Discrimination between pure states and mixed states

Chi Zhang,* Guoming Wang,† and Mingsheng Ying‡
State Key Laboratory of Intelligent Technology and Systems, Department of Computer Science and Technology, Tsinghua University, Beijing 100084, China
(Received 10 February 2007; published 6 June 2007)

In this paper, we discuss the problem of determining whether a quantum system is in a pure state, or in a mixed state. We apply two strategies to settle this problem: the unambiguous discrimination and the maximum confidence discrimination. We prove that the optimal versions of both strategies are equivalent. Furthermore, the scheme also provides a method to estimate the purity of quantum states, and the Schmidt number of composed systems.

DOI: 10.1103/PhysRevA.75.062306 PACS number(s): 03.67.Mn, 03.67.Lx

I. INTRODUCTION

In many applications of quantum information, one of the important elements which affect the result of quantum processes, is the purity of the quantum states produced or utilized. Hence, an interesting and important problem in quantum information is to estimate the purity of a quantum system [1–5]. This problem is also strongly related to the estimation of the entanglement of multiparty systems [6–8].

However, all of the above references considered this problem only in the simplest case of qubits. Estimating the purity of a general quantum system is still open. In this paper, we first consider an extreme situation: given some copies of a quantum state, the task for us is to determine whether the state is pure or mixed. The process is called discrimination between pure states and mixed states. Then, by counting different results obtained in a set of the discriminations, we offer an effective method to estimate the purity of quantum states. The idea of discrimination between pure states and mixed states, was first mentioned in Ref. [8]. However, they did not study the problem formally and systematically, which is our aim in this paper. There are two different strategies to design the discrimination: the unambiguous discrimination [11], and the maximum confidence discrimination [12]. In the strategy of unambiguous discrimination, one can tell whether the quantum system is in a pure state or a mixed state without error, but a nonzero probability of inconclusive answer is allowed. The optimal unambiguous discrimination is the one which minimize the probability of obtaining an inconclusive answer. In this paper, in order to simplify the presentation, we use the term "unambiguous" in a more general sense: it allows the success probability to be zero in some situation. On the other hand, in the maximum confidence discrimination, an inconclusive answer is not allowed, and after each discrimination, one must propose a statement whether the quantum state is pure, or mixed. The discrimination is so named, because the probability of obtaining a correct conclusion is maximized.

It is convenient to introduce some notations here. The symmetric tensor product of states $|\varphi_1\rangle,|\varphi_2\rangle,\ldots,|\varphi_n\rangle$ in a Hilbert space H is defined as

$$|\varphi_1\rangle \vee |\varphi_2\rangle \vee \cdots \vee |\varphi_n\rangle = \frac{1}{\sqrt{n!}}\sum_{\sigma\in S(n)} |\varphi_{\sigma_1}\rangle|\varphi_{\sigma_2}\rangle\cdots|\varphi_{\sigma_n}\rangle, \tag{1}$$

where $S(n)$ is the symmetric (or permutation) group of degree n. The span of all symmetric tensors $|\varphi_1\rangle\vee|\varphi_2\rangle\vee\cdots\vee|\varphi_n\rangle$ in $H^{\otimes n}$ is called the symmetric subspace of $H^{\otimes n}$, and denoted as $H_{\mathrm{sym}}^{\otimes n}$. If the dimension of H is m, then the dimension of $H_{\mathrm{sym}}^{\otimes n}$ is $\binom{n+m-1}{n}$ [13]. The orthogonal complement of $H_{\mathrm{sym}}^{\otimes n}$ is called the asymmetric subspace of $H^{\otimes n}$, and denoted as $H_{\mathrm{asym}}^{\otimes n}$. We use $\Phi(H_{\mathrm{sym}}^{\otimes n})$ and $\Phi(H_{\mathrm{asym}}^{\otimes n})$ to represent the projectors of these two subspaces, respectively. In this paper, we prove that, given n copies of a quantum state ρ in Hilbert space H, the optimal unambiguous discrimination and the maximum confidence discrimination can be carried out by the same measurement $\{\Pi_0=\Phi(H_{\mathrm{sym}}^{\otimes n}),\Pi_1=\Phi(H_{\mathrm{asym}}^{\otimes n})\}$. The difference between these two discriminations comes only from the different explanations of the outcomes. In the optimal unambiguous discrimination, the outcome "0" is an inconclusive answer, and the outcome "1" indicates that the system must be in a mixed state. The drawback of the unambiguous discrimination is that, if the quantum system is in a pure state, people always fail to confirm the answer. However, in the maximum confidence discrimination, the outcome "0" indicates that the quantum system is likely to be in a pure state, and the outcome "1" means a mixed state.

