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Two-dimensional Laguerre-Gaussian soliton family in strongly nonlocal nonlinear media

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We have studied Laguerre-Gaussian spatial solitary waves in strongly nonlocal nonlinear media analytically and numerically. An exact analytical solution of two-dimensional self-similar waves is obtained. Furthermore, a family of different spatial solitary waves has been found. It is interesting that the spatial soliton profile and its width remain unchanged with increasing propagation distance. The theoretical predictions may give new insights into low-energetic spatial soliton transmission with high fidelity.

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The interest in properties of self-similar waves in complex nonlinear optical systems has grown greatly during recent years [1–6]. Although self-similar solutions have been extensively studied in many fields such as plasma physics and nuclear physics [7,8], surprisingly, they have only lately attracted the attention of the nonlinear optics community, and relatively few optical self-similar phenomena have been investigated to date [1–6]. In particular, exact one-dimensional self-similar solitary waves have been found in optical fibers, whose dispersion, nonlinearity, and gain profile are allowed to change with the propagation distance, but the functional forms of these parameters cannot be chosen independently [1–5].

The nonlocal spatial solitons are modeled by the nonlocal nonlinear Schrödinger equation (NNLSE) [9–11]. In general, the nonlinear term has the nonlocal form associated with a symmetric and real-valued response kernel. Moreover, the NNLSE also describes several other physical situations in the literature [12–15]. Snyder and Mitchell [9] simplified the NNLSE to a linear model in the strongly nonlocal case, and their work was highly appreciated by Shen [16]. So far, more properties of solitons have been described by the NNLSE, and the related phenomena have theoretically been clarified [17–19]. The experimental observations [20,21] can be interpreted in the framework of nonlocal nonlinearity [22]. It has been demonstrated that the localized wave packets in cubic nonlinear materials with a symmetric nonlocal nonlinear response of arbitrary shape and degree of nonlocality can be described by a general NNLSE, and the nonlocality of the nonlinearity prevents collapse in optical Kerr media in all physical dimensions, resulting in stable solitary waves under the proper conditions [13–15,18,19].

In this Rapid Communication, we study the propagation of a beam in two-dimensional nonlocal nonlinear media. We show that there exists a class of $LG_{nm}(n,m=0,1,2,\ldots)$ soliton waves, propagating in a self-similar manner. It is found that the predicted self-similar waves can be regarded as a family of LG_{nm} spatial solitons.

In the strongly nonlocal limit [9–11], the wave equation governing beam propagation in two-dimensional nonlocal nonlinear media can be written as [13,17]

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\nabla_{\perp}^{2}u-sr^{2}u+ig(z)u=0, \tag{1}$$

where s is a normalized unit corresponding to the beam in the transverse plane, and $g(z)$ in the final term is the loss coefficient, depending on the propagation distance z. The second term of Eq. (1) represents diffraction, while the third term accounts for nonlinear effects. $\nabla_{\perp}^{2}=\partial^2/\partial r^2+(1/r)\partial/\partial r+(1/r^2)\partial^2/\partial\varphi^2$ is the transverse Laplacian operator in polar coordinates. Here, φ is the azimuthal angle and we consider only $s>0$.

Following Refs. [1–6], the complex field is defined as $u(z,r,\varphi)=A(z,r,\varphi)e^{iB(z,r)}$, where $A(z,r,\varphi)$ and $B(z,r)$ are real functions with $r=\sqrt{x^2+y^2}$. Substituting $u(z,r,\varphi)$ into Eq. (1), and requiring that the real and imaginary parts of each term be separately equal to zero, we obtain the following coupled equations:

$$-\frac{\partial B}{\partial z}+\frac{1}{2}\left[\frac{1}{A}\frac{\partial^2 A}{\partial r^2}-\left(\frac{\partial B}{\partial r}\right)^2+\frac{1}{rA}\frac{\partial A}{\partial r}+\frac{1}{r^2A}\frac{\partial^2 A}{\partial\varphi^2}\right]-sr^2=0, \tag{2}$$

$$\frac{1}{A}\frac{\partial A}{\partial z}+\frac{1}{2}\left(\frac{2}{A}\frac{\partial B}{\partial r}\frac{\partial A}{\partial r}+\frac{\partial^2 B}{\partial r^2}+\frac{1}{r}\frac{\partial B}{\partial r}\right)+g=0. \tag{3}$$

