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Strong superadditivity conjecture holds for the quantum depolarizing channel in any dimension

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Given a quantum channel Φ in a Hilbert space H, set $\hat{H}_{\Phi}(\rho)=\min_{\rho_{av}=\rho}\sum_{j=1}^{k}\pi_j S(\Phi(\rho_j))$, where $\rho_{av}=\sum_{j=1}^{k}\pi_j\rho_j$, the minimum is taken over all probability distributions $\pi=\{\pi_j\}$ and states ρ_j in H, and $S(\rho)=-\mathrm{Tr}\rho\log\rho$ is the von Neumann entropy of a state ρ. The strong superadditivity conjecture states that $\hat{H}_{\Phi\otimes\Psi}(\rho)\geqslant\hat{H}_{\Phi}(\mathrm{Tr}_K(\rho))+\hat{H}_{\Psi}(\mathrm{Tr}_H(\rho))$ for two channels Φ and Ψ in Hilbert spaces H and K, respectively. We have proved the strong superadditivity conjecture for the quantum depolarizing channel in any dimensions.

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I. INTRODUCTION

A linear trace-preserving map Φ on the set of states (positive unit-trace operators) $\mathcal{G}(H)$ in a Hilbert space H is said to be a quantum channel if Φ^* is completely positive [1]. The channel Φ is called bistochastic if $\Phi(\frac{1}{d}I_H)=\frac{1}{d}I_H$. Here and in the following we denote by d and I_H the dimension of H, $\dim H=d<+\infty$, and the identity operator in H, respectively.

Given a quantum channel Φ in a Hilbert space H, set [2]

$$\hat{H}_{\Phi}(\rho)=\min_{\rho_{av}=\rho}\sum_{j=1}^{k}\pi_j S(\Phi(\rho_j)), \tag{1}$$

where $\rho_{av}=\sum_{j=1}^{k}\pi_j\rho_j$ and the minimum is taken over all probability distributions $\pi=\{\pi_j\}$ and states $\rho_j\in\mathcal{G}(H)$. Here and in the following $S(\rho)=-\mathrm{Tr}(\rho\log\rho)$ is the von Neumann entropy of a state ρ. The strong superadditivity conjecture states that

$$\hat{H}_{\Phi\otimes\Psi}(\rho)\geqslant\hat{H}_{\Phi}(\mathrm{Tr}_K(\rho))+\hat{H}_{\Psi}(\mathrm{Tr}_H(\rho)), \tag{2}$$

and $\rho\in\mathcal{G}(H\otimes K)$ for two channels Φ and Ψ in Hilbert spaces H and K, respectively.

The infimum of the output entropy of a quantum channel Φ is defined by the formula

$$S_{min}(\Phi)=\inf_{\rho\in\mathcal{G}(H)}S(\Phi(\rho)). \tag{3}$$

The additivity conjecture for the quantity $S_{min}(\Phi)$ states [3]

$$S_{min}(\Phi\otimes\Psi)=S_{min}(\Phi)+S_{min}(\Psi) \tag{4}$$

for an arbitrary quantum channel Ψ. It was shown in [2] that, if the strong superadditivity conjecture holds, then the additivity conjecture for the quantity S_{min} holds too. Nevertheless the conjecture (2) is stronger than (3).

In the present paper we shall prove the strong superadditivity conjecture for the quantum depolarizing channel for all dimensions of H.

II. THE ESTIMATION OF THE OUTPUT ENTROPY

Our approach is based upon the estimate of the output entropy proved in [4]. Combining formulas (111) and (112) in [4] we get the lemma formulated below.

Lemma. Let $\Phi_{dep}(\rho)=(1-p)\rho+(p/d)I_H$, $\rho\in\mathcal{G}(H)$, $0\leqslant p\leqslant d^2/(d^2-1)$, be the quantum depolarizing channel in the Hilbert space H of the dimension d. Then, for any quantum channel Ψ there exist, an orthonormal basis $\{e_s,\ 1\leqslant s\leqslant d\}$ in H and d states, $\rho_s\in\mathcal{G}(K)$, $1\leqslant s\leqslant d$, such that

$$S((\Phi_{dep}\otimes\Psi)(\rho))\geqslant-\left(1-\frac{d-1}{d}p\right)\log\left(1-\frac{d-1}{d}p\right)-\frac{d-1}{d}p\log\frac{p}{d}+\frac{1}{d}\sum_{s=1}^{d}S(\Psi(\rho_s)) \tag{5}$$

and

$$\frac{1}{d}\sum_{s=1}^{d}\rho_s=\mathrm{Tr}_H(\rho),$$

where $\rho\in\mathcal{G}(H\otimes K)$, $\rho_s=d\mathrm{Tr}_H[(|e_s\rangle\langle e_s|\otimes I_K)\rho]\in\mathcal{G}(K)$, $1\leqslant s\leqslant d$.