There are two assumptions in this paper. First, the purity of quantum states is invariant under any unitary operation. Suppose the purity of a quantum state ρ is represented by $\mu(\rho)$, it must satisfy that $\mu(\rho)=\mu(U\rho U^{\dagger})$, for any unitary operator U. We also assume that, when ρ is a pure state, $\mu(\rho)=1$, otherwise $0\leq\mu(\rho)<1$. For instance, the usually used definition for the purity of quantum states, $\mu(\rho)=\mathrm{Tr}^2(\rho)$, clearly satisfies the conditions. Second, the *a priori* probability distributions of quantum states are also assumed invariant under unitary operations. Let us denote the *a priori* probability density function as $\eta(\rho)$, then $\eta(\rho)=\eta(U\rho U^{\dagger})$, for any unitary operator U. The first assumption is easy to understand, however the second assumption needs some explanation. It may be not true in some situations, when the quantum process is known or partly known. However, we make this assumption based on the situation that there is no

*Electronic address: zangcy00@mails.tsinghua.edu.cn
†Electronic address: wgm00@mails.tsinghua.edu.cn
‡Electronic address: yingmsh@tsinghua.edu.cn

classical information available to design the discrimination, i.e., the only resource one can utilize is the multiple copies of the quantum state. In this case, the second assumption is reasonable. In some applications, people may know the *a priori* distribution of the quantum state, for instance in Ref. [4], then one can design more efficient discriminations by utilizing the classical information offered. However, it should be pointed out that even the *a priori* distribution breaks the unitary invariant assumption, the conclusion about optimal unambiguous discrimination is still correct. Furthermore, the strategy to estimate the purity of quantum states and the method to estimate the Schmidt number of bipartite systems are still available.

The remainder of our present paper is organized as follows. In Sec. II we provide the optimal unambiguous discrimination between pure states and mixed states. In Sec. IV we provide the maximum confidence discrimination between pure states and mixed states. We also generalize the unambiguous discrimination between pure states and mixed states to a "semiunambiguous" estimation for ranks of quantum states in Sec. III, which can also be used to estimate the Schmidt number [14] of bipartite quantum systems. Finally, in Sec. V, we provide a strategy to estimate the purity of quantum states.

II. OPTIMAL UNAMBIGUOUS DISCRIMINATION

In this section, we consider the unambiguous discrimination between pure states and mixed states. Suppose we are given n copies of a quantum state, which is in the Hilbert space H. The unambiguous discrimination is described by a positive operator-valued measure (POVM) on the Hilbert space $H^{\otimes n}$. The measurement is comprised by three positive operators, Π_p, Π_m, and $\Pi_?$, satisfying that

$$\mathrm{Tr}(\Pi_p \rho^{\otimes n}) = 0, \tag{2}$$

for any mixed state ρ,

$$\mathrm{Tr}(\Pi_m \rho^{\otimes n}) = 0, \tag{3}$$

for any pure state $\rho = |\psi\rangle\langle\psi|$, and

$$\Pi_? = I - \Pi_m - \Pi_p. \tag{4}$$

Therefore, if the outcome is p, the system is assured to be in a pure state; if the outcome is m, the system is in a mixed state; and outcome "?" denotes an inconclusive answer.

The efficiency of the discrimination is defined as the possibility of successfully receiving a conclusive answer, which can be expressed as

$$P = \int_{0 \le \mu(\rho) < 1} \mathrm{Tr}(\rho^{\otimes n}\Pi_m)\,\eta(\rho)d\rho + \int_{\mu(\rho)=1} \mathrm{Tr}(\rho^{\otimes n}\Pi_p)\,\eta(\rho)d\rho. \tag{5}$$

The optimal unambiguous discrimination is the one with the maximum efficiency, and we have the following theorem.

Theorem 1. The optimal unambiguous discrimination between pure states and mixed states is a POVM of $\{\Pi_p, \Pi_m, \Pi_?\}$, such that

$$\Pi_p = 0,$$

$$\Pi_m = \Phi(H_{\mathrm{asym}}^{\otimes n}),$$

$$\Pi_? = \Phi(H_{\mathrm{sym}}^{\otimes n}), \tag{6}$$

where $\Phi(H_{\mathrm{sym}}^{\otimes n})$ and $\Phi(H_{\mathrm{asym}}^{\otimes n})$ are the projectors of symmetric subspace and asymmetric subspace of $H^{\otimes n}$, respectively.