To search for a self-similar solution in Eqs. (2) and (3), we introduce a set of transformations [6]

$$A(z,r,\varphi)=\frac{kP(z)\phi(\varphi)}{w(z)}F(\theta), \tag{4a}$$

$$B(z,r)=a(z)+b(z)r+c(z)r^2, \tag{4b}$$

where $w(z)$ is the beam width, $P(z)=P_0\exp[-\int_0^z g(z')dz']$ is the power of the beam, $\theta(z,r)$ is a self-similar variable, $a(z)$ is the phase offset, $b(z)$ is the frequency shift, and $c(z)$ represents the wave front curvature [6]. These variables are all allowed to vary with propagation distance z. Inserting Eqs. (4a) and (4b) into Eq. (3), and making the coefficient of each power of r equal to zero, we obtain $\theta(z,r)=r^2/w^2$, $b(z)=0$, and $c(z)=(1/2w)dw(z)/dz$. By using Eqs. (4) a nonlinear differential equation for $F(\theta)$ is readily derived from Eq. (2),

$$-\frac{Fw^2}{2}\frac{da}{dz}+\frac{dF}{d\theta}+\theta\left[\frac{d^2F}{d\theta^2}+\frac{Fw^2}{2}\left(-\frac{w}{2}\frac{d^2w}{dz^2}-sw^2\right)\right]+\frac{F}{4\theta\phi}\frac{d^2\phi}{d\varphi^2}=0. \quad (5)$$

From Eq. (5), we can obtain

$$-\frac{1}{\phi}\frac{d^2\phi}{d\varphi^2}=m^2, \quad (6)$$

$$-\frac{Fw^2}{2}\frac{da}{dz}+\frac{dF}{d\theta}+\theta\left[\frac{d^2F}{d\theta^2}+\frac{Fw^2}{2}\left(-\frac{w}{2}\frac{d^2w}{dz^2}-sw^2\right)\right]-\frac{m^2F}{4\theta}=0, \quad (7)$$

where $\phi=\cos(m\varphi)+iq\sin(m\varphi)$. The integer m ($=0,1,2,\ldots$) denotes the quantum number of ϕ and the parameter q ($0\leqslant q\leqslant 1$) is the modulation depth of the beam intensity. We note that in the limit $q\to 1$, ϕ becomes radially symmetric. For $q\to 0$, ϕ describes a multipole soliton [22]. In order to simplify Eq. (7), we further introduce another transformation $F(\theta)=\theta^{m/2}e^{-\theta/2}V(\theta)$, we have

$$-\frac{w}{2}\frac{d^2w}{dz^2}-sw^2+\frac{1}{2w^2}=0, \quad (8a)$$

$$-\frac{w^2}{2}\frac{da}{dz}-\frac{1+m}{2}=n. \quad (8b)$$

The combination of Eqs. (8a) and (8b) with Eq. (7) results in

$$\theta\frac{d^2V}{d\theta^2}+(1+m-\theta)\frac{dV}{d\theta}+nV=0, \quad (9)$$

where n ($=0,1,2,\ldots$) is called the radial direction quantum number. Equation (9) is the generalized Laguerre differential equation and its solution $L_n^{(m)}(\theta)$ is the generalized Laguerre polynomial. The amplitude of self-similar waves is given by

$$A_{nm}(z,r,\varphi)=\frac{kP(z)}{w(z)}[\cos(m\varphi)+iq\sin(m\varphi)]\times\left(\frac{r}{w}\right)^m L_n^{(m)}\left(\frac{r^2}{w^2}\right)e^{-r^2/2w^2}, \quad (10)$$

