_{ABC}=0$ for product pure states. π_{ABC} is invariant under permutations of the qubits allowing us to use the proof outlined in [13] to confirm the third condition. Let us consider local positive operator valued measures (POVMs) for one qubit only. Let A_1 and A_2 be two POVM elements such that $A_1^\dagger A_1 + A_2^\dagger A_2 = I$. We can write $A_i = U_i D_i V$, with U_i and V being unitary matrices, and D_i being diagonal matrices with entries (a,b) and $(\sqrt{1-a^2},\sqrt{1-b^2})$, respectively. Consider an arbitrary initial state $|\psi\rangle$ of qubits A, B, and C with $\pi_{ABC}(\psi)$. After the POVM, $|\phi'\rangle = A_i|\psi\rangle$. Normalizing them gives $|\phi_i\rangle = |\phi'_i\rangle/\sqrt{p_i}$ with $p_i=\langle\phi'_i|\phi'_i\rangle$ and $p_1+p_2=1$. Therefore $\langle\pi_{ABC}\rangle = p_1\pi_{ABC}(\phi_1)+p_2\pi_{ABC}(\phi_2)$. Taking into account both the fact that $\pi_{ABC}(U_iD_iV\psi)=\pi_{ABC}(D_iV\psi)$ due to its LU invariance and the key fact that the three-π is also a quartic function of its coefficients in the standard basis, which can be seen from the calculation for the state of Eq. (10), we have $\pi_{ABC}(\phi_1)=(a^2b^2/p_1^2)\pi_{ABC}(\psi)$ and $\pi_{ABC}(\phi_2)=[(1-a^2)^2(1-b^2)^2/p_2^2]\pi_{ABC}(\psi)$. After simple algebraic calculations, we obtain $\langle\pi_{ABC}\rangle\leqslant\pi_{ABC}(\psi)$; thus the third condition that the three-π should be an entanglement monotone is satisfied.

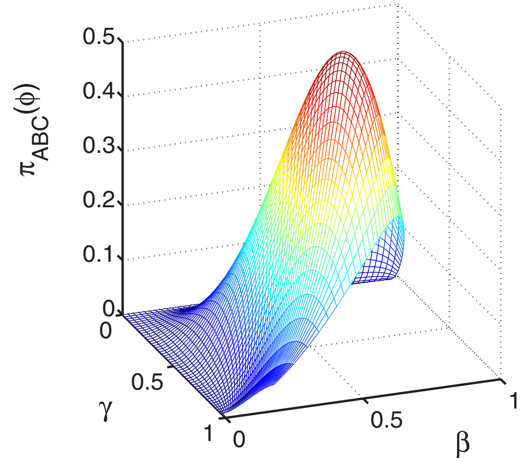

FIG. 1. (Color online) Three-π for the state in Eq. (9) as a function of the coefficients β and γ. Only two coefficients are independent since $|\alpha|^2+|\beta|^2+|\gamma|^2=1$. $\pi_{ABC}(\phi)$ is always greater than zero and reaches the maximal value $(4/9)(\sqrt{5}-1)$ when $\alpha=\beta=\gamma=1/\sqrt{3}$, which, to a certain degree, demonstrates why the W state is maximally entangled.

V. DEMONSTRATION OF TWO EXAMPLES OF THE THREE-π

In order to explicitly see the difference between the three-π and the three-tangle we present the following two examples.

Example 1: The different classes by SLOCC. For the state in Eq. (10) belonging to the W class we get

$$\begin{aligned}\pi_{ABC}(\phi) &= \frac{4}{3}(\alpha^2\sqrt{\alpha^4+4\beta^2\gamma^2}+\beta^2\sqrt{\beta^4+4\alpha^2\gamma^2}\\ &\quad+\gamma^2\sqrt{\gamma^4+4\alpha^2\beta^2}-\alpha^4-\beta^4-\gamma^4)\\ &> \tau_{ABC}(\phi)=0.\end{aligned} \tag{15}$$

We also have performed extensive numerical calculations on the three-π of the other states in the W class, and found that it is always greater than zero [i.e., $\pi_{ABC}(W)>0$] as shown in Eq. (15) for the state $|\phi\rangle$ (see also Fig. 1), implying that these states have three-way entanglement, also. Taking into account that $\tau_{ABC}(W)=0$, the conclusion that the three-tangle underestimates three-way entanglement can be drawn. For the GHZ class we have the property that $\pi_{ABC}(\mathrm{GHZ})\geqslant\tau_{ABC}(\mathrm{GHZ})>0$, while $\pi_{ABC}(\Phi)=\tau_{ABC}(\Phi)=0$ for the states $|\Phi\rangle_{ABC}$ belonging to the classes excluding the W and GHZ classes.