Proof. For a mixed state ρ, whose spectrum decomposition is $\rho = \Sigma_{i=1}^m \lambda_i |\phi_i\rangle\langle\phi_i|$, and for any n-tuple chosen from $\{1,\ldots,m\}$, $\pi = (\pi_1,\ldots,\pi_n)$, where repetition is allowed, let us introduce the following two definitions:

$$\lambda_\pi = \prod_{j=1}^{n} \lambda_{\pi_j} \tag{7}$$

and

$$|\phi_\pi\rangle = \otimes_{j=1}^n |\phi_{\pi_j}\rangle. \tag{8}$$

Then,

$$\rho^{\otimes n} = \sum_\pi \lambda_\pi |\phi_\pi\rangle\langle\phi_\pi|, \tag{9}$$

where π ranges over all n-tuples chosen from $\{1,\ldots,m\}$.

Because Π_p is a positive operator, from Eq. (9) and Eq. (2),

$$\langle\phi_\pi|\Pi_p|\phi_\pi\rangle = 0, \tag{10}$$

for any product state $|\phi_\pi\rangle$. Therefore, when the system is in a pure state $|\psi\rangle$, it satisfies that

$$\langle\psi|^{\otimes n}\Pi_p|\psi\rangle^{\otimes n} = 0, \tag{11}$$

i.e., for any situation, the probability of getting the p result is always zero, which means that without loss of generality, we can simply let $\Pi_p = 0$.

Then, from Eq. (3), we know that Π_m is orthogonal to any $|\psi\rangle^{\otimes n}$, where $|\psi\rangle \in H$. It is known that the span space of all $|\psi\rangle^{\otimes n}$ is just the symmetric subspace of $H^{\otimes n}$, which has been denoted as $H_{\mathrm{sym}}^{\otimes n}$ [15]. Thus, the support space of Π_m must be in the asymmetric subspace $H_{\mathrm{asym}}^{\otimes n}$, i.e., $\Pi_m \le \Phi(H_{\mathrm{asym}}^{\otimes n})$. Hence, the probability of determining a mixed state ρ is

$$P(m|\rho) = \mathrm{Tr}(\rho^{\otimes n}\Pi_m) \le \mathrm{Tr}[\rho^{\otimes n}\Phi(H_{asym}^{\otimes n})]. \tag{12}$$

Therefore, the optimal unambiguous discrimination is the measurement $\{\Pi_p, \Pi_m, \Pi_?\}$ given in Eq. (6). ■

Under the optimal unambiguous discrimination, when the quantum system is in a pure state, the result is sure to be inconclusive. Moreover, if the quantum system is in a mixed state ρ, the probability of receiving an inconclusive answer is

$$P(?|\rho) = \mathrm{Tr}(\rho^{\otimes n}\Pi_?) = \sum_\pi \lambda_\pi \langle\phi_\pi|\Phi(H_{\mathrm{sym}}^{\otimes n})|\phi_\pi\rangle$$

$$= \sum_\pi \frac{\lambda_\pi}{n!}\mathrm{per}(\Gamma_\pi), \tag{13}$$

where Γ_π is the Gram matrix derived from $\{|\phi_{\pi_1}\rangle,\ldots,|\phi_{\pi_n}\rangle\}$, and $\mathrm{per}(A)$ denotes the permanent of the matrix A, i.e.,

$$\mathrm{per}(A) = \sum_{\sigma} \prod_{i} A(i,\sigma(i)), \tag{14}$$

where σ ranges over all permutation on n symbols [13].

Let π be an n-tuple valued in $\{1,\ldots,m\}$. We use n_i^π to denote the number of occurrences of i in π, where $i=1,\ldots,m$. Because for any two eigenvectors of ρ with nonzero eigenvalues, $\langle\phi_i|\phi_j\rangle=\delta_{i,j}$,

$$\Gamma_\pi = \bigoplus_{i=1}^{m} I_{n_i^\pi}, \tag{15}$$

where $I_{n_i^\pi}$ is the n_i^π-dimensional identity matrix. Consequently, from Eq. (13),

$$P(?|\rho) = \sum_{\pi} \frac{\lambda_\pi}{n!}\prod_{i=1}^{m} n_i^\pi ! = \sum_{\Sigma_{i=1}^m n_i=n} \frac{n!}{\prod_{i=1}^m n_i!}\prod_{i=1}^m \lambda_i^{n_i}\frac{\prod_{i=1}^m n_i!}{n!}$$

