Influence of pump-phase fluctuations on entanglement generation using a correlated spontaneous-emission laser

Shahid Qamar,[1,*] Han Xiong,[2] and M. Suhail Zubairy[1,2,3,†]

[1]*Centre for Quantum Physics, COMSATS Institute of Information Technology, Islamabad, Pakistan*

[2]*Institute for Quantum Studies and Department of Physics, Texas A& M University, College Station, Texas 77843, USA*

[3]*Texas A&M University at Qatar, Education City, P.O. Box 23874, Doha, Qatar*

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In this paper, we study the effect of phase fluctuations of the pump field upon the entanglement generation in a two-photon correlated emission laser (CEL). We consider initial vacuum and coherent state for the two-cavity modes. In both cases, we find reduction in the entanglement due to the phase fluctuations. However, our results indicate that entanglement generation is highly sensitive to phase fluctuations when we have initial coherent state in the two modes.

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I. INTRODUCTION

Generation of macroscopic entangled states of atoms and photons have attracted considerable attention in recent years. The prime motivation for these studies come from the field of quantum information. Schemes for the realization of macroscopic entangled atomic ensemble have been demonstrated via quantum state transfer from nonclassical light to atom by Polzik and co-workers [1]. The generation of such states for photons have also been studied. For example, the generation of bright two-mode quadrature squeezed light from a narrow-band nondegenerate optical parametric amplifier (NOPA), polarization entangled light from parametric down conversion driven by an intense pulsed pump field inside a cavity and others [2–6].

In a recent study, we considered a two-photon correlated emission laser (CEL) [7] as a source of an entangled radiation [8,9]. The system essentially consists of a three-level atomic scheme inside a doubly resonant cavity. It is shown that when the amplitude of the Rabi frequency (associated with the strong driving field) is much larger than the atomic decay rates, the system approaches towards a nondegenerate parametric amplifier. Interestingly, entanglement can be obtained for any arbitrary initial state of the two modes, even in the presence of cavity losses.

Essential to our earlier scheme is the strong driving field and the important question is how the fluctuations in the pump affect the system. In some earlier studies, effects of pump fluctuations on squeezing properties of the field in a degenerate parametric amplifier [10,11], two-photon phase sensitive amplifier [12], and nondegenerate multiwave mixing have been studied [13]. It is shown that these fluctuations tend to decrease the amount of squeezing. In a recent study, we considered the effects of amplitude and phase fluctuations upon entanglement generation in a nondegenerate parametric amplifier. It is shown that the entanglement between the signal and idler modes in NOPA is more sensitive to phase fluctuations than to amplitude fluctuations [14].

In this paper, we study the role of pump-phase fluctuations on entanglement generation in a correlated emission laser. In our earlier scheme for the entanglement generation using CEL, it is assumed that the pump is a perfectly coherent monochromatic source. However, in a realistic situation, driving field has always a finite bandwidth associated with the phase fluctuations of the pump field. Here we restrict ourselves to phase fluctuations of the pump beam which is typically an intense driving field with a well stabilized amplitude. We use the classical phase diffusion model to describe the phase of the driving-pump field by following the method discussed in our earlier paper [11]. The phase of the driving field undergoes Brownian motion that may be approximated by a Wiener-Levy diffusion process. In order to estimate the entanglement generation in a CEL, we apply the criterion of Duan *et al.* [18] that requires the measurement of quadratures for the two modes of the cavity field. The field quadratures can be measured employing balanced homodyne detection scheme (BHDS). We consider the same pump source as a local oscillator (LO) in the BHDS. In this configuration, it is expected that the effect of the finite bandwidth of the driving field will no longer occur. However, our results show the suppression of the entanglement due to the phase fluctuations associated with the driving field. This can be understood in terms of the time lag in the response of the system to an instantaneous change in the pump phase. We consider initial vacuum and coherent state for the two-cavity modes. In both cases, we find the suppression of entanglement between the two modes in a CEL due to the phase fluctuations. However, the entanglement generation is more sensitive to the pump phase fluctuations when we have initial coherent state in the two modes of the cavity. This suggests the requirement of a highly stabilized driving source to generate the entanglement.

II. MODEL

We consider a system of three-level atoms as shown in Fig. 1 in which atoms interact with the two modes of the

*Permanent address: Department of Physics and Applied Mathematics, Pakistan Institute of Engineering and Applied Sciences, Nilore, Islamabad, Pakistan. Electronic address: shahid_qamar@pieas.edu.pk

†Electronic address: zubairy@tamu.edu

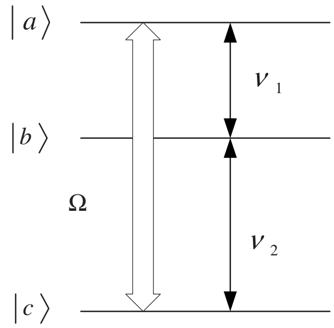

FIG. 1. Schematic for the entanglement amplifier. A three-level atomic system in a cascade configuration. The transition between levels $|a\rangle$-$|b\rangle$ and $|b\rangle$-$|c\rangle$ at frequencies ν_1 and ν_2 are resonant with the cavity. The transition $|a\rangle$-$|c\rangle$ is dipole forbidden and can be induced by a strong magnetic field.

field inside a doubly resonant cavity. The dipole allowed transitions $|a\rangle$-$|b\rangle$ and $|b\rangle$-$|c\rangle$ are resonantly coupled with the two non-degenerate modes ν_1 and ν_2 of the cavity, while the forbidden transition $|a\rangle$-$|c\rangle$ is induced by applying a strong magnetic field for magnetic dipole allowed transition. The Hamiltonian for the system in the rotating wave approximation is given by

$$H_I = \hbar g_1(a_1|a\rangle\langle b| + a_1^\dagger|b\rangle\langle a|) + \hbar g_2(a_2|b\rangle\langle c| + a_2^\dagger|c\rangle\langle b|) - \frac{1}{2}\hbar\Omega(e^{-i\phi(t)}|a\rangle\langle c| + e^{i\phi(t)}|c\rangle\langle a|), \tag{1}$$

where a_1 ($a_1^\dagger$) and a_2 ($a_2^\dagger$) are the annihilation (creation) operators for the two nondegenerate modes of the cavity field and $g_1(g_2)$ are the associated vacuum Rabi frequency. The Rabi frequency induced by the strong driving field is represented by $\Omega e^{-i\phi(t)}$.