Example 2: Superpositions of GHZ and W states. Quantifying of multipartite mixed states is also a fundamental issue in quantum-information theory. An optimal decompositions for the three-tangle of mixed three-qubit states composed of a GHZ state and a W state is obtained [19]. In order to further explore the relationship between the three-π and three-tangle, we first write down the superposed state of GHZ and W states:

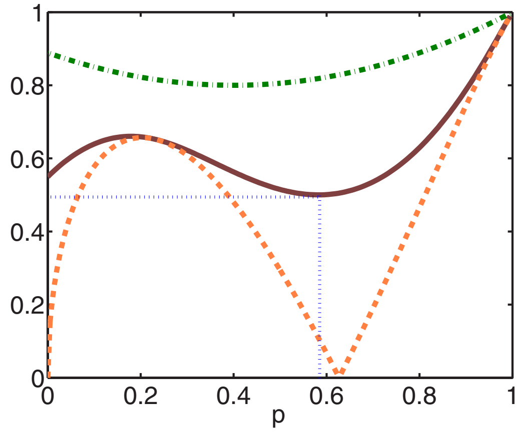

FIG. 2. (Color online) Plot of $\pi_{ABC}^{(-)}$ (solid line), $\tau_{ABC}^{(-)}$ (dashed line), and squared $\mathcal{N}_{A(BC)}^{(-)}$ (dash-dotted line) for the state $|\Psi^{(-)}\rangle_{ABC}$ in Eq. (15) as a function of p. The dotted line indicates the minimal value of $\pi_{ABC}^{(-)}$. The two measures match at $p=0.2$ and 1.

$$|\Psi^{(\pm)}\rangle_{ABC} = \sqrt{p}|\mathrm{GHZ}\rangle \pm \sqrt{1-p}|W\rangle. \tag{16}$$

The three-tangle for $|\Psi^{(\pm)}\rangle_{ABC}$ is known as $\tau_{ABC}^{(\pm)} = |p^2 \pm (8\sqrt{6}/9)\sqrt{p(1-p)^3}|$ and with Eqs. (11)–(14) we plot $\pi_{ABC}^{(\pm)}$ (see Figs. 2 and 3). The two measures show similar trends and the fact that $\pi_{ABC} \geqslant \tau_{ABC}$ is shown. Notice that a similar result was obtained also in [20]; however, their defined residual entanglement $E=\mathcal{N}_{A(BC)}-\mathcal{N}_{AB}-\mathcal{N}_{AC}$ is not an entanglement measure [21]. On the other hand, for the state $|\Psi^{(-)}\rangle_{ABC}$ the location of p of the minimal value of the two measures does not match (see Fig. 2), i.e., when $p \approx 0.58$ the extremely minimal $\pi_{ABC}^{(-)} \approx 0.5$, which is smaller than $\pi_{ABC}(W) = \frac{4}{9}(\sqrt{5}-1) \approx 0.55$, being equal to $\pi_{ABC}^{(-)}$ when $p=0$. But $\tau_{ABC}^{(-)}=0$ when $p=4\sqrt[3]{2}/(3+\sqrt[3]{2}) \approx 0.63$ for the state $|\Psi^{(-)}\rangle_{ABC}$, which provides a basis for the optimal decomposition of mixtures of the GHZ and W states [19]. In a similar way, we can also achieve optimal decomposition of such

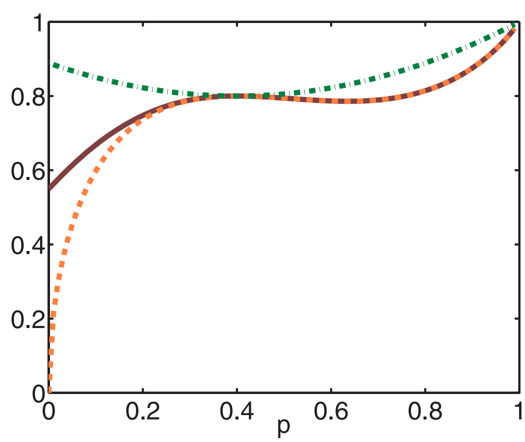

FIG. 3. (Color online) Plot of $\pi_{ABC}^{(+)}$ (solid line), $\tau_{ABC}^{(+)}$ (dashed line), and squared $\mathcal{N}_{A(BC)}^{(+)}$ (dash-dotted line) for the state $|\Psi^{(+)}\rangle_{ABC}$ in Eq. (15) as a function of p. The two measures match at $p \in [0.4,1]$; together with the squared $\mathcal{N}_{A(BC)}^{(+)}$, they match at $p=0.4$.

