Four-level entangled quantum heat engines

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Based on a two-qubit one-dimensional isotropic spin-1/2 Heisenberg spin chain in a constant external magnetic field, we construct a four-level entangled quantum heat engine (QHE) and investigate the influence of entanglement between two qubits on basic thermodynamic quantities, i.e., the heat transferred and the work done in a cycle, and the efficiency of the QHE. The validity of the second law of thermodynamics is confirmed in the entangled system. We also find several interesting features of the variation of the efficiency with the entanglement of two different thermal equilibrium states in a work cycle in zero and finite magnetic field. An abrupt transition of efficiency is found in zero field.

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I. INTRODUCTION

Carnot showed that every heat engine that operates between two heat baths has an efficiency less than or equal to the Carnot efficiency $\eta_c = 1 - T_l/T_h$, where T_h and T_l are the temperature of the high-temperature energy source and the low-temperature energy sink, respectively. It is equivalent to the second law of thermodynamics and supported by numerous experimental evidences without exception so far in classical systems [1,2].

Recently, many investigations have been carried out to explore various possible improvement of the heat engine by virtue of rapidly developing quantum mechanics, and many surprising properties [3–14] are reported. Scully and collaborators proposed quantum heat engines (QHEs) that can extract work from a single thermal bath via quantum negentropy [8] or vanishing quantum coherence [9,10]. They also obtained laser action in hot exhaust gases of an Otto heat engine by using a laser and maser in tandem [6,7]. Kieu [5] constructed a QHE which is a two-level quantum system and undergoes quantum adiabatic process and energy exchanges with heat baths at different stages in a work cycle. Armed with this class of QHE and the interpretations of heat transferred and work performed at the quantum level, he clarified some important aspects of the second law of thermodynamics.

However, none of the QHEs mentioned above involve the most extraordinary phenomenon in quantum mechanics, i.e., quantum entanglement [15,16]. To enrich research in QHEs, we introduce an entangled QHE and investigate the influence of entanglement on its thermodynamic characteristics. The second law of thermodynamics is first shown to be valid in an entangled system. Some interesting connections between entanglement and basic thermodynamic quantities, i.e., the heat transferred and the work done in a work cycle, and the efficiency of the QHE, are also given analytically or graphically.

II. ENTANGLED QHE MODEL

Our QHE is a two-qubit one-dimensional (1D) isotropic spin-1/2 Heisenberg model [17–19]. The Hamiltonian for the system in a constant external magnetic field B is given by

$$\mathbf{H} = J(\vec{\sigma}^1 \cdot \vec{\sigma}^2 + \vec{\sigma}^2 \cdot \vec{\sigma}^1) + B(\sigma_z^1 + \sigma_z^2), \tag{1}$$

where $\vec{\sigma}^i = (\sigma_x^i, \sigma_y^i, \sigma_z^i)$ are the Pauli matrices for the ith ($i = 1,2$) spin. J is the exchange constant while $J > 0$ and $J < 0$ correspond to the antiferromagnetic and the ferromagnetic cases, respectively. The four eigenvalues of Eq. (1) are [20]

$$E_1 = -6J,$$

$$E_2 = 2J - 2B,$$

$$E_3 = 2J,$$

$$E_4 = 2J + 2B. \tag{2}$$

When the system is in the thermal equilibrium state (temperature T), it is described by $\rho(T) = e^{-\mathbf{H}/kT}/Z$, where $Z = \mathrm{Tr}(e^{-\mathbf{H}/kT})$ is the partition function and k is Boltzmann's constant. The entanglement of formation between two qubits is already known as [20]

$$E = -\left(\frac{1+\sqrt{1-c^2}}{2}\right)\log_2\left(\frac{1+\sqrt{1-c^2}}{2}\right) - \left(\frac{1-\sqrt{1-c^2}}{2}\right)\log_2\left(\frac{1-\sqrt{1-c^2}}{2}\right), \tag{3}$$

where c is the concurrence [21] given by

$$c = \begin{cases} 0 & \text{if } e^{8J/kT} \leqslant 3, \\ \dfrac{e^{8J/kT} - 3}{1 + e^{-2B/kT} + e^{2B/kT} + e^{8J/kT}} & \text{if } e^{8J/kT} > 3. \end{cases} \tag{4}$$

The concurrence or entanglement apparently vanishes when $J \leqslant 0$ from the above expressions; therefore we focus on $J > 0$ in the paper. In addition, we use concurrence directly as the measurement of entanglement in the following discussion for simplicity, since there is a one-to-one correspondence between the entanglement of formation and concurrence.