In our earlier study, we consider similar configuration for entanglement generation by taking the phase of the driving field to be constant. Here we assume a realistic situation in which the phase undergoes a diffusion process which leads to the finite bandwidth of the driving field. In the above Hamiltonian, $\phi(t)$ represents the fluctuating phase of the pump field. This fluctuating phase can be written as

$$\phi(t) = \phi_0 + \phi_1(t), \tag{2}$$

where ϕ_0 is a constant corresponding to the average value of the fluctuating phase and $\phi_1(t)$ is the random phase described by a Gaussian random process which performs a Brownian motion described by a Wiener-Levy stochastic process such that $\langle\phi(t)\rangle=0$. For such a process, the two-time correlation function is given by

$$\langle\phi_1(t)\phi_1(t')\rangle = D(t + t' - |t - t'|), \tag{3}$$

where D is the diffusion coefficient. The derivative of this diffusion process is a white noise with

$$\langle\dot{\phi}_1(t)\dot{\phi}_1(t')\rangle = 2D\delta(t - t'), \tag{4}$$

from which follows

$$\langle \mathbf{e}^{-i\phi(t)}\rangle = \mathbf{e}^{-i\phi(0)}\mathbf{e}^{-Dt}. \tag{5}$$

This leads to a Lorentizian power spectrum for the pump field, whose linewidth (half width at half maximum) is D.

The estimation of the entanglement for a mixed state remains a challenging problem [15]. Formally, a system is considered to be entangled if it is nonseparable, i.e., the density matrix for the state ρ cannot be written as a convex combination of product states

$$\rho\sum_j p_j\rho^{(1)} \otimes \rho_j^{(2)}, \tag{6}$$

with $p_j \geq 0$ and $\Sigma_j p_j=1$. For pure state there exist sufficient and necessary conditions for entanglement. For example, the Schmidt decomposition or the von Neumann entropy of the reduced density matrix of a state can be used to estimate the entanglement of a pure state. For an arbitrary mixed state there exist no sufficient and necessary criteria for the estimation of entanglement. However, several sufficient entanglement criteria have been proposed in some recent studies [16,17]. In a recent study, Hillery and Zubairy [15] provide a class of inequalities whose violation show the presence of entanglement in two-mode systems which can be extended to detect entanglement in systems consisting of more than two modes.

In order to estimate the entanglement for the two modes of the cavity field in a correlated emission laser, here we use the criterion which is due to Duan *et al.* [18]. In some recent experiments this criterion has been tested to demonstrate the entanglement between the twin beams produced in an optical parametric oscillator [6]. According to this criterion, a state of the system is entangled if the sum of the quantum fluctuations of the two Einstein-Podolsky-Rosen (EPR)-like operators $\hat{u}$ and $\hat{v}$ of the two modes satisfy the inequality

$$(\Delta\hat{u})^2 + (\Delta\hat{v})^2 < 2. \tag{7}$$

For a general state this criteria provides a sufficient condition for entanglement. It is also shown in Ref. [18] that for a two-mode continuous variable Gaussian states, this criteria turns out to be a necessary and sufficient condition for inseparability. In Eq. (7),

$$\hat{u} = \hat{x}_1 + \hat{x}_2,$$

$$\hat{v} = \hat{p}_1 - \hat{p}_2. \tag{8}$$

Here $\hat{x}_1(\hat{x}_2)$ and $\hat{p}_1(\hat{p}_2)$ are the quadratures for the two modes of the cavity field which can be measured using the balanced homodyne detection (BHD) scheme [19].

In homodyne detection, the signal which in our case comes from the cavity field interferes with a beam from the local oscillator at a lossless beam splitter. For a realistic beam splitter, fluctuations add into the system, which can introduce noise in BHD. However, a detailed analysis is required to study their influence. Typically, the same source is used for the local oscillator whose phase fluctuations are correlated with those of the pump. The reference frame for the two Hermitian, mutually conjugate quadratures in BHD is defined by the phase of the local oscillator. In our case the phase of the local oscillator is time dependent and undergoes a diffusion process. In order to accommodate it the factor $\phi(t)$ is introduced such that

$$\hat{x}_j = \frac{(a_j e^{i\phi(t)/2} e^{i\theta} + a_j^\dagger e^{-i\phi(t)/2} e^{-i\theta})}{\sqrt{2}},$$

$$\hat{p}_j = \frac{(a_j e^{i\phi(t)/2} e^{i\theta} - a_j^\dagger e^{-i\phi(t)/2} e^{-i\theta})}{\sqrt{2}i} \quad (9)$$

(with $j=1,2$) are the quadrature operators for the two cavity modes 1 and 2, respectively. Here θ is the reference phase between the field quadratures. In some earlier studies effects of phase fluctuations on squeezing is considered. It is shown that the phase fluctuations degrade the amount of squeezing even when both local oscillator and pump field is considered from the same source. The dependence of the observed squeezing on the phase diffusion is due to the time lag in the response of the system to the instantaneous change in pump phase [11].

If we substitute the definition of $\hat{u}$ and $\hat{v}$ into Eq. [7], we obtain

$$\begin{aligned}(\Delta\hat{u})^2 + (\Delta\hat{v})^2 = \{ & \langle\langle a_1^\dagger a_1\rangle\rangle + \langle\langle a_1 a_1^\dagger\rangle\rangle + \langle\langle a_2 a_2^\dagger\rangle\rangle + \langle\langle a_2^\dagger a_2\rangle\rangle \\ & + 2(\langle\langle a_1 a_2 e^{i\phi_1(t)}\rangle\rangle + \langle\langle a_1^\dagger a_2^\dagger e^{-i\phi_1(t)}\rangle\rangle \\ & - \langle\langle a_1 e^{i\phi_1(t)/2}\rangle\rangle\langle\langle a_2 e^{i\phi_1(t)/2}\rangle\rangle \\ & - \langle\langle a_1^\dagger e^{-i\phi_1(t)/2}\rangle\rangle\langle\langle a_2^\dagger e^{-i\phi_1(t)/2}\rangle\rangle) \\ & - \langle\langle a_1 e^{i\phi_1(t)/2}\rangle\rangle\langle\langle a_1^\dagger e^{-i\phi_1(t)/2}\rangle\rangle - \langle\langle a_1^\dagger e^{-i\phi_1(t)/2}\rangle\rangle \\ & \times\langle\langle a_1 e^{i\phi_1(t)/2}\rangle\rangle - \langle\langle a_2 e^{i\phi_1(t)/2}\rangle\rangle\langle\langle a_2^\dagger e^{-i\phi_1(t)/2}\rangle\rangle \\ & - \langle\langle a_2^\dagger e^{-i\phi_1(t)/2}\rangle\rangle\langle\langle a_2 e^{i\phi_1(t)/2}\rangle\rangle\}. \end{aligned} \quad (10)$$

Here the reference phase θ is taken as $-\pi/4$ to cancel out the constant phase term, i.e., $\phi_0=\pi/2$, from our expression for $(\Delta\hat{u})^2+(\Delta\hat{v})^2$. The total photon numbers $\langle\hat{N}\rangle=\langle\langle a_1^\dagger a_1\rangle\rangle + \langle\langle a_2^\dagger a_2\rangle\rangle$.

