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Self-stabilized spatiotemporal dynamics of dissipative light bullets generated from inputs without spherical symmetry in three-dimensional Ginzburg-Landau systems

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In order to meet experimental conditions, the generation, evolution, and self-stabilization of optical dissipative light bullets from non-spherically-symmetric input pulses is studied. Steady-state solutions of the (3+1)-dimensional complex cubic-quintic Ginzburg-Landau equation are computed using the variational approach with a trial function asymmetric with respect to three transverse coordinates. The analytical stability criterion is extended to systems without spherical symmetry, allowing determination of the domain of dissipative parameters for stable solitonic solutions. The analytical predictions are confirmed by numerical evolution of the asymmetric input pulses toward stable dissipative light bullets. Once established, the dissipative light bullet remains surprisingly robust.

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Complex nonlinear dissipative systems are now a subject of very broad interest [1]. A wide class of such systems, ranging from nonlinear optics, plasma physics, and fluid dynamics to superfluidity, superconductivity, and Bose-Einstein condensates, can be modeled by complex Ginzburg-Landau equations [2]. Stable exact analytical solutions of multidimensional complex cubic-quintic Ginzburg-Landau equations (CQGLEs) do not exist [3]. Such nonintegrable systems can be solved only numerically. The domains of parameters where the solutions are stable, giving fully localized spatiotemporal dissipative solitons, have been obtained only by numerical simulations [4,5]. However, an analytical approach, even though approximate, is needed in order to guide simulations and to avoid the tedious numerical computations necessary to determine the stability domain point by point [6]. In a recent publication [7] we used the variational method extended to dissipative systems, to establish this stability domain of parameters for a spherically symmetric CQGLE. Indeed, an analytical stability criterion for dissipative one-, two-, and three-dimensional solitons was established and confirmed by exhaustive numerical simulations. Such a criterion provides analytically a broad domain of input parameters for generation of stable (3+1)-dimensional solitons called dissipative light bullets.

The objective of the present work is to extend the synergy of our analytical and numerical approach in order to study a much broader class of Ginzburg-Landau systems involving no spherical symmetry. In spherically symmetric CQGLEs, second-order derivatives are made with respect to the light bullet radius $r=\sqrt{x^2+y^2+t^2}$, imposing constraints on the independent transverse space (x and y) and time (t) variables (see Eq. (1) in Refs. [6,7]). Such a constraint on input pulses seriously limits the class of considered systems and their experimental realization [8,9]. Therefore, here we study the (3+1)-dimensional complex cubic-quintic Ginzburg-Landau equations for the normalized field envelope E, describing separately the diffractions along the x and y coordinates and the anomalous group velocity dispersion in time t without such a constraint,

$$i\frac{\partial E}{\partial z}+\frac{\partial^2 E}{\partial x^2}+\frac{\partial^2 E}{\partial y^2}+\frac{\partial^2 E}{\partial t^2}+|E|^2E-\nu|E|^4E=\mathbb{Q}. \tag{1}$$

In order to prevent wave collapse a saturating nonlinearity is required [6]. As a consequence, cubic and quintic nonlinearities have to have opposite signs, i.e., the parameter ν is positive. Terms denoted by $\mathbb{Q}$ are all dissipative

$$\mathbb{Q}=i\left[\delta E+\varepsilon|E|^2E+\mu|E|^4E+\beta\left(\frac{\partial^2 E}{\partial x^2}+\frac{\partial^2 E}{\partial y^2}+\frac{\partial^2 E}{\partial t^2}\right)\right]. \tag{2}$$

The stability of the pulse background involves linear loss, thus the parameter δ must be negative [3,7]. The parameters ε and μ are associated, respectively, with cubic and quintic gain-loss terms. The parabolic gain ($\beta>0$) is taken with respect to each transverse coordinate separately. A simultaneous balance of diffraction and dispersion with self-focusing and gain with loss is the prerequisite for generation of dissipative light bullets. As a consequence, for a given set of parameters the continuous family of solutions reduces to a fixed one representing an isolated attractor [3,5].