where $k=[n!/\Gamma(n+m+1)]^{1/2}$ and $L_n^{(m)}(\theta)=[\Gamma(m+1+n)/n!\Gamma(m+1)]F(-n,m+1,\theta)$. Here $F(-n,m+1,\theta)$ is the Kummer function. Now we solve Eq. (8a). For simplicity, Eq. (8a) is reexpressed as

$$\frac{1}{2}\left(\frac{dy}{dz}\right)^2+\frac{2sw_0^4(y^2-1)(y^2-\lambda)}{y^2}=0, \quad (11)$$

where $y=\frac{w}{w_0}$ and $\lambda=\frac{1}{2sw_0^4}$. Taking $y(z)|_{z=0}=1$ and $dy(z)/dz|_{z=0}=0$, integrating Eq. (11) yields

$$w^2=w_0^2[\cos^2(2\sqrt{s}w_0^2z)+\lambda\sin^2(2\sqrt{s}w_0^2z)]. \quad (12)$$

It can be seen from Eq. (12) that beam diffraction initially overcomes beam-induced refraction when $\lambda>1$, and the beam initially expands, with $\frac{w^2}{w_0^2}$ vibrating between a maximum λ and a minimum 1; whereas, when $\lambda<1$, the reverse happens and the beam initially contracts, with $\frac{w^2}{w_0^2}$ breathing between a maximum 1 and a minimum λ. When $\lambda=1$, diffraction is exactly balanced by nonlinearity. In this way, the generalized Laguerre-Gaussian beam preserves its width as it travels in a straight path along the z axis [9]. Furthermore, by using (8b) and (12), the phase offset and the wave front curvature of the beam can be obtained readily:

$$a(z)=a_0-\frac{(2n+m+1)\arctan[\sqrt{\lambda}\tan(2\sqrt{s}w_0^2z)]}{2\sqrt{s\lambda}w_0^4}, \quad (13)$$

$$c(z)=\frac{\sqrt{s}w_0^2(\lambda-1)\sin(4\sqrt{s}w_0^2z)}{1+\lambda-(\lambda-1)\cos(4\sqrt{s}w_0^2z)}. \quad (14)$$

Using Eqs. (4a) and (4b), we get the exact self-similar solution of Eq. (1),

$$u_{nm}(z,r,\varphi)=\frac{kP(z)}{w(z)}[\cos(m\varphi)+iq\sin(m\varphi)]\times\left(\frac{r}{w}\right)^m L_n^{(m)}\left(\frac{r^2}{w^2}\right)e^{-r^2/2w^2+ia(z)+ic(z)r^2}, \quad (15)$$

where $w(z)$, $a(z)$, and $c(z)$ are determined by Eqs. (12)–(14).

It has been noted that the beam in Eq. (15) is influenced by the degrees (n,m) of the generalized Laguerre polynomials s, the angle distribution, the beam width, and the loss. It should be pointed out that the beam width is determined by s and w_0. The energy of the beam $E=\int_{-\infty}^{\infty}|u(z,r,\varphi)|^2dr\,d\varphi=p^2(1+q^2)\leqslant 2p^2$ is calculated according to the orthogonal relation of $L_n^{(m)}(\theta)$. As seen, the energy of the solitons in general is less than that of the sec h soliton and has nothing to do with the degrees (n,m) of the generalized Laguerre polynomials. Since Eq. (1) is a linear wave equation, the highly nonlinear effect of collapse cannot occur [13–15,18,19]. Hence, by constructing the appropriate system parameters and initial values, the higher-order LG_{nm} beam can transmit in the same way as the lower one in the media regardless of any other conditions.

Figure 1 shows a comparison of the analytical solution with numerical simulation for the initial LG_{11} beam with different λ parameters ($w_0=1$, $p_0=1$, $g=0$, $q=0$). Figure 1(a) displays the self-similar evolution of an initial LG_{11} beam $u(0,r,\varphi)=[P_0r\cos(\varphi)/\sqrt{2}w_0^2](2-r^2/w_0^2)\exp(-r^2/2w_0^2)$ propagating in lossless media.