In our notations, the single bracket stands for stochastic averages and the double bracket stands for the quantum mechanical average as well as the stochastic average. It is clear from Eq. (10) that entanglement in our system can be measured by evaluating the values of the averaged moments $\langle\langle a_1^\dagger a_1\rangle\rangle$, $\langle\langle a_2^\dagger a_2\rangle\rangle$, $\langle\langle a_1 a_2 e^{i\phi_1(t)}\rangle\rangle$, $\langle\langle a_1 e^{i\phi_1(t)/2}\rangle\rangle$, and $\langle\langle a_2 e^{i\phi_1(t)/2}\rangle\rangle$ (and their respective conjugates). We need an equation of motion for the stochastically averaged reduced density operator $\langle\rho\rangle$ to calculate the expression for $\langle\langle a_1^\dagger a_1\rangle\rangle$, whereas equations for the transformed phase shifted density operators, i.e., $\langle\rho\exp[-i\phi(t)]\rangle$ and $\langle\rho\exp[-i\phi(t)/2]\rangle$(and their respective conjugates) are needed to evaluate the expressions for $\langle\langle a_1^\dagger a_2^\dagger e^{-i\phi_1(t)}\rangle\rangle$, $\langle\langle a_1 a_2 e^{i\phi_1(t)}\rangle\rangle$, $\langle\langle a_1^\dagger e^{-i\phi_1(t)/2}\rangle\rangle$, $\langle\langle a_1 e^{i\phi_1(t)/2}\rangle\rangle$, $\langle\langle a_2^\dagger e^{-i\phi_1(t)/2}\rangle\rangle$, and $\langle\langle a_2 e^{i\phi_1(t)/2}\rangle\rangle$.

Here $\langle\langle a_1 a_2 e^{i\phi_1(t)}\rangle\rangle$ and its complex conjugate terms are the correlation between the two quantum-mechanical modes averaged over the stochastic process. This is also known as the combination tone [13], which is responsible for the generation of entanglement between the two modes. The correlation between the two modes should be greater than the correlation between the same modes to ensure the entanglement generation.

III. EFFECT OF PUMP-PHASE FLUCTUATIONS UPON ENTANGLEMENT GENERATION

In this section we discuss the effect of phase fluctuations upon the entanglement generation in a correlated emission laser. Our system is modified as compared to the system discussed in Ref. [8] due to the introduction of phase fluctuations in the driving field. Hence we are interested in the master equation for the reduced density matrix for the two-quantum-mechanical cavity modes averaged over the random phase of the driving field. Here we are considering a linear theory of two-photon CEL, which is sufficient for the discussion of entanglement generation. Therefore, we treat the transition $|a\rangle$-$|b\rangle$ and $|b\rangle$-$|c\rangle$ quantum mechanically up to the second order in the coupling constant g. Physically, this means that the cavity field cannot saturate the atoms. However, for strong driving field, we consider all orders in the Rabi frequency Ω. We also assume that the atoms are injected in the cavity in the lower level $|c\rangle$ at a rate r_a.

A similar situation for the master equation of the cavity field for a single-mode two-photon phase sensitive linear amplifier is considered in Ref. [12]. A detailed procedure for the derivation of the equation of motion for stochastic variables averaged over the stochastic process is also presented there. We follow the same procedure to obtain the stochastically averaged equation of motion for the two mode of the cavity field. The resulting equation is given by

$$\begin{aligned}\langle\dot{\rho}\rangle = & -[B_{11}^* a_1 a_1^\dagger\langle\rho\rangle + B_{11}\langle\rho\rangle a_1 a_1^\dagger - (B_{11}+B_{11}^*)a_1^\dagger\langle\rho\rangle a_1 \\ & + B_{22}^* a_2^\dagger a_2\langle\rho\rangle + B_{22}\langle\rho\rangle a_2^\dagger a_2 - (B_{22}+B_{22}^*)a_2\langle\rho\rangle a_2^\dagger] \\ & - [B_{12}^* a_1 a_2\langle\rho e^{i\phi(t)}\rangle + B_{21}\langle\rho e^{i\phi(t)}\rangle a_1 a_2 \\ & - (B_{12}^* + B_{21})a_2\langle\rho e^{i\phi(t)}\rangle a_1] - [B_{21}^* a_1^\dagger a_2^\dagger\langle\rho e^{-i\phi(t)}\rangle \\ & + B_{12}\langle\rho e^{-i\phi(t)}\rangle a_1^\dagger a_2^\dagger - (B_{12}+B_{21}^*)a_1^\dagger\langle\rho e^{-i\phi(t)}\rangle a_2^\dagger] \\ & - \kappa_1(a_1^\dagger a_1\rho - 2a_1\rho a_1^\dagger + \rho a_1^\dagger a_1) \\ & - \kappa_2(a_2^\dagger a_2\rho - 2a_2\rho a_2^\dagger + \rho a_2^\dagger a_2), \end{aligned} \quad (11)$$

where the cavity damping term is included in the usual way. Here we have assumed that the two cavity modes are coupled to two independent vacuum reservoir with κ_1 and κ_2 being the cavity decay rates of modes 1 and 2, respectively. The coefficients B_{11}, B_{12}, B_{21}, and B_{22} in Eq. (11) are the same as defined in Appendix A. For simplicity, it is assumed that the atomic decay rate Γ is the same for all three levels. Here terms proportional to B_{11} and B_{22} correspond to the emission from level $|a\rangle$ and absorption from level $|c\rangle$, respectively, and the terms proportional to B_{12} and B_{21} corresponds to the atomic coherence generated by the coupling field Ω. The equations of motion for the stochastically averaged reduced phase shifted density operators $\langle\rho e^{-i\phi(t)}\rangle$and $\langle\rho e^{-i\phi(t)/2}\rangle$ are presented in Appendix B. Equations (11), (B1), and (B6), which incorporate the effects of the finite bandwidth associated with the strong driving pump field, reduce to the results of Xiong, Scully, and Zubairy [8] for $D=0$. In the limit when $\Omega \gg \Gamma$ and $D \ll \Gamma$, the system describes a nondegenerate parametric amplifier.