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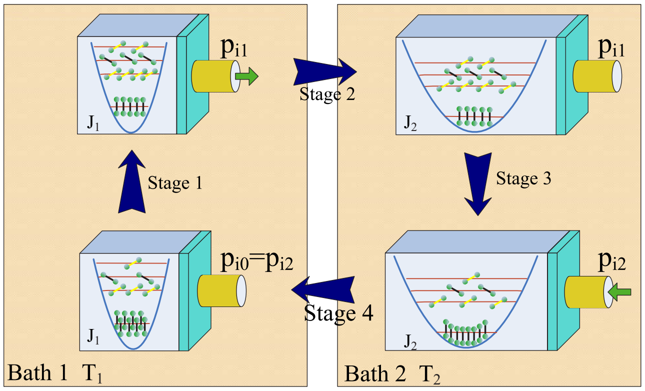

FIG. 1. (Color online) Sketch of the entangled four-level quantum heat engine. Two spin-1/2 qubits together play the role of working fluid.

We adopt Kieu's explanations of heat transferred and work performed at the quantum level [5] in this paper. The infinitesimal heat transferred $\bar{d}Q$ and work done $\bar{d}W$ are identified as

$$\bar{d}Q \coloneqq \sum_i E_i dp_i,$$

$$\bar{d}W \coloneqq \sum_i p_i dE_i. \tag{5}$$

Mathematically speaking, these are not total differentials but are path dependent. These expressions interpret heat as a change in the occupation probabilities but not in the distribution of the energy eigenvalues themselves; and work done as the redistribution of the energy eigenvalues but not of the occupation probabilities of each energy level [22].

A cycle of the QHE has four stages which are presented below and sketched in Fig. 1. It is supposed that the external magnetic field is kept constant throughout.

(1) Stage 1. The system has the probability p_{i0} ($i=1,2,3,4$) to be in each of its four eigenstates, respectively. By contacting with a heat bath at temperature T_1 for some time, the occupation probability of each eigenstate becomes p_{i1}. Only heat is transferred in this stage due to the change in the occupation probability.

(2) Stage 2. The system is then isolated from the heat bath and undergoes a quantum adiabatic process (provided the process is sufficiently slow) while the exchange constant changes from J_1 to J_2. The probability of each eigenstate is maintained as p_{i1} throughout according to the quantum adiabatic theorem [23]. An amount of work is thus performed and no heat is transferred.

(3) Stage 3. The system, with the probability p_{i1} to be in each eigenstate, is brought into some contact with another heat bath at temperature T_2 until the occupation probability becomes p_{i2}. Still only heat is transferred.

(4) Stage 4. The system is removed from the heat bath and again undergoes a quantum adiabatic process. The exchange constant changes from J_2 back to J_1 and the occupation probability of each eigenstate is maintained, that is, $p_{i2}=p_{i0}$. Some work is then performed.

If the system contacts with two heat baths sufficiently in stage 1 and stage 3 to arrive at thermal equilibrium with them, then the occupation probabilities p_{i1} and p_{i2} depend only on the temperatures of the heat baths and the energy levels of the system, i.e.,

$$p_{ij} = \frac{e^{E_{ij}/kT_j}}{\sum_i e^{E_{ij}/kT_j}}$$

$$\text{for } i = 1,2,3,4$$

$$\text{and } j = 1,2, \tag{6}$$

where E_{i1} and E_{i2} are defined in Eq. (2) by taking $J=J_1$ and $J=J_2$, respectively.

According to the quantum interpretations of heat transferred and work done in Eq. (5), the heat transferred in stage 1 (Q_1) and in stage 3 (Q_2) is given by

$$Q_1 = \sum_i E_{i1}(p_{i1} - p_{i2}), \tag{7a}$$

$$Q_2 = \sum_i E_{i2}(p_{i2} - p_{i1}). \tag{7b}$$

$Q>0$ and $Q<0$ correspond to absorption and release of heat from and to the heat baths, respectively. From the law of conservation of energy, the net work done by the QHE in two quantum adiabatic processes, i.e., stage 2 and stage 4, is

$$W = Q_1 + Q_2 = \sum_i (E_{i1} - E_{i2})(p_{i1} - p_{i2}). \tag{8}$$

$W>0$ and $Q<0$ correspond to work performed by and on the QHE, respectively.