mixed states for the three-π [22]. Note that, for mixed three-qubit states of ABC, the monogamy inequality Eq. (5) turns out to be

$$\mathcal{N}_{AB}^2 + \mathcal{N}_{AC}^2 \leqslant \min[\mathcal{N}_{A(BC)}^2], \tag{17}$$

which has to be minimized for all possible decomposition of ρ_{ABC}. The other inequalities in Eqs. (8) and (9) need the same manipulations.

VI. EXTENSION TO PURE MULTIQUBIT STATES

The generalized CKW inequality for the case of n qubits is proved [11] analogously to Eq. (1), which can be expressed as

$$\mathcal{C}_{A_1A_2}^2 + \mathcal{C}_{A_1A_3}^2 + \cdots + \mathcal{C}_{A_1A_n}^2 \leqslant \mathcal{C}_{A_1(A_2A_3\cdots A_n)}^2, \tag{18}$$

where we denote n qubits by $A_1, A_2, \ldots, A_n$, and $\mathcal{C}_{A_1(A_2A_3\cdots A_n)}^2$ is the bipartite quantum entanglement across the bipartition $A_1{:}A_2A_3\cdots A_n$. Provided that the concurrence $\mathcal{C}$ is replaced with the negativity $\mathcal{N}$, it is desirable to know whether Eq. (18) holds. As we will show in the following, it does hold. For the bipartition $A_1{:}A_2A_3\cdots A_n$ Eq. (4) may generalize to

$$\mathcal{N}_{A_1(A_2A_3\cdots A_n)} = \mathcal{C}_{A_1(A_2A_3\cdots A_n)}. \tag{19}$$

In the previous section we have shown that $\mathcal{N} \leqslant \mathcal{C}$ for mixed two-qubit states, so in this case we have

$$\mathcal{N}_{A_1A_2} \leqslant \mathcal{C}_{A_1A_2},$$

$$\mathcal{N}_{A_1A_3} \leqslant \mathcal{C}_{A_1A_3},$$

$$\vdots$$

$$\mathcal{N}_{A_1A_n} \leqslant \mathcal{C}_{A_1A_n}. \tag{20}$$

Thus Eqs. (18)–(20) result in

$$\mathcal{N}_{A_1A_2}^2 + \mathcal{N}_{A_1A_3}^2 + \cdots + \mathcal{N}_{A_1A_n}^2 \leqslant \mathcal{N}_{A_1(A_2A_3\cdots A_n)}^2, \tag{21}$$

which may also be used to study the entanglement for a wide class of complex quantum systems [11].

In order to show the applicability of the generalized monogamy inequality in Eq. (21) we present two examples. Before continuing, we denote by $\tau_{A_1A_2\cdots A_n}$ the residual entanglement based on Eq. (18), and by $\pi_{A_1A_2\cdots A_n}$ the residual entanglement based on Eq. (21). The first example is the generalized GHZ state [25]

$$|\varphi\rangle_{A_1A_2\cdots A_n} = \frac{1}{\sqrt{2}}(|0^{\otimes n}\rangle + |1^{\otimes n}\rangle). \tag{22}$$

It is easy to check that $\pi(|\varphi\rangle_{A_1A_2\cdots A_n}) = \tau(|\varphi\rangle_{A_1A_2\cdots A_n}) = 1$. The second example is the generalized multiqubit state

$$|\psi\rangle_{A_1A_2\cdots A_n} = \alpha_1|100\cdots 0\rangle + \alpha_2|010\cdots 0\rangle + \cdots + \alpha_n|000\cdots 1\rangle. \tag{23}$$

For this state, one obtains that $\mathcal{C}_{A_1A_2}^2 = 4\alpha_1^2\alpha_2^2$, $\mathcal{C}_{A_1A_3}^2$