We can evaluate the time evolution of the quantum fluctuations of the EPR operators $\hat{u}$ and $\hat{v}$ and the mean number of photons from Eqs. (11), (B1), and (B6). In particular we calculate the time evolution of the various moments involved in the quantities $(\Delta\hat{u})^2+(\Delta\hat{v})^2$ and the total photon number $\langle\hat{N}\rangle$ which are given in Appendix C. The resulting expressions for an arbitrary state are rather complicated and we do not reproduce them here. We present here the exact analytical results only for the initial vacuum state, which is given by the following:

$$[(\Delta u)^2+(\Delta v)^2](t)=\frac{2A_3}{A_1}e^{A_1t/2}\sinh\frac{A_1t}{2}-A_2B_1A_3\left\{\frac{(A_1+C_2)e^{A_1t}}{A_1\prod_i(A_1-\lambda_i)}+\frac{C_2}{A_1\prod_i\lambda_i}-\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_it}(C_2+\lambda_i)}{(A_1-\lambda_i)\lambda_i(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\right\}$$
$$+A_2B_4\left\{\frac{C_2}{\prod_i\lambda_i}-\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_it}(C_2+\lambda_i)}{\lambda_i(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\right\}+C_1B_1A_3\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_it}}{\lambda_i(\lambda_i-\lambda_j)(\lambda_i-\lambda_j)}-\frac{1}{\prod_i\lambda_i}\right\}$$
$$+C_1B_4\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_it}(\lambda_i-A_1)}{\lambda_i(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}+\frac{A_1}{\prod_i\lambda_i}\right\}-B_1A_3\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_it}(C_2+\lambda_i)}{\lambda_i(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}-\frac{C_2}{\prod_i\lambda_i}\right\}$$
$$-B_4\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_it}(\lambda_i-A_1)}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}+\frac{C_2A_1}{\prod_i\lambda_i}\right\}+C_2\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_it}(\lambda_i-A_1)}{\lambda_i(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\right\}, \tag{12}$$

$$\langle\hat{N}\rangle(t)=\frac{2A_3}{A_1}e^{A_1t/2}\sinh\frac{A_1t}{2}-A_2B_1A_3\left\{\frac{(A_1+C_2)e^{A_1t}}{A_1\prod_i(A_1-\lambda_i)}+\frac{C_2}{\prod_i\lambda_i}-\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_it}}{(A_1-\lambda_i)(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\right\}$$
$$+A_2B_4\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_it}(C_2+\lambda_i)}{\lambda_i(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\right\}+C_1B_1A_3\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_it}}{\lambda_i(\lambda_i-\lambda_j)(\lambda_i-\lambda_j)}\right\}$$
$$+C_1B_4\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_it}(\lambda_i-A_1)}{\lambda_i(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\right\}. \tag{13}$$

In Fig. 2, we show the time development of $[(\Delta u)^2+(\Delta v)^2](t)$ and $\langle\hat{N}\rangle(t)$ against the dimensionless interaction time gt. We choose, $\Omega=400$ kHz, $\Gamma=20$ kHz, $\kappa_1=\kappa_2=3.85$ kHz, $r_a=22$ kHz, $g=g_1=g_1=43$ kHz (these parameters values are such that they correspond to the micromaser experiment in Garching [20]) for $D/\Gamma=0$, $D/\Gamma=0.01$, and $D/\Gamma=0.02$. Here the initial state of the two modes is considered to be the vacuum state. For $D=0$, we obtain the same results as discussed in Ref. [8], i.e., the two modes evolve into the entangled state and remains entangled for a long time unless it is destroyed by some other dissipation channels. For $D\neq 0$, we obtain a decrease in the time for which the two modes remain entangled. This is due to decoherence that is introduced into the system as a result of the finite linewidth of the pump source. Further increase in D shows that entanglement remains only for a very limited period of time. It is clear from the results that the entanglement will be finally eliminated due to the phase fluctuations. This elimination becomes faster due to the increase in pump-phase fluctuations. The other important quantity is the mean number of photons in the two modes which is also influenced by the phase fluctuations. If we increase the diffusion coefficient D then the average number of photons (corresponding to entangled state) in the two modes significantly decrease.

In Fig. 3, we consider the initial coherent state with 10^2 photons in each mode. All the other parameters are the same as in Fig. 2. The choice of the phase for the coherent state is such that $\alpha_1\alpha_2=-|\alpha_1\alpha_2|$ (which is the best choice for the growth of the mean photon numbers in two modes of the cavity provided $\alpha>\kappa$, as discussed in Ref. [8]. It may be mentioned that the analytical expressions for $(\Delta\hat{u}^2)+(\Delta\hat{v}^2)$ and $\langle\hat{N}\rangle$ are cumbersome when we have an initial coherent state in the two modes of the cavity field. We present here our numerical results. For $D=0$, we reproduce the results presented in Ref. [8]. It is interesting to note that even for $D/\Gamma=0.0001$, we obtain a significant decrease in the time for which the two modes remain entangled. When we choose, $D/\Gamma=0.0012$, entanglement almost vanishes. The average number of photons also show a significant reduction due to the increase in D/Γ. The dotted points here show the truncation when $(\Delta\hat{u}^2)+(\Delta\hat{v}^2)=2$ and the state is not necessarily entangled at that point. The temporal evolution of the $\langle\hat{N}\rangle$ is almost the same as in Fig. 2 for $D/\Gamma=0$, 0.0001, and

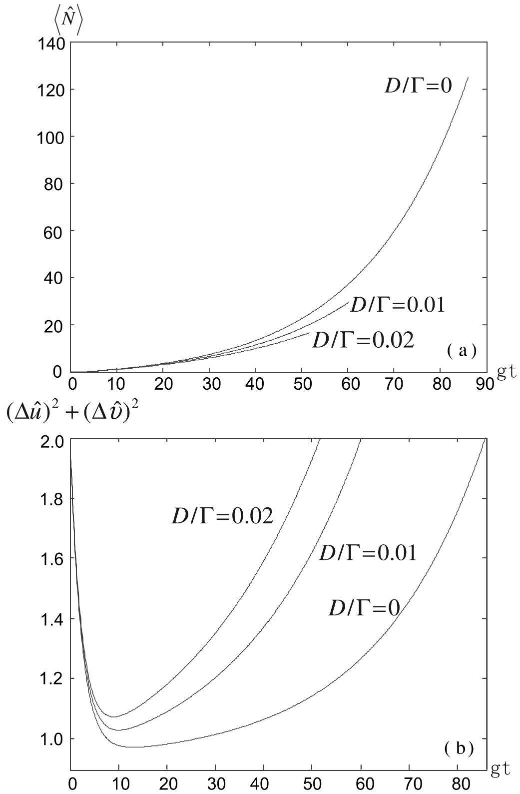

FIG. 2. Time development of (a) $\langle \hat{N} \rangle$ and (b) $(\Delta \hat{u})^2 + (\Delta \hat{v})^2$ for the initial vacuum states for the two modes (in terms of normalized interaction time gt) with $\Omega = 400$ kHz, $\Gamma = 20$ kHz, $\kappa_1 = \kappa_2 = 3.85$ kHz, $r_a = 22$ kHz, $g = g_1 = g_2 = 43$ kHz (parameters are chosen such that they correspond to the micromaser experiments [20]) for $D/\Gamma = 0, 0.01$, and 0.02.