It is worthy of mention that for a qualified heat engine

$$T_1 > T_2, \tag{9a}$$

$$W = Q_1 + Q_2 > 0 \tag{9b}$$

must be satisfied. Equation (9a) is actually our presupposition and (9b) means some work done by the heat engine. Equation (9b) straightforwardly indicates three possible cases, i.e., (i) $Q_1 > -Q_2 > 0$, (ii) $Q_2 > -Q_1 > 0$, and (iii) $Q_1 > 0$ and $Q_2 > 0$. (i) is physically acceptable while (ii) and (iii) apparently violate the second law of thermodynamics. In the latter part of the paper, we will prove (ii) and (iii) impossible analytically for null magnetic fields and graphically for finite fields. Now the efficiency of the QHE can be defined as

$$\eta = \frac{W}{Q_1} = 1 + \frac{Q_2}{Q_1}. \tag{10}$$

The entanglement under our consideration is that of two thermal equilibrium states at the end of stage 1 and stage 3, denoted by c_1 and c_2, respectively. They are

$$c_1 = \begin{cases} 0 & \text{if } e^{8J_1/kT_1} \leqslant 3, \\ \dfrac{e^{8J_1/kT_1} - 3}{1 + e^{-2B/kT_1} + e^{2B/kT_1} + e^{8J_1/kT_1}} & \text{if } e^{8J_1/kT_1} > 3. \end{cases} \tag{11a}$$

$$c_2 = \begin{cases} 0 & \text{if } e^{8J_2/kT_2} \leqslant 3, \\ \dfrac{e^{8J_2/kT_2} - 3}{1 + e^{-2B/kT_2} + e^{2B/kT_2} + e^{8J_2/kT_2}} & \text{if } e^{8J_2/kT_2} > 3. \end{cases} \tag{11b}$$

III. QHE AND ENTANGLEMENT

Now we start to explore the relation between entanglement and basic thermodynamics quantities and the efficiency of the QHE. From Eq. (11) we find

$$J_1 = \frac{1}{8}kT_1 \ln\left(\frac{c_1(1 + e^{-2B/kT_1} + e^{2B/kT_1}) + 3}{1 - c_1}\right),$$

$$J_2 = \frac{1}{8}kT_2 \ln\left(\frac{c_2(1 + e^{-2B/kT_2} + e^{2B/kT_2}) + 3}{1 - c_2}\right), \tag{12}$$

when $e^{8J_i/kT_i} > 3$ $(i=1,2)$ are satisfied. Substituting Eq. (12) into Eqs. (7), (8), and (10), we obtain the expressions for Q_1, Q_2, W, and η as functions of c_1, c_2, T_1, T_2, and B. We first investigate the simplest case of a vanishing field. When $B=0$, Eq. (12) becomes

$$J_1 = \frac{1}{8}kT_1 \ln\left(\frac{6}{1 - c_1} - 3\right),$$

$$J_2 = \frac{1}{8}kT_2 \ln\left(\frac{6}{1 - c_2} - 3\right). \tag{13}$$

By simple deduction, Eq. (7) becomes

$$Q_1 = 8J_1(p_{12} - p_{11}) = \frac{1}{2}kT_1(c_2 - c_1)\ln\left(\frac{6}{1 - c_1} - 3\right), \tag{14a}$$

$$Q_2 = -8J_2(p_{12} - p_{11}) = -\frac{1}{2}kT_2(c_2 - c_1)\ln\left(\frac{6}{1 - c_2} - 3\right). \tag{14b}$$

Q_1 and Q_2 obviously have contrary signs, so case (iii) is excluded. The net work done in a cycle now is given by

$$\begin{aligned} W &= (8J_1 - 8J_2)(p_{12} - p_{11}) \\ &= \frac{1}{2}k(c_2 - c_1)\left[T_1 \ln\left(\frac{6}{1 - c_1} - 3\right) - T_2 \ln\left(\frac{6}{1 - c_2} - 3\right)\right]. \end{aligned} \tag{15}$$

Therefore the positive net work condition (9b) indicates two possible situations, i.e.,