$$= \sum_{\Sigma_{i=1}^m n_i=n} \prod_{i=1}^m \lambda_i^{n_i}. \tag{16}$$

From the above analysis, it is easy to see that the optimality of the unambiguous discrimination given in Theorem 1 is not dependent on the *a priori* probability distribution. More clearly, for any pure state, any unambiguous discrimination will fail to give a conclusive answer. On the other hand, for any mixed state ρ, the unambiguous discrimination offered in Theorem 1 always has the maximum success possibility among all unambiguous discriminations, and the success possibility is

$$P(m|\rho) = 1 - P(?|\rho) = 1 - \sum_{\Sigma_{i=1}^m n_i=n} \prod_{i=1}^m \lambda_i^{n_i}, \tag{17}$$

where $\lambda_1,\ldots,\lambda_m$ are nonzero eigenvalues of ρ.

III. SEMIUNAMBIGUOUS ESTIMATION FOR THE SCHMIDT NUMBER

As we know, the entanglement of a bipartite quantum system is closely related to the purity of one of its subsystems [8–10]. Especially, the entanglement of a bipartite pure state sometimes can be well characterized by its Schmidt number [16]. For simplicity, in the rest of this section, we assume that the total quantum system is in a pure state. Then, whether a subsystem is in a pure state is equivalent to whether the total quantum system is in a product state. Hence, the measurement given in Sec. II also provides an unambiguous estimation for the entanglement of bipartite quantum systems. Moreover, in this section, we will provide a natural generalization, which can be called semiunambiguous estimation of the Schmidt number of bipartite systems.

The Schmidt number of a bipartite system is equal to the rank of the quantum state in each of its subsystems [14]. Hence, estimating the Schmidt number is equivalent to estimating the rank of quantum states. First, let us reconsider the discrimination between pure states and mixed states. In the discrimination, the m result means that the rank of the state is no less than 2, while the inconclusive answer can also be considered as a trivial conclusion that the rank of the state is no less than 1. Although the discrimination does not offer the exact value of the rank of the quantum state, it offers a lower bound for the rank. Moreover, the lower bound is assured to be correct. In this mean, we can consider the discrimination between pure states and mixed states, also a "semiunambiguous" estimation for the rank of quantum states. A more general "semiunambiguous" estimation of the rank of quantum states can be defined as a POVM on $H^{\otimes n}$ with operators $\{\Pi_1,\Pi_2,\ldots,\Pi_m\}$, where m is the dimension of H. The measurement satisfies that for any quantum state ρ whose rank is k, $\mathrm{Tr}(\Pi_i\rho^{\otimes n})=0$, for any $i>k$. Thus, whenever the outcome k is observed, we can make sure that the rank of ρ is no less than k.

Before providing the semiunambiguous estimation of the rank of quantum states, we first introduce some fundamental knowledge about group representation theory needed here. For details, please see Ref. [17].

A Young diagram $[\lambda]=[\lambda_1,\ldots,\lambda_k]$, where $\Sigma\lambda_i=n$ and $\lambda_1\geq\lambda_2\geq\cdots\lambda_k>0$, is a graphical representation of a partition of a natural number n. It consists of n cells, arranged in left-justified rows, where the number of cells in the ith row is λ_i.

A Young tableau is obtained by placing the numbers $1,\ldots,n$ in the n cells of a Young diagram. If the numbers form an increasing sequence along each row and each column, the Young tableau is called a standard Young tableau. For a given Young diagram $[\lambda]$, the number of standard Young tableaus can be calculated with the hook length formula, and denoted by $f^{[\lambda]}$. In this paper, we use $T_r^{[\lambda]}$ to denote the rth standard Young tableau, where $r=1,\ldots,f^{[\lambda]}$.

The Hilbert space $H^{\otimes n}$, where the dimension of H is m, can be decomposed into a set of invariant subspaces under operation $U^{\otimes n}$, where U can be any unitary operation on H. Each of the subspaces corresponds to a standard Young tableau $T_r^{[\lambda]}$, where the number of rows in $[\lambda]$ is no more than m. So, we can denote the subspaces as $H_r^{[\lambda]}$, and denote its projector as $\Phi(H_r^{[\lambda]})$. Then, we have

$$H^{\otimes n} = \bigoplus_{[\lambda],r} H_r^{[\lambda]}. \tag{18}$$

For instance, the symmetric subspace $H_{\mathrm{sym}}^{\otimes n}$ is just one of these subspaces, $H_1^{\square\square\cdots\square}$.