It is clearly seen that the beam changes from four speckles at the initial position to two speckles at $2\sqrt{s}w_0^2z=\frac{\pi}{2}$. That is, the beam contracts as it propagates in such lossless media when $\lambda<1$. On the contrary, the beam expands when $\lambda>1$; the number of speckles remains unchanged at four.

Furthermore, we find that when $\lambda=1$ it can be deduced that $w=w_0$, $c=0$, and $a(z)=a_0-(2n+m+1)z/w_0^2$. This is an accessible LG_{nm} soliton [9]. If $g(z)=0$, Eq. (15) is simplified to the LG_{nm} soliton expression

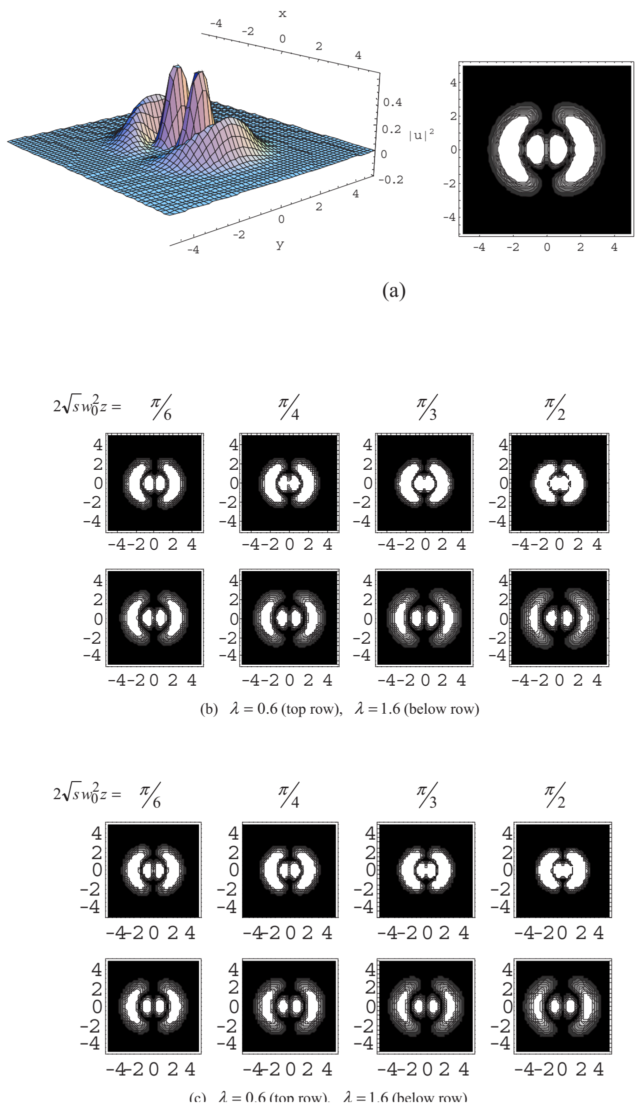

FIG. 1. (Color online) Comparison of analytical solution with numerical simulation for the initial LG_{11} beam in lossless media with different parameters λ. (a) Initial intensity distribution; (b) analytical solution of Eq. (15) for $\lambda = 0.6$ (top) and 1.6 (bottom), and (c) numerical simulation of Eq. (1) for $\lambda = 0.6$ (top) and 1.6 (bottom).

$$u_{nm}^s(z,r,\varphi) = \frac{kP_0}{w_0}[\cos(m\varphi) + iq\sin(m\varphi)] \times \left(\frac{r}{w_0}\right)^m L_n^{(m)}\left(\frac{r^2}{w_0^2}\right) e^{-r^2/2w_0^2 + ia(z)}. \qquad (16)$$

A spatial soliton of this kind exists in strongly nonlocal media and is often called a strongly nonlocal optical spatial soliton [9,13]. It is interesting that this spatial soliton with any width can propagate in the media as long as $2sw_0^4$ equals exactly 1 (namely, $\lambda = 1$). It should be noted that s is a parameter determined by the material properties and w_0.