In order to generalize the spherically symmetric variational approach established in Ref. [7], we construct for the system of Eqs. (1) and (2) the total Lagrangian $\mathbb{L}=\mathbb{L}_c+\mathbb{L}_Q$ containing not only a conservative part,

$$\mathbb{L}_c=\frac{i}{2}\left(E\frac{\partial E^*}{\partial z}-E^*\frac{\partial E}{\partial z}\right)+G-\frac{|E|^4}{2}+\frac{\nu|E|^6}{3}, \tag{3}$$

but also a dissipative one [7],

$$\mathbb{L}_Q=i\delta|E|^2+i\frac{\varepsilon|E|^4}{2}+i\frac{\mu|E|^6}{3}-i\beta G, \tag{4}$$

where $G=|\partial E/\partial x|^2+|\partial E/\partial y|^2+|\partial E/\partial t|^2$. The independent treatment of all three transverse coordinates involves an asymmetric trial function

$$E = A \exp\left(i(Cx^2 + Dy^2 + Ft^2 + \Psi) - \frac{x^2}{2X^2} - \frac{y^2}{2Y^2} - \frac{t^2}{2T^2} \right) \tag{5}$$

as a functional of amplitude A, temporal (T) and spatial (X and Y) pulse widths, anisotropic wave front curvatures C and D, chirp F, and phase Ψ. Following Kantorovitch, the constant parameters of the Rayleigh-Ritz approach are substituted by the functions $\eta = A, X, Y, T, C, D, F, \Psi$ of the independent variable z [7]. Each of these functions is optimized giving one of eight Euler-Lagrange equations averaged over transverse coordinates x, y, and t,

$$\frac{d}{dz}\left(\frac{\partial L_c}{\partial \eta'} \right) - \frac{\partial L_c}{\partial \eta} = 2 \, \text{Re} \int \int \int dx\, dy\, dt\, \mathbb{Q} \frac{\partial E^*}{\partial \eta}, \tag{6}$$

containing already the averaged conservative Lagrangian $L_c = \int\int\int \mathbb{L}_c dx\, dy\, dt$. The real part is labeled Re. The differentiation with respect to z is labeled by a prime.

Within the variational approximation, to the partial differential CQGLE corresponds a set of eight coupled first-order differential equations (FODEs) resulting from the variations in amplitude

$$\frac{dA}{dz} = \delta A + \frac{7\varepsilon A^3}{8\sqrt{2}} + \frac{2\mu A^5}{3\sqrt{3}} - \frac{\beta A}{W^2} - 2SA; \tag{7}$$

asymmetric widths

$$\frac{dX}{dz} = 4CX - \frac{\varepsilon X A^2}{4\sqrt{2}} - \frac{2\mu X A^4}{9\sqrt{3}} + \frac{\beta}{X} - 4\beta C^2 X^3, \tag{8}$$

$$\frac{dY}{dz} = 4DY - \frac{\varepsilon Y A^2}{4\sqrt{2}} - \frac{2\mu Y A^4}{9\sqrt{3}} + \frac{\beta}{Y} - 4\beta D^2 Y^3, \tag{9}$$

and

$$\frac{dT}{dz} = 4FT - \frac{\varepsilon T A^2}{4\sqrt{2}} - \frac{2\mu T A^4}{9\sqrt{3}} + \frac{\beta}{T} - 4\beta F^2 T^3; \tag{10}$$

anisotropic wave front curvatures

$$\frac{dC}{dz} = \frac{1}{X^4} - \frac{A^2}{4\sqrt{2}X^2} + \frac{2\nu A^4}{9\sqrt{3}X^2} - 4C^2 - \frac{4\beta C}{X^2}, \tag{11}$$

$$\frac{dD}{dz} = \frac{1}{Y^4} - \frac{A^2}{4\sqrt{2}Y^2} + \frac{2\nu A^4}{9\sqrt{3}Y^2} - 4D^2 - \frac{4\beta D}{Y^2}, \tag{12}$$

and

$$\frac{dF}{dz} = \frac{1}{T^4} - \frac{A^2}{4\sqrt{2}T^2} + \frac{2\nu A^4}{9\sqrt{3}T^2} - 4F^2 - \frac{4\beta F}{T^2}; \tag{13}$$

and phase

$$\frac{d\Psi}{dz} = 2\beta S - \frac{1}{W^2} + \frac{7A^2}{8\sqrt{2}} - \frac{2\nu A^4}{3\sqrt{3}}; \tag{14}$$

with $W^{-2} = X^{-2} + Y^{-2} + T^{-2}$ and $S = C + D + F$. For convenience reasons, all dissipative parameters, considered as small, are divided by $\delta_0 = |\delta|$: $\varepsilon_0 = \varepsilon/\delta_0$, $\mu_0 = \mu/\delta_0$, and $\beta_0 = \beta/\delta_0$. The