$=4\alpha_1^2\alpha_3^2,\ldots,\ \mathcal{C}^2_{A_1A_n}=4\alpha_1^2\alpha_n^2$, and $\mathcal{C}^2_{A_1(A_2A_3\cdots A_n)}=4\alpha_1^2(\alpha_2^2+\alpha_3^2+\cdots+\alpha_n^2)$, and these quantities make Eq. (18) an equality. As a result, we get $\tau(|\psi\rangle_{A_1A_2\cdots A_n})=0$, which was also shown in [10]. On the other hand, one can obtain

$$\mathcal{N}^2_{A_1A_2}=4\alpha_1^2\alpha_2^2+2(1-\alpha_1^2-\alpha_2^2)^2 - 2(1-\alpha_1^2-\alpha_2^2)\sqrt{(1-\alpha_1^2-\alpha_2^2)^2+4\alpha_1^2\alpha_2^2},$$

$$\mathcal{N}^2_{A_1A_3}=4\alpha_1^2\alpha_3^2+2(1-\alpha_1^2-\alpha_3^2)^2 - 2(1-\alpha_1^2-\alpha_3^2)\sqrt{(1-\alpha_1^2-\alpha_3^2)^2+4\alpha_1^2\alpha_3^2},\quad\ldots,$$

$$\mathcal{N}^2_{A_1A_n}=4\alpha_1^2\alpha_n^2+2(1-\alpha_1^2-\alpha_n^2)^2 - 2(1-\alpha_1^2-\alpha_n^2)\sqrt{(1-\alpha_1^2-\alpha_n^2)^2+4\alpha_1^2\alpha_n^2},$$

and

$$\mathcal{N}^2_{A_1(A_2A_3\cdots A_n)}=4\alpha_1^2(\alpha_2^2+\alpha_3^2+\cdots+\alpha_n^2),$$

and these quantities make Eq. (21) a strict inequality. Thus we get $\pi(|\psi\rangle_{A_1A_2\cdots A_n})>0=\tau(|\psi\rangle_{A_1A_2\cdots A_n})$.

These two examples show that the residual entanglement defined by Eq. (21) is always greater than zero for genuine multiqubit entangled states, which makes us believe that the measure of entanglement based on this residual entanglement would be a powerful candidate for a good measure of entanglement for multipartite pure states.

VII. CONCLUSIONS

Summarizing, we proved a monogamy inequality in terms of negativity such that the three-π is defined so as to quantify the residual entanglement for three-qubit states. The three-π is shown to be a natural entanglement measure and can be extended to mixed states and general pure n-qubit states. The three-way entanglement for the W and GHZ classes quantified by the three-π always exists, while the three-tangle is zero for the W class. Compared to the three-π, the three-tangle generally underestimates the entanglement. Note that the monogamy inequality for distributed Gaussian entanglement in terms of negativity was also established [23] and the information-theoretic measure of genuine multiqubit entanglement based on bipartite partitions was introduced [24]. Therefore, further investigation by using the results in this paper will help us deeply understand genuine multipartite entanglement.

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[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, U.K., 2000).
[2] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[3] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
[4] S. Hill and W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).
[5] A. Osterloh, L. Amico, G. Falci, and R. Fazio Nature (London) **416**, 608 (2002).
[6] L. A. Wu, M. S. Sarandy, and D. A. Lidar, Phys. Rev. Lett. **93**, 250404 (2004).
[7] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).
[8] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[9] C. Lunkes, Č. Brukner, and V. Vedral, Phys. Rev. Lett. **95**, 030503 (2005).
[10] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).
[11] T. J. Osborne and F. Verstraete, Phys. Rev. Lett. **96**, 220503 (2006).
[12] Y. C. Ou, Phys. Rev. A **75**, 034305 (2007).
[13] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
[14] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
[15] K. Chen, S. Albeverio, and S. M. Fei, Phys. Rev. Lett. **95**, 040504 (2005).
[16] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).
[17] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[18] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[19] R. Lohmayer, A. Osterloh, J. Siewert, and A. Uhlmann, Phys. Rev. Lett. **97**, 260502 (2006).
[20] S. Shelly Sharma and N. K. Sharma, e-print arXiv:quant-ph/0609012.
[21] For example, when $\alpha^2=0.9$ and $\beta^2=\gamma^2=0.05$ for the state in Eq. (10), we get $\mathcal{N}_{AB}+\mathcal{N}_{AC}\approx 0.75>0.6=\mathcal{N}_{A(BC)}$, leading to $E<0$.
[22] Y. C. Ou and H. Fan (unpublished).
[23] T. Hiroshima, G. Adesso, and F. Illuminati, Phys. Rev. Lett. **98**, 050503 (2007).
[24] J. M. Cai, Z. W. Zhou, X. X. Zhou, and G. C. Guo, Phys. Rev. A **74**, 042338 (2006).
[25] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 12 (1990).