For a quantum state ρ in H, whose rank is k, the support space of $\rho^{\otimes n}$ is in the sum of subspaces $H_r^{[\lambda]}$, where the number of rows in the Young diagram $[\lambda]$ is no greater than k. Therefore, a semiunambiguous estimation of the rank of quantum states can be designed as a POVM of $\{\Pi_1,\ldots,\Pi_m\}$, such that

$$\Pi_i = \sum_{h([\lambda])=i} \sum_{r} \Phi(H_r^{[\lambda]}), \tag{19}$$

where $h([\lambda])$ is the number of rows in $[\lambda]$. As said above, if the rank of ρ is k, $\mathrm{Tr}(\Pi_i\rho^{\otimes n})=0$, for any $i>k$. Thus, once an

"i" result is observed, we can assert that the rank of ρ is no less than i. For n copies of a bipartite quantum system, through measuring any of its subsystems with the measurement given in Eq. (19), we can semiunambiguously estimate the Schmidt number of the whole system.

IV. MAXIMUM CONFIDENCE DISCRIMINATION

In this section, we consider a different strategy for determining whether the quantum system is in a pure state, which is called "maximum confidence discrimination" [12].

The discrimination is still a POVM of $\{\Pi_p,\Pi_m\}$. But when the outcome is m, the quantum system is believed in a mixed state; otherwise, the outcome is p, and the quantum state is considered to be pure. A maximum confidence strategy is to maximize the reliability of the conclusion, i.e., let the following two probabilities be maximized:

$$P(\text{pure}|p)=\frac{\int_{\mu(\rho)=1}\eta(\rho)\mathrm{Tr}(\rho^{\otimes n}\Pi_p)d\rho}{\int_{\mu(\rho)\le 1}\eta(\rho)\mathrm{Tr}(\rho^{\otimes n}\Pi_p)d\rho} \tag{20}$$

and

$$P(\text{mixed}|m)=\frac{\int_{\mu(\rho)<1}\eta(\rho)\mathrm{Tr}(\rho^{\otimes n}\Pi_m)d\rho}{\int_{\mu(\rho)\le 1}\eta(\rho)\mathrm{Tr}(\rho^{\otimes n}\Pi_m)d\rho}. \tag{21}$$

Clearly, Eq. (20) and Eq. (21) do not always get maximum values at the same time. However, on the assumptions about unitary invariance of $\eta(\rho)$ and $\mu(\rho)$, we can prove that there exists a measurement $\{\Pi_p,\Pi_m\}$ maximizing both Eq. (20) and Eq. (21), as the following theorem states.

Theorem 2. The maximum confidence discrimination between pure states and mixed states is a POVM of $\{\Pi_p,\Pi_m\}$, such that

$$\Pi_p=\Phi(H_{\text{sym}}^{\otimes n}),$$
$$\Pi_m=\Phi(H_{\text{asym}}^{\otimes n}), \tag{22}$$

where $\Phi(H_{\text{sym}}^{\otimes n})$ and $\Phi(H_{\text{asym}}^{\otimes n})$ are as in Theorem 1.

Proof. First, we consider the construction of Π_p. From the assumptions that $\eta(\rho)=\eta(U\rho U^\dagger)$ and $\mu(\rho)=\mu(U\rho U^\dagger)$,

$$P(\text{pure}|p)=\frac{\int_{\mu(\rho)=1}\eta(\rho)\mathrm{Tr}(\rho^{\otimes n}\Pi_p)d\rho}{\int_{\mu(\rho)\le 1}\eta(\rho)\mathrm{Tr}(\rho^{\otimes n}\Pi_p)d\rho}=\frac{\int_{\mu(\rho)=1}\eta(\rho)\mathrm{Tr}[\rho^{\otimes n}U^{\otimes n}\Pi_p(U^\dagger)^{\otimes n}]d\rho}{\int_{\mu(\rho)\le 1}\eta(\rho)\mathrm{Tr}[\rho^{\otimes n}U^{\otimes n}\Pi_p(U^\dagger)^{\otimes n}]d\rho}, \tag{23}$$

for any unitary operation U. Hence, if Π_p maximizes Eq. (20), so does $\int U^{\otimes n}\Pi_p(U^\dagger)^{\otimes n}dU$ with respect to the normalized invariant measure dU of the unitary group $U(m)$. Hence, we can choose the operator Π_p to satisfy that