We find that the spatial solitons in Eq. (16) are determined by two parameters n and m. For a fixed number n and different m (or a fixed number m and different n), the LG_{nm} solitons form a family, having some common characteristics.

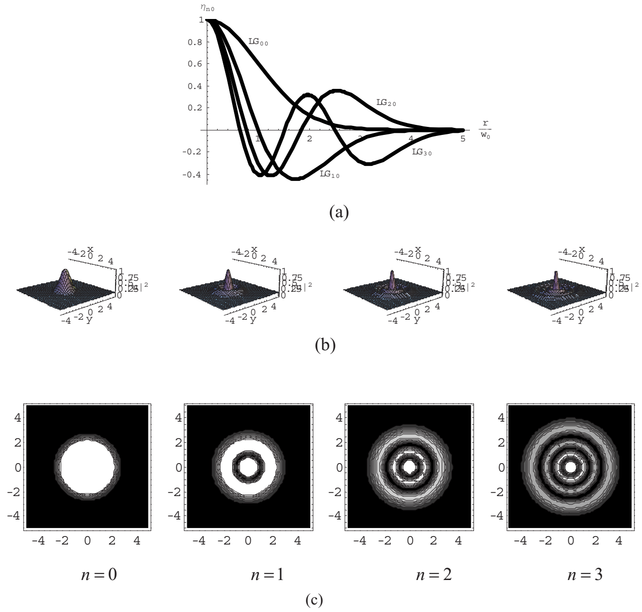

FIG. 2. (Color online) (a) Amplitude comparison of radial distribution of LG_{0n} soliton, corresponding to LG_{00}, LG_{10}, LG_{20}, LG_{30} from top to bottom. (b) Optical field distribution of LG_{0n} soliton with $w_0 = 1$ and $P_0 = 1$. (c) Intensity distribution of LG_{0n} soliton with $w_0 = 1$ and $P_0 = 1$.

Now, we turn to discuss the distribution of the amplitude, the intensity, and the positions of the zero point ($\eta_{nm} = 0$) and extreme point ($d\eta_{nm}/dr = 0$) of LG_{nm} solitons, where $\eta_{nm} = \phi(r/w_0)^m L_n^{(m)}(r^2/w_0^2)e^{-r^2/2w_0^2}$.

Figure 2(a) is a plot of the radial distribution of the low-order LG_{n0} solitons for different n values, but $m = 0$. Strikingly, there exist n zero points and $n+1$ extreme points along with the radial direction. Figures 2(b) and 2(c) show the distribution of optical field and intensity of LG_{n0} ($n = 0,1,2,3$) solitons. Their maximum optical intensity is located at the center of the propagation axis.

Physically, when $m = 0$, it indicates that the nonlinear polarization of the medium has the symmetry of the electric field due to an additional strongly nonlocal condition [13]; the distribution of the optical field and intensity is obviously irrelevant for the azimuthal angle distribution [15,18,19].

Figure 3 shows some properties of the LG_{n2} soliton for $m > 0$ and $q = 0$. We see that η_{n2} has $n+1$ zero points and $n+1$ extreme points along the radial direction. Meanwhile, there exist $2m$ zero points and $2m$ extreme points along the azimuthal angle distribution. In contrast with the situation when $m = 0$, the optical intensity is zero at the center of the propagation axis.

As seen from Figs. 2(a) and 3(a), the LG_{nm} soliton family is an evanescent field on the decline. The physical origin of the phenomenon can qualitatively be understood from the "nonlocality." Nonlinear nonlocality means that the nonlinear polarization of a medium with a small volume of radius r_0 ($r_0 \ll$ any wavelength involved) depends not only on the value of the electric field inside this volume (at the present time and in the past), but also on the electric field outside the volume under consideration [13–15,18,19]. The stronger the