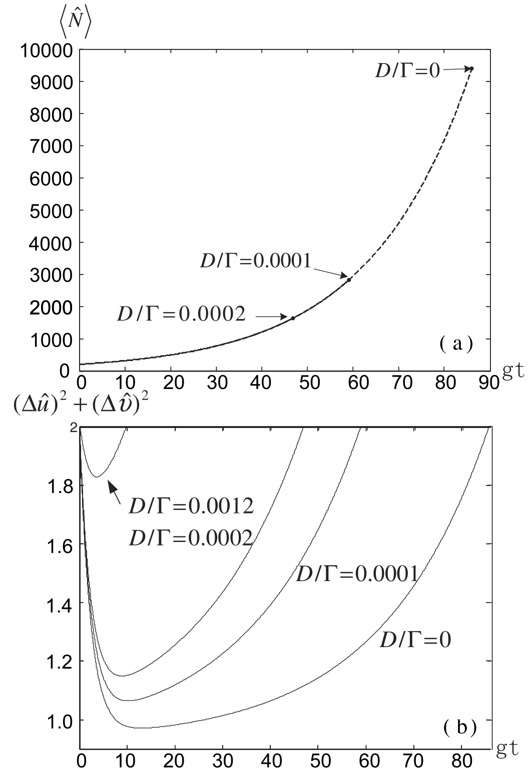

FIG. 3. Time development of (a) $\langle \hat{N} \rangle$ and (b) $(\Delta \hat{u})^2 + (\Delta \hat{v})^2$ for the initial coherent states $|10, -10\rangle$ in terms of normalized time gt for different values of D/Γ. All the other parameters are the same as in Fig. 2.

0.0002. It is just due to the small difference in D/Γ that they appear to be coincided. Our results show that the effects of phase diffusion on entanglement generation is drastic when we have initial coherent state in the two modes of the cavity field. An insight in these results can be obtained from Eq. (10) that shows the various moments required to estimate the entanglement of the system. It is clear that for the initial vacuum state all the first-order moments, which are phase sensitive, never show up, however, for the initial coherent state they contribute. For $D = 0$ all the first-order moments contribute coherently, however, in the presence of phase fluctuations decoherence introduces, which results in the significant reduction of entanglement even for very small values of D/Γ. Due to this fact, for the coherent state we assumed the value for the ratio of D/Γ that is dropped by a factor of 100 as compared to its value for initial vacuum state. Finally, in Fig. 4, we show the same plots as in Fig. 3 for a fixed value of the diffusion coefficient, i.e., $D/\Gamma = 0.01$. Here we consider the initial vacuum and coherent states with 1, 4, and 9 photons in each mode of the cavity. All the other parameters are the same as in Fig. 2. It is interesting to note that the time scale for the two modes to remain entangled reduces as we go from the initial vacuum state to the different coherent state with increasing amplitude. Our results clearly show that the decoherence effects due to the finite bandwidth of the driving field become more and more important as we increase the amplitude of the initial coherent state in the two modes.

These results are interesting because in BHDS we measure the field quadratures with respect to a reference phase whose fluctuations depend upon the phase of the pump field. Therefore, we expect no effect of the pump bandwidth on the measured quadratures. The question then is what causes the suppression of the entanglement in our system. An insight into these results can be obtained by considering the fact that the atomic coherence which is produced at any time by the driving field with a certain phase decays exponentially in time because of the atomic decay rate Γ. As a result, at a particular time the atomic response not only depends upon the phase of the driving field at that particular moment, but also keep track of its previous interaction until it decays. The interaction of the atom with the fluctuating phases of the driving field during this time is responsible for the suppression of the entanglement. In fact there is a delay in the response of the system to an instantaneous change in the phase of the local oscillator.

It may be pointed out that in our earlier paper, we also predicted the reduction in the entanglement due to the phase diffusion in a nondegenerate parametric amplifier, Ref. [14]. However, in contrast to the present system, there we have defined the quadratures with respect to a fixed reference phase.

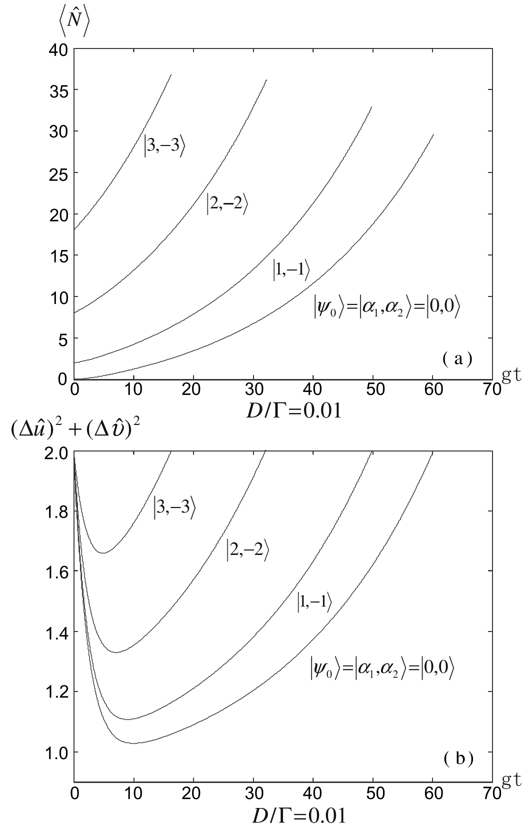

FIG. 4. Time development of (a) $\langle \hat{N} \rangle$ and (b) $(\Delta \hat{u})^2 + (\Delta \hat{v})^2$ for the initial vacuum states and coherent states $|1,-1\rangle$, $|2,-2\rangle$, and $|3,-3\rangle$ for $D/\Gamma = 0.01$. All the other parameters are the same as in Fig. 2.

IV. CONCLUSION

We have studied the generation of entanglement using correlated emission laser by including the effects of a finite bandwidth associated with the driving pump field. The interaction of three-level atoms with two modes of the cavity field inside a doubly resonant cavity is considered. Atomic coherence is introduced by driving the system with a strong classical field whose phase fluctuations are modeled by a classical phase-diffusion process. The master equation for the cavity field modes averaged over the stochastic process is derived. In order to estimate the entanglement, we apply the criterion of Duan *et al.*, which requires the measurement of quadratures for the two modes of the cavity field. In balanced homodyne detection, we consider the interference of the cavity field with a beam whose phase fluctuations are correlated with the pump. By doing so, we expect that the quadrature measurement will be independent of the phase fluctuation, however, it is not true. A clear difference with the case $D = 0$ is that the time for which two modes remain entangled decreases. It is due to the exponential decay of the atomic coherence, which leads to a delayed atomic response as compared to the instantaneous change in the phase of the local oscillator. We consider initial vacuum and coherent state for the two modes of the cavity field. It is interesting to note that the condition to obtain entanglement in the presence of phase diffusion is more stringent when the two modes are initially considered in a coherent state as compared to the vacuum state. For example, for initial coherent state, the two modes evolve into entangled state only under the condition when $D \ll \Gamma$.

Our results clearly show that the effect of phase fluctuations on the generation of the macroscopic entangled state using two-photon CEL can be minimized by considering a gain medium with large atomic dipole decay rate Γ as compared to the diffusion constant D of the driving field. Physically, this means the generation of bright entangled beams requires a highly stabilized laser source such that the linewidth of the driving field is much less than the atomic linewidth; otherwise, the driving field is going to completely destroy the entanglement generation.