$$\Pi_p=\int U^{\otimes n}\Pi_p(U^\dagger)^{\otimes n}dU, \tag{24}$$

which shows that Π_p commutes with any unitary operator of the form $U^{\otimes n}$. Thus, from the representation theory of classical groups in Ref. [18], Π_p can be expressed as a linear combination of permutation operators,

$$\Pi_p=\sum_\sigma \alpha_\sigma V_\sigma, \tag{25}$$

where $\alpha_\sigma\in C$, σ ranges over all permutations of n elements, and V_σ is the permutation operator derived from σ, i.e.,

$$V_\sigma|\psi_1\rangle|\psi_2\rangle\cdots|\psi_n\rangle=|\psi_{\sigma_1}\rangle|\psi_{\sigma_2}\rangle\cdots|\psi_{\sigma_n}\rangle. \tag{26}$$

For any state $|\varphi\rangle$ in the symmetric subspace $H_{\text{sym}}^{\otimes n}$, $V_\sigma|\varphi\rangle=|\varphi\rangle$, so $\Pi_p|\varphi\rangle=(\Sigma_\sigma\alpha_\sigma)|\varphi\rangle$, which indicates that

$$\Pi_p=\alpha\Phi(H_{\text{sym}}^{\otimes n})\oplus\Pi_p', \tag{27}$$

where Π_p' is a positive operator whose support space is in $H_{\text{asym}}^{\otimes n}$, and $\alpha=\Sigma_\sigma\alpha_\sigma$. Because for any pure state $\rho=|\psi\rangle\langle\psi|$, the support space of $\rho^{\otimes n}$ is in the symmetric subspace $H_{\text{sym}}^{\otimes n}$, $\mathrm{Tr}(\rho^{\otimes n}\Pi_p')=0$ for any $\mu(\rho)=1$. Therefore, the numerator of Eq. (20) does not change if we substitute Π_p with $\alpha\Phi(H_{\text{sym}}^{\otimes n})$, and the denominator diminishes or remains the same. So, the optimal Π_p has the form of $\alpha\Phi(H_{\text{sym}}^{\otimes n})$ for any constant α.

On the other hand, if we choose $\Phi(H_{\text{asym}}^{\otimes n})$ as Π_m, then for any pure state ρ, whose purity $\mu(\rho)=1$, we have $\mathrm{Tr}(\rho^{\otimes n}\Pi_m)=0$, and Eq. (21) has the maximum value 1. To satisfy the condition $\Pi_p+\Pi_m=I$, let $\alpha=1$, $\Pi_p=\Phi(H_{\text{sym}}^{\otimes n})$. This completes the proof. ■

It is easy to see that the optimal unambiguous discrimination and the maximum confidence discrimination are the same measurement $\{\Pi_0=\Phi(H_{\text{sym}}^{\otimes n}),\Pi_1=\Phi(H_{\text{asym}}^{\otimes n})\}$. The difference between the two discriminations is the meaning of the "0" result. In the former discrimination, the "0" result means an inconclusive answer; however, in the latter one, if a "0" result is obtained, the quantum system is considered to be in a pure state.

From Eq. (16), for n copies of a quantum state ρ, under the measurement of $\{\Pi_0,\Pi_1\}$ given above, the probability of receiving a "0" result is

$$P_0(n)=\sum_{\Sigma_{i=1}^m n_i=n}\prod_{i=1}^m \lambda_i^{n_i}, \tag{28}$$

where $\lambda_1,\ldots,\lambda_m$ are the eigenvalues of ρ. As we know, the above quantity is the complete symmetric polynomial of degree n for $\{\lambda_1,\ldots,\lambda_m\}$, which is usually denoted by $h_n(\lambda_1,\ldots,\lambda_m)$. From Ref. [19], the complete symmetric polynomials can be derived from a generating function

$$H_m(t)=\sum_{k\ge 0}h_k(\lambda_1,\ldots,\lambda_m)t^k=\frac{1}{\prod_{i=1}^m(1-t\lambda_i)}. \tag{29}$$

Let λ^* stand for the maximum eigenvalue of ρ, then, if we have n copies of the states, the probability of judging it to be pure can be evaluated as

$$P_0(n) = \sum_{\sum_{i=1}^{m} n_i = n} \prod_{i=1}^{m} \lambda_i^{n_i} \leq \binom{n+m-1}{n} (\lambda^*)^n. \qquad (30)$$

Then, if the quantum system is in a pure state, $P_0(n)$ will always be 1, otherwise $\lambda^* < 1$, and $P_0(n)$ will converge to zero with exponential convergence rate. Hence, when n is large enough, the maximum confidence discrimination offers high reliability.