In the present paper our goal is to prove the following theorem.

Theorem. Let Φ_{dep} be the quantum depolarizing channel in the Hilbert space of the dimension d. Then, for an arbitrary quantum channel Ψ in a Hilbert space K the strong superadditivity conjecture holds, i.e.,

$$\hat{H}_{\Phi_{dep}\otimes\Psi}(\rho)\geqslant\hat{H}_{\Phi_{dep}}(\mathrm{Tr}_K(\rho))+\hat{H}_{\Psi}(\mathrm{Tr}_H(\rho)). \tag{6}$$

Proof. Suppose that

$$\rho=\sum_{j=1}^{k}\pi_j\rho_j \tag{7}$$

and the states ρ_j, $1\leqslant j\leqslant k$, form the optimal ensemble for (1) in the sense that

$$\hat{H}_{\Phi_{dep}\otimes\Psi}(\rho)=\sum_j\pi_j S((\Phi_{dep}\otimes\Psi)(\rho_j)). \tag{8}$$

Applying (5) to each element of the sum in (8), we get

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$$\hat{H}_{\Phi_{dep}\otimes\Psi}(\rho) \geqslant -\left(1-\frac{d-1}{d}p\right)\log\left(1-\frac{d-1}{d}p\right) - \frac{d-1}{d}p\log\frac{p}{d} + \frac{1}{d}\sum_{j=1}^{k}\pi_j\sum_{s=1}^{d}S(\Psi(\rho_{js})), \tag{9}$$

where $\rho_{js}=d\mathrm{Tr}_H[(|e_{js}\rangle\langle e_{js}|\otimes I_K)\rho_j]\in\mathcal{G}(K)$, $1\leqslant j\leqslant d$, and each set $\{e_{js}, 1\leqslant s\leqslant d\}$ forms the orthonormal basis of H for $1\leqslant j\leqslant k$.

It follows from the lemma that

$$\frac{1}{d}\sum_{j=1}^{k}\pi_j\sum_{s=1}^{d}\Psi(\rho_{js}) = \sum_{j=1}^{k}\pi_j\Psi(\mathrm{Tr}_H(\rho_j)) = \Psi(\mathrm{Tr}_H(\rho)). \tag{10}$$

The equality (10) results in

$$\frac{1}{d}\sum_{j=1}^{k}\pi_j\sum_{s=1}^{d}S(\Psi(\rho_{js})) \geqslant \hat{H}_{\Psi}(\mathrm{Tr}_H(\rho)). \tag{11}$$

Notice that the quantity (1) is always bounded from below by the quantity (3). For the quantum depolarizing channel Φ_{dep} (1) coincides with (3) for any state because (3) is achieved on any pure input state due to the covariance property of Φ_{dep}. Thus, we get

$$\hat{H}_{\Phi_{dep}}(\rho) = -\left(1-\frac{d-1}{d}p\right)\log\left(1-\frac{d-1}{d}p\right) - \frac{d-1}{d}p\log\frac{p}{d} = S_{min}(\Phi_{dep}) \tag{12}$$

for any state $\rho\in\mathcal{G}(H)$. Taking into account (9), (11), and (12) we get

$$\hat{H}_{\Phi_{dep}\otimes\Psi}(\rho) \geqslant \hat{H}_{\Phi_{dep}}(\mathrm{Tr}_K(\rho)) + \hat{H}_{\Psi}(\mathrm{Tr}_H(\rho)),$$

$\rho\in\mathcal{G}(H\otimes K)$. Thus, the strong superadditivity conjecture for the quantum depolarizing channel is proved. ■

III. CONCLUSION

At the first time the additivity conjecture (4) for the quantum depolarizing channel was proved for the first time in [4]. The method was based upon the estimation of l_p-norms of the channel. On the other hand, in the papers [5–7] it was shown that the decreasing property of the relative entropy can also be used to prove the additivity conjecture for some partial cases at least [8–13]. In the present paper we have proved that the estimation of the output entropy obtained in [4] allows a proof of the strong superadditivity conjecture (2) for the quantum depolarizing channel. One possible basis for considering the strong superadditivity conjecture can be drawn from the paper [2]. There was presented a proof of the global equivalence of the additivity conjecture for constrained channels and the strong superadditivity conjecture.

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