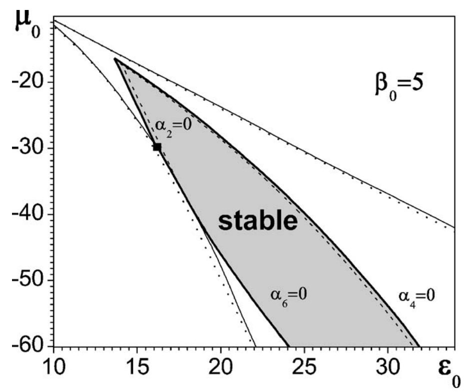

FIG. 1. Stability domain of A^- solutions computed exactly (full curves) and up to δ_0 (dashed curves).

exact steady-state solutions are obtained from Eqs. (7)–(13) for z derivatives of amplitude, widths, and curvatures equal zero. The only possible steady-state solutions are symmetric with equal widths $X = Y = T$ and curvatures $C = D = F$. In the dissipative case, the power $P = A^2 XYT$ is no longer a constant [6,7]. However, in steady state the power $P = 21.76A^{-1}(1.38 - \nu A^2)^{-3/2}$, the width $X = 2.79A^{-1}(1.38 - \nu A^2)^{-1/2}$, and the propagation constant $\Psi' = 0.09A^2$ depend, up to δ_0, only on the amplitude as in the conservative case [6,9]. Variationally obtained families of conservative steady-state solutions reduces to a fixed double solution for a given set of dissipative parameters. Indeed, the steady-state amplitude has two discrete values A^+ and A^-,

$$A^{\pm} = 1.17 \sqrt{\frac{\gamma \pm \sqrt{\gamma^2 + 98.53(\mu_0 + \beta_0 \nu)}}{6(\mu_0 + \beta_0 \nu)}} \tag{15}$$

where $\gamma = 3\beta_0 - 4\varepsilon_0$. A double solution ($A^- > A^+$) exists for a cubic gain ($\varepsilon_0 > 0$) and a quintic loss ($\mu_0 < 0$) in the (ε_0, μ_0) domain between the dotted parabola $\gamma^2 + 98.53(\mu_0 + \beta_0 \nu) = 0$ and straight line $A^- = 1.175$ in Fig. 1. The striking difference from conservative systems is the nonzero wave front curvature $C = 0.032A^2[1.38(\varepsilon_0 - \beta_0) + A^2(\mu_0 + \nu\beta_0)]\delta_0$ [6,9].

To be a soliton a steady-state solution must be stable. In order to check the stability of solutions our stability criterion based on the variational approach and the method of Lyapunov exponents has to be generalized for non-spherically-symmetric conditions [7]. A Jacobi matrix is built from the derivatives of the right-hand sides of Eqs. (7)–(13) with respect to amplitude, widths, and curvatures taken in a symmetric steady state. The steady-state solutions of the seven coupled FODEs are stable if and only if the real parts of the solutions λ of the equation

$$(\lambda^3 + \alpha_1 \lambda^2 + \alpha_2 \lambda + \alpha_3)(\lambda^2 + \alpha_4 \lambda + \alpha_5)^2 = 0 \tag{16}$$

are nonpositive [10]. That is fulfilled when the Hurwitz conditions are satisfied. The stability criterion for steady-state solutions of the CQGLE is explicitly expressed up to δ_0 as

$$\alpha_2 = 0.07A^4(1.38 - \nu A^2)(4\nu A^2 - 1.38) > 0, \tag{17}$$

$$\alpha_3 = 0.02A^4(1.38 - \nu A^2)^2[(4\varepsilon_0 - 3\beta_0)A^2 - 22.63]\delta_0 > 0, \tag{18}$$