In Sec. III we discuss the semiunambiguous estimation of the rank of quantum states, which is given in Eq. (19). An open problem is whether this measurement also offers a maximum confidence estimation of the rank of quantum states, if we consider the result "i" as a claim that the rank of the quantum state is i.

V. ESTIMATING THE PURITY OF QUANTUM STATES

The maximum confidence discrimination between pure states and mixed states provides a natural intuition for the purity of a quantum system, i.e., the greater the probability of getting a "0" result, the closer it is to a pure state. Hence, by repetitively performing the measurement, and counting the proportion of "0" results, we can estimate the probability of judging the system being pure, which, in some sense, reflects some information about the purity of the system. However, a more interesting conclusion is that, no matter how people define the purity of quantum states, as long as it satisfies the condition of unitary invariant, it can be well estimated through a set of maximum confidence discriminations.

On the assumption of unitary invariant, the purity of a quantum state ρ, $\mu(\rho) = \mu(\mathrm{diag}(\lambda_1, \ldots, \lambda_m))$, where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of ρ. Hence, $\mu(\rho)$ is a function of its eigenvalues. Estimating the purity of a quantum state ρ can be reduced to estimating the eigenvalues of ρ [20]. The characteristic polynomial of ρ is a polynomial, whose roots are the eigenvalues, i.e.,

$$\det(xI - \rho) = \prod_{i=1}^{m} (x - \lambda_i) = \sum_{j=0}^{m} a_j x^{m-j}. \qquad (31)$$

If we can successfully estimate every coefficient a_j, $j = 0, \ldots, m$, the eigenvalues can be estimated by solving the equation $\sum_{j=0}^{m} a_j x^{m-j} = 0$.

Recall the famous Viete's theorem, it is easy to know

$$a_0 = e_0(\lambda_1, \ldots, \lambda_m) = 1,$$

$$a_1 = -e_1(\lambda_1, \ldots, \lambda_m) = -\sum_{i=1}^{m} \lambda_i,$$

$$a_2 = e_2(\lambda_1, \ldots, \lambda_m) = \sum_{1 \leq i_1 < i_2 \leq n} \lambda_{i_1} \lambda_{i_2},$$

$$\cdots,$$

$$a_k = (-1)^k e_k(\lambda_1, \ldots, \lambda_m) = (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

$$\cdots,$$

$$a_m = (-1)^m e_m(\lambda_1, \ldots, \lambda_m) = (-1)^m \lambda_1 \lambda_2 \cdots \lambda_m. \qquad (32)$$

Here, the polynomial $e_k(\lambda_1, \ldots, \lambda_m)$ is the mth elementary symmetric polynomial of $\{\lambda_1, \ldots, \lambda_m\}$ [19], whose generating function is

$$E_m(t) = \sum_{i=0}^{m} e_i(\lambda_1, \ldots, \lambda_m) t^i = \prod_{i=1}^{m} (1 + t\lambda_i). \qquad (33)$$

Combined with Eq. (29), we have that $H(t)E(-t) = 1$, so

$$\sum_{r=0}^{k} (-1)^r e_r h_{m-r} = 0, \qquad (34)$$

for any $k \geq 1$, if we set $e_r(\lambda_1, \ldots, \lambda_m) = 0$, when $r > m$. Here, for simplicity, we use e_k, h_l to denote $e_k(\lambda_1, \ldots, \lambda_m)$, $h_l(\lambda_1, \ldots, \lambda_m)$, respectively. Then, it is not hard to see that

$$e_k = \begin{vmatrix} h_1 & h_2 & h_3 & \cdots & h_{k-1} & h_k \\ 1 & h_1 & h_2 & \cdots & h_{k-2} & h_{k-1} \\ 0 & 1 & h_1 & \cdots & h_{k-3} & h_{k-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & h_1 & h_2 \\ 0 & 0 & 0 & \cdots & 1 & h_2 \end{vmatrix}. \qquad (35)$$

Clearly, $h_1 = \sum_{i=1}^{m} \lambda_i = \mathrm{Tr}(\rho) = 1$. As stated in Sec. IV, for any $k \geq 2$, h_k is the probability of receiving the "0" result, when we measure $\rho^{\otimes k}$ by the measurement $\{\Pi_0 = \Phi(H_{\mathrm{sym}}^{\otimes k}), \Pi_1 = \Phi(H_{\mathrm{asym}}^{\otimes k})\}$. Therefore, if we have N copies of quantum state ρ, where N is much larger than m, we can estimate the eigenvalues of ρ in the following strategy.