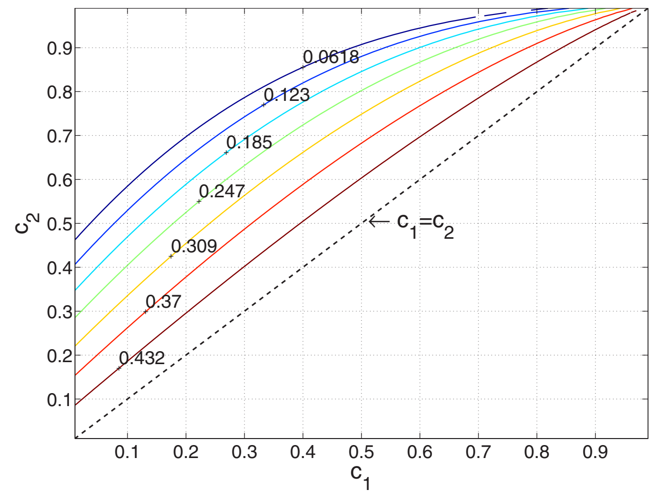

FIG. 2. (Color online) Variation of the efficiency of the quantum heat engine with the entanglement of two different thermal equilibrium states at the end ofstage 1 and stage 3 in an isoline map of efficiency for $kT_1=2$, $kT_2=1$, and $B=0$.

$$c_2 > c_1 \quad \text{and} \quad T_1 \ln\left(\frac{6}{1 - c_1} - 3\right) - T_2 \ln\left(\frac{6}{1 - c_2} - 3\right) > 0 \tag{16a}$$

or

$$c_2 < c_1 \quad \text{and} \quad T_1 \ln\left(\frac{6}{1 - c_1} - 3\right) - T_2 \ln\left(\frac{6}{1 - c_2} - 3\right) < 0. \tag{16b}$$

Equation (16b) can be easily proved incompatible with Eq. (9a), that is, case (ii) is impossible. So far we have clarified case (i), i.e., $Q_1 > -Q_2 > 0$ is the only possible case. The efficiency is now given by

$$\eta_0 = 1 - \frac{J_2}{J_1} = 1 - \frac{\ln\left(\dfrac{6}{1 - c_2} - 3\right)}{\ln\left(\dfrac{6}{1 - c_1} - 3\right)}\frac{T_2}{T_1} \quad \text{for } c_2 > c_1. \tag{17}$$

When $c_2=c_1$, $\eta_0=0$ since $Q_1=Q_2=W=0$ from Eqs. (14) and (15). Equations (14), (15), and (17) analytically give the expressions for basic thermodynamic quantities and the efficiency in terms of two concurrences, respectively.

In order to intuitively investigate how entanglement affects η_0, we sketch the variation of η_0 with c_1 and c_2 in Fig. 2 where an isoline map is used. We define $\gamma = c_1/c_2$ as a measurement of the difference of c_1 and c_2. Three features of the isolines of efficiency can be found from Fig. 2. First, the Carnot efficiency defined as $\eta_c = 1 - T_2/T_1$ is never achievable. Combined with the above explanations where neither case (ii) nor (iii) is possible, the second law of thermodynamics is proved to hold all the time even when entanglement is indeed involved. Second, η_0 increases almost monotonically with γ, approaching unity when either c_1 or c_2 is fixed. Finally, Fig. 2 shows that $\gamma \to 1$ leads to $\eta_0 \to \eta_c$. However, a unit γ yields zero η as mentioned before, i.e., an

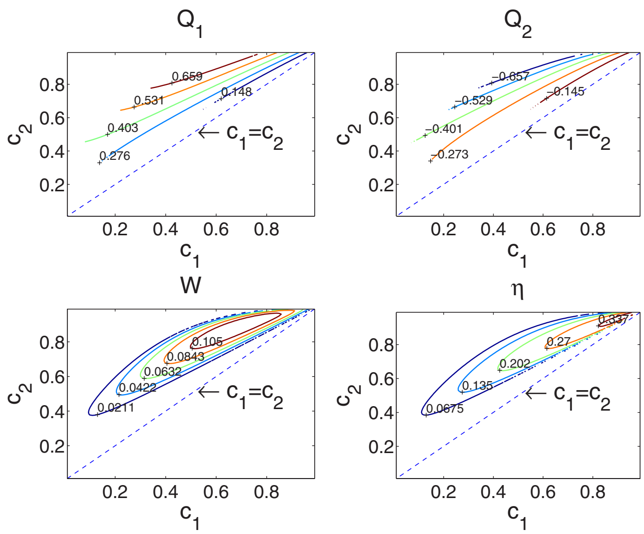

FIG. 3. (Color online) Variation of Q_1, Q_2, W, and η with c_1 and c_2 in an isoline map of efficiency for $kT_1=2$, $kT_2=1$, and $B=1$.

abrupt transition of the efficiency occurs. In fact, both the heat exchanged with two heat baths (Q_1,Q_2) and the net work done by the QHE (W) in Eqs. (14) and (15) vanish at the critical point $\gamma=1$, i.e., the heat engine undergoes a trivial cycle.