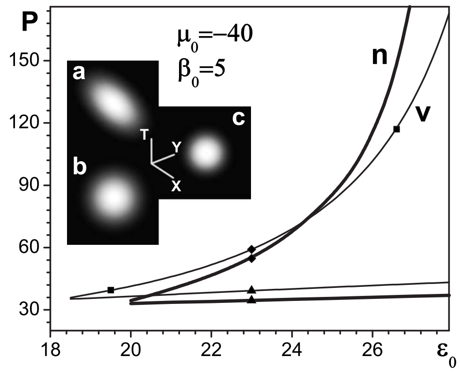

FIG. 2. Upper (between squares) stable (diamonds) and lower unstable (triangles) branches of variational (v) and numerical (n) curves with insets corresponding to asymmetric (a) and symmetric (b) inputs as well as to the soliton (c).

$$\alpha_4 = [0.35(\varepsilon_0 + 2\beta_0)A^2 + 0.29(\mu_0 - 2\nu\beta_0)A^4]\delta_0 > 0, \quad (19)$$

$\alpha_6 = \alpha_1\alpha_2 - \alpha_3 > 0$, with

$$\alpha_1 = (0.06\varepsilon_0 A^2 - 0.77\mu_0 A^4 - 2.67)\delta_0, \quad (20)$$

and $\alpha_5 = O(\delta^2)$. As a consequence, in the (ε_0, μ_0) domain in Fig. 1 only the A^- solution is stable in the shaded region between the curves $\alpha_2=0$ and $\alpha_6=0$ (separated by a square), as well as $\alpha_4=0$. The dashed and dotted curves are obtained from Eqs. (15)–(20) taken up to δ_0. The full curves correspond to the exact solution of the same set of equations solved parametrically. The input pulse chosen in the stable domain of parameters is not yet a light bullet since the variationally obtained v curve in Fig. 2 corresponding to the power P as a function of the parameter ε_0 is only a good approximation of the exact n curve obtained by numerical solution of Eq. (1). Numerical simulations are performed using the Crank-Nicholson integration scheme with a Gauss-Seidel iteration procedure. The integration step is $\Delta z = 0.01$. The number of sampling points is 201 following each transverse dimension. Following Nicolis and Prigogin's theory of dissipative structures and self-organization, the curves in Fig. 2 correspond to bifurcations with the upper stable branch and lower unstable branch controlled by the parameter ε_0 [10]. The analytically predicted domain of stability is exhaustively checked point by point using numerical simulations of Eq. (1); a stable soliton is generated from each point, so that each point can be taken as representative. As a consistent illustration, throughout the paper we use the same representative set of dissipative parameters $\delta_0=0.01$, $\varepsilon_0=23$, $\mu_0=-40$, $\nu=1$, and $\beta_0=5$; the triangle corresponding to A^+ is on the lower unstable branch of the v curve. The diamond associated with the corresponding A^- is on the upper branch, stable between the squares. If this solution is taken as the input in numerical simulations [inset (b) in Fig. 2], it will evolve by shrinking toward the stable dissipative soliton [inset (c)] represented by the diamond on the exact n curve. However, the same final soliton can be obtained starting from an asymmetric, i.e., ellipsoidal input pulse [inset (a)] with the same set of dissipative parameters belonging to the stable domain, as numerical simulations demonstrate [see Figs. 3(a)–3(c)]. An initial pulse with strong asymmetry, having the ratio of spatial and temporal widths $X:Y:T = 1.3:1:0.7$ in Fig. 3(a) [see also inset (a) in Fig. 2], will oscillate, evolving to a spherical light bullet with identical widths [inset (c) in Fig. 2]. In the beginning of the evolution, after a few oscillations [on a larger scale in Fig. 3(b)], the amplitude slightly decreases in order to adjust to the exact perfectly stable soliton solution (on a compressed scale). The power, after transient oscillations, increases, reaching light bullet power, which remains constant [Fig. 3(c)]. Conservation of power amplitude and widths using extremely long simulations (more than $z = 30\,000$) also confirms the stability of our code and the accuracy of our results. Therefore, following our numerical simulations of the CQGLE, an asymmetric input pulse with dissipative parameters from the established stable domain evolves always toward a stable dissipative light bullet [inset (c) in Fig. 2] situated on the n curve. After a short evolution (after $z \approx 260$), a stable dissipative structure is generated corresponding to a dissipative light bullet, which is permanently self-maintained. We checked also the resistance of such solitons to perturbations by increasing the amplitude by 5% for each z from $z=800$ to 2800 (see Fig. 4). The soliton as a self-organized system adapts in order to resist big successive perturbations. Once systematic increase of amplitude is stopped, the solitonic system continues to compensate by inertia so that a sharp decrease appears. Immediately after, however, the stable light bullet is reestablished, demonstrating an astonishing robustness. Therefore, for fixed steady-state solutions, the balance between diffraction, dispersion, and saturating nonlinearity is not realized independently of the gain-loss compensation. The curvature $C=D=F$ is negative in the stable (ε_0, μ_0) domain considered. As a consequence, the self-focusing dominates both diffraction and dispersion; hence, it would increase the amplitude in the