First, separate the N copies into m groups, the kth group has kN_k copies of the quantum state. Then, operate the measurement $\{\Pi_0 = \Phi(H_{\mathrm{sym}}^{\otimes k}), \Pi_1 = \Phi(H_{\mathrm{asym}}^{\otimes k})\}$ on $\rho^{\otimes k}$ for N_k times in the kth group. Suppose among these results, the number of "0" results is S_k, then we can estimate $P_0(k)$, i.e., h_k by S_k / N_k. Then, through Eq. (35), we can estimate every e_k, where $1 \leq k \leq m$. Hence, from Eq. (32), the characteristic polynomial of ρ, whose roots are the eigenvalues we want to estimate, is known. The task that remains for us is to solve the equation given in Eq. (31).

VI. CONCLUSION

In this paper, we investigate the discrimination between pure states and mixed states, which may play an important role in further study for estimating the purity of quantum states. The discrimination is described by a POVM of $\{\Pi_0 = \Phi(H_{\mathrm{sym}}^{\otimes n}), \Pi_1 = \Phi(H_{\mathrm{asym}}^{\otimes n})\}$ on n copies of the quantum state being discriminated. If the "0" result is considered as an inconclusive answer, the measurement is the optimal unambiguous discrimination. On the other hand, if the "0" result is considered as a hint that the quantum system is in a pure state, the discrimination is the maximum confidence discrimination. We also provide a semiunambiguous estimation for the rank of quantum states, which also can be used to estimate the Schmidt number of bipartite quantum systems. Finally, we give a strategy to estimate the purity of quantum systems.

ACKNOWLEDGMENTS

The authors would like to thank Yuan Feng, Runyao Duan, Zhengfeng Ji, Zhaohui Wei, and Jianxin Chen for previous discussions. This work was partly supported by the Natural Science Foundation of China (Grants Nos. 60621062 and 60503001) and the Hi-Tech Research and Development Program of China (863 project) (Grant No. 2006AA01Z102).

[1] A. G. White, D. F. V. James, P. H. Eberhard, and P. G. Kwiat, Phys. Rev. Lett. **83**, 3103 (1999).
[2] M. Legre, M. Wegmuller, and N. Gisin, Phys. Rev. Lett. **91**, 167902 (2003).
[3] M. G. A. Paris, F. Illuminati, A. Serafini, and S. DeSiena, Phys. Rev. A **68**, 012314 (2003).
[4] E. Bagan, M. A. Ballester, R. Muñoz-Tapia, and O. Romero-Isart, Phys. Rev. Lett. **95**, 110504 (2005).
[5] E. Bagan, M. A. Ballester, R. D. Gill, A. Monras, and R. Munoz-Tapia, Phys. Rev. A **73**, 032301 (2006).
[6] A. Acin, R. Tarrach, and G. Vidal, Phys. Rev. A **61**, 062307 (2000).
[7] P. Horodecki, Phys. Rev. Lett. **90**, 167901 (2003).
[8] L. Aolita and F. Mintert, Phys. Rev. Lett. **97**, 050501 (2006).
[9] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
[10] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
[11] I. Ivanovic, Phys. Lett. A **123**, 257 (1987); A. Chefles, *ibid.* **239**, 339 (1998); T. Rudolph, R. W. Spekkens, and P. S. Turner, Phys. Rev. A **68**, 010301(R) (2003); Y. Feng, R. Duan, and M. Ying, *ibid.* **70**, 012308 (2004); C. Zhang, Y. Feng, and M. Ying, Phys. Lett. A **353**, 300 (2006).
[12] R. Kosut, I. Walmsey, Y. Eldar, and H. Rabitz, e-print arXiv:quant-ph/0403150; S. Croke, E. Andersson, S. M. Barnett, C. R. Gilson, and J. Jeffers, Phys. Rev. Lett. **96**, 070401 (2006).
[13] R. Bhatia, *Matrix Analysis* (Springer, Berlin, 1991).
[14] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[15] A. Chefles, E. Andersson, and I. Jex, J. Phys. A **37**, 7315 (2004).
[16] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic, Dordrecht, 1993).
[17] H. Boerner, *Representations of Groups* (North-Holland, Amsterdam, 1963).
[18] H. Weyl, *The Classical Groups* (Princeton University Press, Princeton, NJ, 1946).
[19] R. P. Stanley, *Enumerative Combinatorics* (Cambridge University Press, Cambridge, 1999).
[20] M. Keyl and R. F. Werner, Phys. Rev. A **64**, 052311 (2001).