Next we study the case of nonzero magnetic field. The analytical expressions for basic thermodynamic quantities and the efficiency of the QHE now are too complicated to give clear results directly, so they are shown graphically. The Carnot efficiency is also appointed as $\eta_c=1-T_2/T_1=1/2$ in advance for simplicity. We choose three representative B values to plot the variation of basic thermodynamic quantities and the efficiency with c_1 and c_2 in Figs. 3–5. The curves are discontinuous because those situations when condition (9b) is not satisfied are excluded.

We find some intriguing features of these figures. First, $Q_1>-Q_2>0$ is always true as long as condition (9b) is satisfied. Although analytical evidence is absent, we have tested this for many randomly chosen B values and found complete agreement. Furthermore, we find that $\eta_c=1/2$ is not achievable in these three figures. Therefore the second law of thermodynamics holds all the while. Second, each isoline of efficiency becomes a loop instead of an open curve in vanishing field. This indicates that η no longer increases monotonically with γ when either c_1 or c_2 is fixed. This is understandable since η is doubtlessly affected by nonzero B. Third, the physically acceptable range of two entanglements varies for different magnetic fields. In a high enough magnetic field, the loops also appear when $c_1>c_2$ whereas in a small field it seems that only $c_2>c_1$ is relevant. This can be roughly explained as follows. Consider the extreme case $J_1=J_2$; since $T_1>T_2$ and entanglement generally decreases with the increase of temperature [21], c_2 should often be greater that c_1. However, it also was shown in Ref. [21] that entanglement could increase with the increase of temperature in certain magnetic fields. Thereby $c_2<c_1$ would possibly occur for some B values.

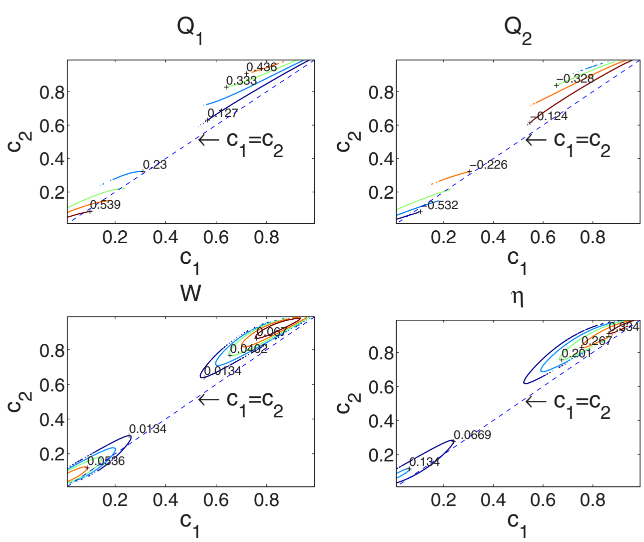

FIG. 4. (Color online) Variation of Q_1, Q_2, W, and η with c_1 and c_2 in an isoline map of efficiency for $kT_1=2$, $kT_2=1$, and $B=3$.

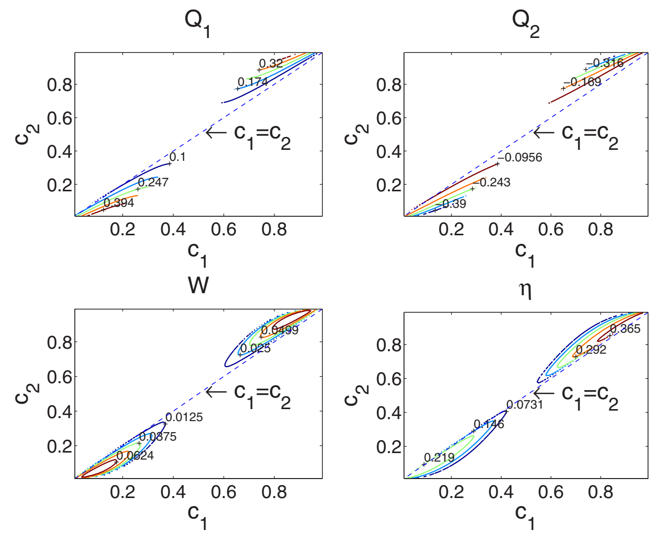

FIG. 5. (Color online) Variation of Q_1, Q_2, W, and η with c_1 and c_2 in an isoline map of efficiency for $kT_1=2$, $kT_2=1$, and $B=5$.