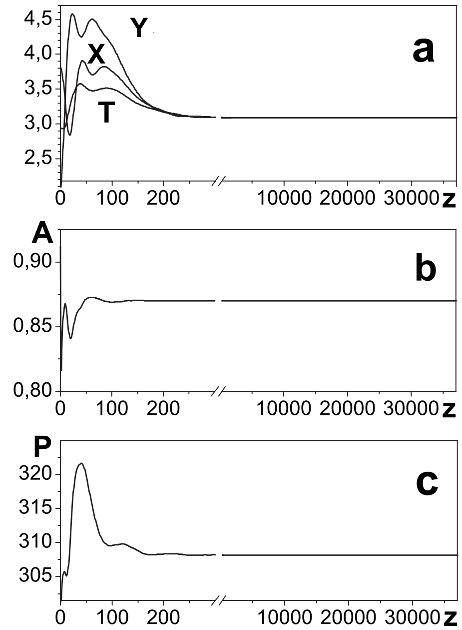

FIG. 3. Numerical evolution of an asymmetric input pulse toward a stable dissipative soliton.

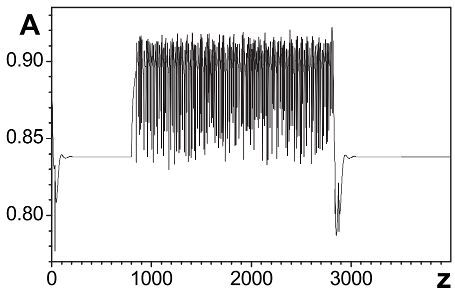

FIG. 4. Light bullet resisting 5% increase of the amplitude for each z from $z=800$ to 2800.

absence of loss and gain terms. Detailed analysis of the numerical contribution of each term in the CQGLE [Eq. (1)] confirms the cross compensation. Indeed, the self-focusing excess resulting from the sum of all real terms on the left-hand side of the CQGLE is exactly compensated by the real part of the parabolic terms on the right-hand side [Eq. (2)]. The cross compensation is achieved by matching of all imaginary terms from the right-hand side with the imaginary parts of the diffraction and dispersion terms. Therefore, in symmetric as well as asymmetric conditions, cross compensation appears as a very efficient mechanism of stabilization through synergy of self-focusing, loss, and gain.

In conclusion, the $(3+1)$-dimensional CQGLE is treated for a non-spherically-symmetric input, using a joint numerical and analytical approach based on extension of the variational method. A stability criterion based on FODEs without spherical symmetry is established in order to select stable steady-state solutions from the domain of dissipative parameters obtained analytically through exact parametric resolution. The stability of the analytically predicted domain is confirmed point by point using numerical simulations. Other (ε_0, μ_0) domains for different sets of parameters δ_0, β_0, and ν are also tested and will be the subject of a forthcoming paper, including different parameters β_0 for each transverse coordinate in the parabolic gain term. Following our numerical simulations, each asymmetric input pulse with dissipative parameters chosen from the stability domain determined by that criterion always evolves in such a way as to generate a stable dissipative light bullet. The limitation of the variation approach to a set of trial functions without the possibility of reshaping does not affect the generality of the analytical stability criterion. For each set of dissipative parameters from the proposed stability domain, a stable steady state corresponding to the approximate solution of the CQGLE taken as input will evolve, attracted by the fixed exact solution in order to self-organize into a dissipative light bullet. It is worthwhile to stress that even very asymmetric input pulses (as in Fig. 3), for the same dissipative parameters from our domain, which are far from stable spherically symmetric steady states, always self-organize into soliton. The analytically obtained stable steady states are in the domain of attraction of the exact solution. As a consequence, bullets are very robust, resisting successive increase of amplitude during evolution. The opportunity to treat analytically and numerically asymmetrical input pulses propagating toward necessarily stable and robust dissipative light bullets opens possibilities for diverse practical applications, including experiments.

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