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APPENDIX A

The coefficients B_{11}, B_{12}, B_{21}, and B_{22} in Eq. (11) are given by

$$B_{11} = \frac{g_1 g_2 r_a \Omega^2 [3\Gamma/2 + D]}{2\Gamma[\Gamma(\Gamma + D) + \Omega^2][\Gamma(\Gamma + D) + \Omega^2/4]}, \tag{A1}$$

$$B_{12} = -\frac{i g_1 g_2 r_a \Omega/2[(2\Gamma + D)(4D + \Gamma)(\Gamma + D) + (2D + \Gamma)\Omega^2/2]}{(\Gamma + D)[\Gamma(\Gamma + 4D)(\Gamma + D) + \Omega^2(\Gamma + 2D)][\Gamma(\Gamma + D) + \Gamma^2/4]}, \tag{A2}$$

$$B_{21} = -\frac{i g_1 g_2 r_a \Omega/2[(\Gamma + D)(-\Gamma - D)(\Gamma + 4D) + (2D + \Gamma)\Omega^2/2]}{(\Gamma + D)[\Gamma(\Gamma + 4D)(\Gamma + D) + \Omega^2(\Gamma + 2D)][\Gamma(\Gamma + D) + \Gamma^2/4]}, \tag{A3}$$

$$B_{22} = \frac{g_1 g_2 r_a [(\Gamma + D)[\Gamma(\Gamma + D) + \Omega^2/2] - \Gamma\Omega^2/4]}{\Gamma[\Gamma(\Gamma + D) + \Omega^2][\Gamma(\Gamma + D) + \Omega^2/4]}. \tag{A4}$$

APPENDIX B

The equations of motion for the reduced density matrix for the fields, i.e., $\langle \rho e^{-i\phi(t)}\rangle$ and $\langle \rho e^{-i\phi(t)/2}\rangle$, averaged over the stochastic processes are obtained by following the same method as discussed in [12] and are given by the following:

$$\begin{aligned}\frac{d}{dt}\langle \rho e^{-i\phi(t)}\rangle = &-[C_{11}^{*}a_1a_1^{\dagger}\langle \rho^{-i\phi(t)}\rangle + C_{11}\langle \rho^{-i\phi(t)}\rangle a_1a_1^{\dagger} - (C_{11}+C_{11}^{*})a_1^{\dagger}\langle \rho^{-i\phi(t)}\rangle a_1 \\ &+ C_{22}^{*}a_2^{\dagger}a_2\langle \rho^{-i\phi(t)}\rangle + C_{22}\langle \rho^{-i\phi(t)}\rangle a_2^{\dagger}a_2 - (C_{22}+C_{22}^{*})a_2\langle \rho^{-i\phi(t)}\rangle a_2^{\dagger}] \\ &- [C_{12}^{*}a_1a_2\langle \rho\rangle + C_{21}\langle \rho\rangle a_1a_2 - (C_{12}^{*}+C_{21})a_2\langle \rho\rangle a_1] \\ &- [C_{21}^{*}a_1^{\dagger}a_2^{\dagger}\langle \rho e^{-2i\phi(t)}\rangle + C_{12}\langle \rho e^{-2i\phi(t)}\rangle a_1^{\dagger}a_2^{\dagger} - (C_{12}+C_{21}^{*})a_1^{\dagger}\langle \rho e^{-2i\phi(t)}\rangle a_2^{\dagger}] \\ &- \kappa_1(a_1^{\dagger}a_1\rho - 2a_1\rho a_1^{\dagger} + \rho a_1^{\dagger}a_1) - \kappa_2(a_2^{\dagger}a_2\rho - 2a_2\rho a_2^{\dagger} + \rho a_2^{\dagger}a_2) - D\langle \rho e^{-i\phi(t)}\rangle,\end{aligned} \tag{B1}$$

where the coefficients C_{11}, C_{12}, C_{21}, and C_{22} are given by

$$C_{11} = \frac{g_1g_2r_a\{\Gamma[\Omega^2/4(\Gamma+4D)+\Gamma\Omega^2/4]+\Omega^2/4(\Gamma+D)(\Gamma+4D)\}}{(\Gamma+D)[\Gamma(\Gamma+D)(\Gamma+4D)+\Gamma\Omega^2+2D\Omega^2][\Gamma(\Gamma+D)+\Omega^2/4]}, \tag{B2}$$

$$C_{12} = -\frac{ig_1g_2r_a\Omega/2\{(\Gamma+D)[\Gamma(D+\Gamma)+\Omega^2/2]+\Gamma^2(D+\Gamma)\}}{\Gamma(\Gamma+D)[\Gamma(\Gamma+D)+\Omega^2][\Gamma(\Gamma+D)+\Gamma^2/4]}, \tag{B3}$$

$$C_{21} = -\frac{ig_1g_2r_a\Omega/2[(\Gamma+4D)(-\Gamma-4D)(\Gamma+9D)+(5D+\Gamma)\Omega^2/2]}{(\Gamma+4D)[(\Gamma+4D)(\Gamma+9D)(\Gamma+D)+\Omega^2(\Gamma+5D)][(\Gamma+D)(\Gamma+4D)+\Gamma^2/4]}, \tag{B4}$$

$$C_{22} = \frac{g_1g_2r_a\{(\Gamma+4D)[(\Gamma+4D)(D+\Gamma)\Gamma+\Gamma\Omega^2/2+D\Omega^2]-(\Omega^2/4)\Gamma(\Gamma+D)\}}{(\Gamma+D)[\Gamma(\Gamma+D)(\Gamma+4D)+\Gamma\Omega^2+2D\Omega^2][(\Gamma+4D)(\Gamma+D)+\Omega^2/4]}. \tag{B5}$$

Also

$$\begin{aligned}\frac{d}{dt}\langle \rho e^{-i\phi(t)/2}\rangle = &-[D_{11}^{*}a_1a_1^{\dagger}\langle \rho^{-i\phi(t)/2}\rangle + D_{11}\langle \rho^{-i\phi(t)/2}\rangle a_1a_1^{\dagger} - (D_{11}+D_{11}^{*})a_1^{\dagger}\langle \rho^{-i\phi(t)/2}\rangle a_1 \\ &+ D_{22}^{*}a_2^{\dagger}a_2\langle \rho^{-i\phi(t)/2}\rangle + D_{22}\langle \rho^{-i\phi(t)/2}\rangle a_2^{\dagger}a_2 - (D_{22}+D_{22}^{*})a_2\langle \rho^{-i\phi(t)/2}\rangle a_2^{\dagger}] \\ &- [D_{12}^{*}a_1a_2\langle \rho^{i\phi(t)/2}\rangle + D_{21}\langle \rho^{i\phi(t)/2}\rangle a_1a_2 - (D_{12}^{*}+D_{21})a_2\langle \rho^{i\phi(t)/2}\rangle a_1] \\ &- [D_{21}^{*}a_1^{\dagger}a_2^{\dagger}\langle \rho e^{-3i\phi(t)/2}\rangle + D_{12}\langle \rho e^{-3i\phi(t)/2}\rangle a_1^{\dagger}a_2^{\dagger} - (D_{12}+D_{21}^{*})a_1^{\dagger}\langle \rho e^{-3i\phi(t)/2}\rangle a_2^{\dagger}] \\ &- \kappa_1(a_1^{\dagger}a_1\rho - 2a_1\rho a_1^{\dagger} + \rho a_1^{\dagger}a_1) - \kappa_2(a_2^{\dagger}a_2\rho - 2a_2\rho a_2^{\dagger} + \rho a_2^{\dagger}a_2) - D/4\langle \rho e^{-i\phi(t)/2}\rangle,\end{aligned} \tag{B6}$$