IV. POSITIVE WORK CONDITION OF THE QHE

Finally, we discuss the positive net work condition (9b) in more detail. When the external magnetic field vanishes, Eq. (9b) becomes

$$\frac{J_1}{T_1}<\frac{J_2}{T_2}\Leftrightarrow T_1>T_2\left(\frac{J_1}{J_2}\right), \tag{18}$$

or in terms of concurrence

$$c_1<c_2 \tag{19}$$

for $J_1>J_2$. When $J_1<J_2$, condition (9b) demands $J_1/T_1>J_2/T_2$ which is apparently incompatible with condition (9a). Equation (18) is just the same as the condition (6) in Ref. [5], also in contradistinction to the classical requirement that T_1 only needs to be larger than T_2.

When the QHE operates in nonzero magnetic field, condition (9b) demands

$$\frac{e^{8J_2/kT_2}}{e^{8J_1/kT_1}} > \frac{1+e^{-2B/kT_2}+e^{2B/kT_2}}{1+e^{-2B/kT_1}+e^{2B/kT_1}} > 1 \quad \text{for } J_1 > J_2, \tag{20a}$$

$$\frac{e^{8J_2/kT_2}}{e^{8J_1/kT_1}} < \frac{1+e^{-2B/kT_2}+e^{2B/kT_2}}{1+e^{-2B/kT_1}+e^{2B/kT_1}} \quad \text{for } J_1 < J_2. \tag{20b}$$

The second inequality in Eq. (20a) results from the monotonic increase of the function $f(x)=1+e^x+e^{-x}$ with increase of x. The lower bound of T_1/T_2 given by Eq. (20a) is higher than that given by Eq. (18) since Eq. (18) equals $e^{8J_2/kT_2}/e^{8J_1/kT_1}>1$.

Moreover, we investigate Eq. (20) in terms of concurrence. Recalling Eq. (12), Eq. (20) becomes

$$c_2 + \frac{3}{\mathcal{Z}_2} > c_1 + \frac{3}{\mathcal{Z}_1} \quad \text{for } J_1 > J_2, \tag{21a}$$

$$c_2 + \frac{3}{\mathcal{Z}_2} < c_1 + \frac{3}{\mathcal{Z}_1} \quad \text{for } J_1 < J_2, \tag{21b}$$

where

$$\mathcal{Z}_1 = 1 + e^{-2B/kT_1} + e^{2B/kT_1} + e^{8J/kT_1} = \frac{4+e^{-2B/kT_1}+e^{2B/kT_1}}{1-c_1},$$

$$\mathcal{Z}_2 = 1 + e^{-2B/kT_2} + e^{2B/kT_2} + e^{8J/kT_2} = \frac{4+e^{-2B/kT_2}+e^{2B/kT_2}}{1-c_2}. \tag{22}$$

Notice that $\mathcal{Z}_2 > \mathcal{Z}_1$ always holds, whatever is the relation between J_1 and J_2 from Eqs. (9a) and (20), so in order to make the heat engine be qualified, the condition (21a) at $B \neq 0$ is stronger than the condition (19) at $B=0$. Equation (21b) or Eq. (20b) provides an extra area of physical parameters of the QHE which does not exist in Ref. [5].

V. CONCLUSION

In conclusion, by introducing an entangled QHE and quoting the quantum interpretations of heat and work from Ref. [5], we have broadened the investigations of quantum heat engine by involving entanglement in the QHE. Based on a two-qubit 1D isotropic spin-1/2 Heisenberg chain in a constant external magnetic field, we construct a four-level entangled QHE and deduce the expressions of Q_1, Q_2, W, and η as functions of concurrences c_1 and c_2. To guarantee that the QHE is qualified, we use a preassumption and a positive net work condition. These conditions are compared with Eq. (6) in Ref. [5] and also expressed in terms of concurrence. Is is shown that the second law of thermodynamics is valid all the time. We investigate the influence of entanglement on the heat transferred (Q_1, Q_2), the net work done (W), and the efficiency (η) for vanishing and finite magnetic field. For zero field, we obtain the analytical expressions for Q_1, Q_2 [Eq. (14)], W [Eq. (15)], and η [Eq. (17)] and find three properties. An interesting phenomenon is the occurrence of an abrupt transition at $c_1=c_2$, which can be explained from the expressions of Q_1, Q_2, and W. For nonzero field, we graphically explore the variation of three thermodynamic quantities and the efficiency with c_1 and c_2, respectively. Some intriguing features and their qualitative explanations are given.

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