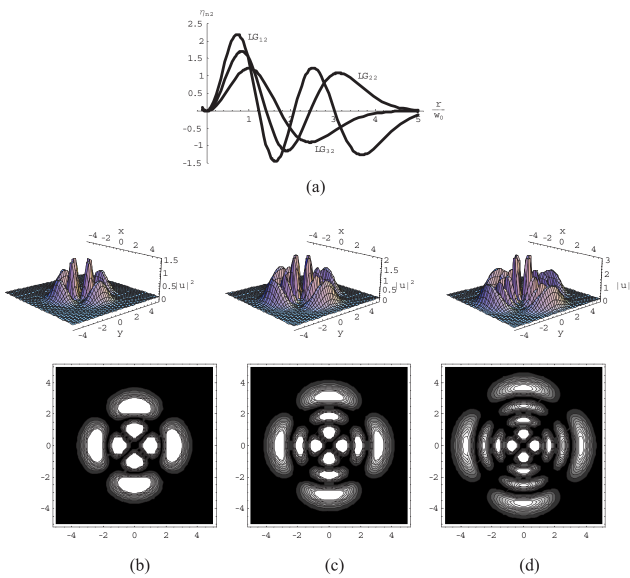

FIG. 3. (Color online) (a) Comparison of amplitude distribution of LG_{n2} soliton, corresponding to LG_{12}, LG_{22}, LG_{32} from bottom to top. (b), (c), (d) Optical field and intensity distribution of LG_{12}, LG_{22}, and LG_{32} solitons with $w_0=1$ and $P_0=1$, respectively.

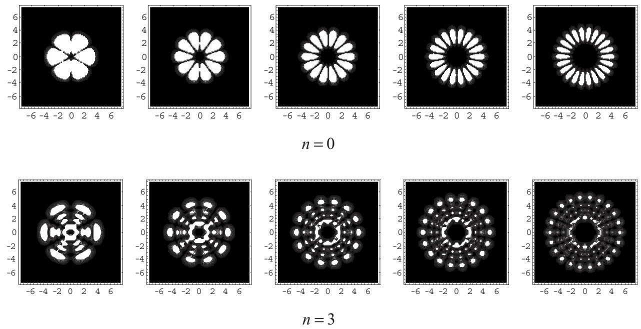

FIG. 4. Variation of speckle number with m for the LG_{nm} ($n=0,3$) soliton family, where the quantum number is taken into account for $m=3,5,7,9,11$ from left to right.

nonlocality, the more fields are involved in contributing to the polarization; hence a larger evanescent field is obtained.

We further study a multipole soliton ($q=0$) when $n=0,3$ with different m. For the LG_{00} soliton, the amplitude η_{nm} has no zero point along the radial direction (see Fig. 2). Other solitons have only one zero point located at $r=\sqrt{m}w_0$. The larger m, the larger r is. Figure 4 shows the intensity distributions with different m when $n=0,3$. We see that the distributions change regularly with azimuthal angle distribution. When m is large enough, the speckles form an optical ring (necklace solitons) [22].

These features can be understood simply and explained easily. In a strongly nonlocal nonlinear medium, the refractive index is determined by the intensity distribution over the entire transverse plane, and under proper conditions the nonlocality can lead to an increase of refractive index in the overlap region r_0 ($r_0 \ll$ any wavelength involved), giving rise to the formation of multipole solitons. Note that, when the nonlocal response function is much wider than the beam itself [14,15,18,19], so that the width of the refractive index distribution greatly exceeds the width of an individual light spot, it makes a very large range of nonlocality in this medium [13].

In summary, the self-similar waves of the Laguerre-Gaussian spatial soliton family in strongly nonlocal nonlinear media has been studied analytically and numerically. An exact analytical solution of two-dimensional self-similar waves has been obtained and an additional family of spatial solitons has been found. It should be pointed out that the nonlinear polarization of the medium has the symmetry of the electric field due to additional strong nonlocality; the distribution of optical field and intensity is obviously irrelevant to the azimuthal angle distribution. The stronger nonlocality, the more fields are involved in contributing to the polarization; hence a larger light field on the decline can be obtained. In a strongly nonlocal nonlinear medium and under proper conditions, the nonlocality leads to an increase of refractive index in the overlap region, giving rise to the formation of multipole solitons. It is expected that the present findings may give new insight into low-energetic spatial soliton transmission with high fidelity.

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