where the coefficients D_{11}, D_{12}, D_{21}, and D_{22} are given by

$$D_{11} = \frac{g_1g_2r_a\{(\Gamma+D/4)\Omega^2/2(\Gamma+5D/4)+\Omega^2/4(\Gamma+D/4)(\Gamma+9D/4)\}}{(\Gamma+D/4)[(\Gamma+D/4)^2(\Gamma+9D/4)+\Omega^2(\Gamma+5D/4)][(\Gamma+D/4)^2+\Omega^2/4]}, \tag{B7}$$

$$D_{12} = -\frac{ig_1g_2r_a\Omega/2[(\Gamma+D/4)^2(9D/4+\Gamma)+\Omega^2/2(\Gamma+5D/4)+(\Gamma+D/4)(\Gamma+9D/4)(D/4+\Gamma)]}{[(\Gamma+D/4)^3(\Gamma+9D/4)+\Omega^2(\Gamma+5D/4)(\Gamma+D/4)][(-\Gamma-D/4)^2+\Gamma^2/4]}, \tag{B8}$$

$$D_{21} = -\frac{ig_1g_2r_a\Omega/2[(\Gamma+9D/4)(-\Gamma-9D/4)(\Gamma+25D/4)+(13D/4+\Gamma)\Omega^2/2]}{(\Gamma+9D/4)[(\Gamma+25D/4)(\Gamma+9D/4)(\Gamma+D/4)+\Omega^2(\Gamma+13D/4)][(\Gamma+D/4)(\Gamma+9D/4)+\Gamma^2/4]}, \tag{B9}$$

$$D_{22} = \frac{g_1g_2r_a\{(\Gamma+9D/4)[(\Gamma+9D/4)(D/4+\Gamma)^2+(\Gamma+5D/4)\Gamma\Omega^2/2]-(\Omega^2/4)(\Gamma+D/4)^2\}}{(\Gamma+D/4)[(\Gamma+9D/4)(\Gamma+D/4)^2+\Omega^2(\Gamma+5D/4)][(\Gamma+D/4)(\Gamma+9D/4)+\Omega^2/4]}. \tag{B10}$$

APPENDIX C

The equations of motion for the various moments required to estimate the entanglement generation in a two-photon CEL can be obtained in a straightforward way from Eqs. (11), (B1), and (B6). For a suitable choice of the average value of the fluctuating phase $\phi_0=\pi/2$, we obtain the following two sets of coupled equations for various moments:

$$\frac{d}{dt}\langle\langle a_1^\dagger a_1\rangle\rangle = A_1\langle\langle a_1^\dagger a_1\rangle\rangle + A_2(\langle\langle a_1 a_2 e^{i\phi_1(t)}\rangle\rangle + \langle\langle a_1^\dagger a_2^\dagger e^{-i\phi_1(t)}\rangle\rangle) + A_3, \tag{C1}$$

$$\frac{d}{dt}(\langle\langle a_1 a_2 e^{i\phi_1(t)}\rangle\rangle + \langle\langle a_1^\dagger a_2^\dagger e^{-i\phi_1(t)}\rangle\rangle) = B_2(\langle\langle a_1 a_2 e^{i\phi_1(t)}\rangle\rangle + \langle\langle a_1^\dagger a_2^\dagger e^{-i\phi_1(t)}\rangle\rangle) + B_3\langle\langle a_2^\dagger a_2\rangle\rangle - B_1\langle\langle a_1^\dagger a_1\rangle\rangle - B_4, \tag{C2}$$

$$\frac{d}{dt}\langle\langle a_2^\dagger a_2\rangle\rangle = -C_2\langle\langle a_2^\dagger a_2\rangle\rangle - C_1(\langle\langle a_1 a_2 e^{i\phi_1(t)}\rangle\rangle + \langle\langle a_1^\dagger a_2^\dagger e^{-i\phi_1(t)}\rangle\rangle), \tag{C3}$$

and

$$\frac{d}{dt}(\langle\langle a_1 e^{i\phi_1(t)/2}\rangle\rangle) = D_1\langle\langle a_1 e^{i\phi_1(t)/2}\rangle\rangle + D_2\langle\langle a_2^\dagger e^{-i\phi_1(t)/2}\rangle\rangle, \tag{C4}$$

$$\frac{d}{dt}(\langle\langle a_2^\dagger e^{-i\phi_1(t)/2}\rangle\rangle) = E_1\langle\langle a_2^\dagger e^{-i\phi_1(t)/2}\rangle\rangle - E_2\langle\langle a_1 e^{i\phi_1(t)/2}\rangle\rangle, \tag{C5}$$

where the coefficients $A_1, A_2, A_3, B_1, B_2, B_3, B_4, C_1, C_2, D_1, D_2, E_1$, and E_2 are defined in Appendix D.

The coupled set of Eqs. (C1)–(C3) and (C4) and (C5) can be solved exactly using, for example, the Laplace transform techniques. The solutions are given by the following:

$$\begin{aligned}\langle\langle a_1^\dagger a_1\rangle\rangle(t) &= \left\{ e^{A_1 t} + A_2 B_1\left(\frac{(A_1+C_2)e^{A_1 t}}{A_1\prod_i (A_1-\lambda_i)} - \sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_i t}(C_2+\lambda_i)}{(A_1-\lambda_i)(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\right)\right\}\langle\langle a_1^\dagger a_1\rangle\rangle_0 \\
&\quad + A_2\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_i t}(C_2+\lambda_i)}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\right\}(\langle\langle a_1 a_2 e^{i\phi_1(t)}\rangle\rangle + \langle\langle a_1^\dagger a_2^\dagger e^{-i\phi_1(t)}\rangle\rangle)_0 \\
&\quad + A_2 B_3\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_i t}}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_j)}\right\}\langle\langle a_2^\dagger a_2\rangle\rangle_0 \\
&\quad - A_2 B_1 A_3\left\{\frac{(A_1+C_2)e^{A_1 t}}{A_1\prod_i (A_1-\lambda_i)} - \sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_i t}(C_2+\lambda_i)}{(A_1-\lambda_i)\lambda_i(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)} + \frac{C_2}{A_1\prod_i \lambda_i}\right\} \\
&\quad - A_2 B_4\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_i t}(C_2+\lambda_i)}{\lambda_i(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)} - \frac{C_2}{\prod_i \lambda_i}\right\} + \frac{2A_3}{A_1}e^{A_1 t/2}\sinh\frac{A_1 t}{2},\end{aligned} \tag{C6}$$

$$\begin{aligned}(\langle\langle a_1 a_2 e^{i\phi_1(t)}\rangle\rangle + \langle\langle a_1^\dagger a_2^\dagger e^{-i\phi_1(t)}\rangle\rangle)(t) &= -B_1\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_i t}(C_2+\lambda_i)}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\right\}\langle\langle a_1^\dagger a_1\rangle\rangle_0 \\
&\quad + \left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_i t}(\lambda_i-A_1)}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)} + C_2\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_i t}(\lambda_i-A_1)}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\right\}\times(\langle\langle a_1 a_2 e^{i\phi_1(t)}\rangle\rangle \\
&\quad + \langle\langle a_1^\dagger a_2^\dagger e^{-i\phi_1(t)}\rangle\rangle)_0 + B_3\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_i t}(\lambda_i-A_1)}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_j)}\right\}\langle\langle a_2^\dagger a_2\rangle\rangle_0 \\
&\quad - B_1 A_3\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_i t}(C_2+\lambda_i)}{\lambda_i(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)} - \frac{C_2}{\prod_i \lambda_i}\right\} \\
&\quad - B_4\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_i t}(\lambda_i-A_1)}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)} + C_2\left(\frac{A_1}{\prod_i \lambda_i} + \sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_i t}(\lambda_i-A_1)}{\lambda_i(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\right)\right\},\end{aligned} \tag{C7}$$

$$\begin{aligned}\langle\langle a_2^\dagger a_2\rangle\rangle(t) &= C_1B_1\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_i t}}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\right\}\langle\langle a_1^\dagger a_1\rangle\rangle_0\\ &+C_1\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_i t}(\lambda_i-A_1)}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\right\}(\langle\langle a_1a_2e^{i\phi_1(t)}\rangle\rangle+\langle\langle a_1^\dagger a_2^\dagger e^{-i\phi_1(t)}\rangle\rangle)_0\\ &+\left\{e^{-C_2t}+C_1B_3\left(\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_i t}(\lambda_i-A_1)}{(C_2-\lambda_i)(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}+\frac{e^{-C_2t}(A_1+C_2)}{\prod_i(C_2+\lambda_i)}\right)\right\}\langle\langle a_2^\dagger a_2\rangle\rangle_0\\ &+C_1B_1A_3\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_i t}}{\lambda_i(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}-\frac{1}{\prod_i\lambda_i}\right\}+C_1B_4\left\{\sum_{i,j,k,i\neq j,j\neq k}\frac{e^{\lambda_i t}(\lambda_i-A_i)}{\lambda_i(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}+\frac{A_1}{\prod_i\lambda_i}\right\},\end{aligned} \quad (C8)$$

where λ_i's are the roots of the following cubic equation:

$$\lambda^3+\alpha\lambda^2-\beta\lambda+\gamma=0, \quad (C9)$$

and

$$\alpha=C_2-A_1-B_2, \quad (C10a)$$

$$\beta=A_1C_2-B_1A_2+B_2C_2-A_1B_2-B_3C_1, \quad (C10b)$$

$$\gamma=B_1A_2C_2+A_1B_2C_2-A_1B_3C_1. \quad (C10c)$$

The roots of the cubic equation can be obtained exactly by, for example, using the Cardano formula. The solutions of $\langle\langle a_1e^{i\phi_1(t)/2}\rangle\rangle$ and $\langle\langle a_2^\dagger e^{-i\phi_1(t)/2}\rangle\rangle$ are given by

$$\begin{aligned}\langle\langle a_1e^{i\phi_1(t)/2}\rangle\rangle(t) &= \frac{1}{(\lambda^+-\lambda^-)}\{[e^{\lambda^+t}(E_1+\lambda^+)-e^{\lambda^-t}(E_1+\lambda^-)]\\ &\times\langle\langle a_1\rangle\rangle_0+(e^{\lambda^+t}-e^{\lambda^-t})D_2\langle\langle a_2^\dagger\rangle\rangle_0t\},\end{aligned} \quad (C11)$$

$$\begin{aligned}\langle\langle a_2^\dagger e^{-i\phi_1(t)/2}\rangle\rangle(t) &= -\frac{1}{(\lambda^+-\lambda^-)}\{e^{\lambda^+t}-e^{\lambda^-t}\}E_2\langle\langle a_1\rangle\rangle_0\\ &+\left\{e^{-E_1t}-D_2E_2\left(\frac{e^{\lambda^+t}}{(E_1+\lambda^+)(\lambda^+-\lambda^-)}\right.\right.\\ &+\frac{e^{E_1t}}{(E_1+\lambda^+)(E_1+\lambda^-)}\\ &\left.\left.+\frac{e^{\lambda^-t}}{(E_1+\lambda^-)(\lambda^--\lambda^+)}\right)\right\}\langle\langle a_2^\dagger\rangle\rangle_0,\end{aligned} \quad (C12)$$

where

$$\lambda^\pm=-\frac{(E_1-D_1)\pm\sqrt{(E_1-D_1)^2+4(D_1E_1-D_2E_2)}}{2}. \quad (C13)$$

APPENDIX D

The coefficients $A_1,A_2,A_3,B_1,B_2,B_3,B_4,C_1,C_2,D_1,D_2$, E_1, and E_2 in Eqs. (C1)–(C5) are given by the following:

$$A_1=2B_{11}-\kappa_1-\kappa_2, \quad (D1)$$

$$A_2=iB_{12}^*=-iB_{12}, \quad (D2)$$

$$A_3=B_{11}+B_{11}^*, \quad (D3)$$

$$B_1=iC_{21}-iC_{21}^*, \quad (D4)$$

$$B_2=C_{11}^*-C_{22}-D-\kappa_1-\kappa_2, \quad (D5)$$

$$B_3=iC_{12}-iC_{12}^*, \quad (D6)$$

$$B_4=iC_{21}-iC_{21}^*, \quad (D7)$$

$$C_1=iB_{21}=-iB_{21}^*, \quad (D8)$$

$$C_2=B_{22}^*+B_{22}-\kappa_1-\kappa_2, \quad (D9)$$

$$D_1=D_{11}-D/4-\kappa_1, \quad (D10)$$

$$D_2=-iD_{12}, \quad (D11)$$

$$E_1=D_{22}+D/4+\kappa_2, \quad (D12)$$

$$E_2=iD_{21}. \quad (D13)$$

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