One-step implementation of an *N*-qubit controlled-phase gate with neutral atoms trapped in an optical cavity

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We present a one-step scheme for direct implementation of an *N*-qubit controlled-phase gate with atoms trapped in a high-*Q* optical cavity without resorting to a sequence of single- and two-qubit gates. The interaction time that is required to implement the scheme does not rise with increasing number of qubits. This might lead to more efficient construction of quantum circuits and quantum algorithms.

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The recent development of quantum-information processing has shed new light on complexity and communication theory. The existence of quantum algorithms for specific problems shows that a quantum computer can in principle provide a tremendous speedup compared to a classical computer [1,2]. This fact has triggered in recent years a lot of studies on the theoretical and practical aspects of quantum computation. It is known that the two-qubit controlled-phase gate and the one-qubit gate are universal in constructing a quantum computer. In other words, any multiqubit gate can be decomposed into these elementary gates [3]. On the other hand, quantum algorithms [1,2] and quantum error-correction protocols [4] require multiqubit gates, but the procedure of decomposing multiqubit gates into the elementary gates becomes complicated as the number of qubits increases [2,5,6]. Therefore, the direct implementation of multiqubit phase gates is of importance for reducing the complexity of the physical realization of practical quantum computation and quantum algorithms.

During the past few years there have been tremendous advances in experiments realizing cavity quantum electrodynamics (QED), which open new prospects in the implementation of large-scale quantum computation and generation of nonclassical states with the atoms trapped in an optical cavity. Experimentally, a sequence of single photons has been observed on demand from a single atom strongly coupled to a high-finesse optical cavity [7,8], and single cesium atoms have been cooled and trapped inside a small optical cavity in the strong-coupling regime [9,10]. Theoretically, some schemes have been proposed for realizing two-qubit gates [11–14], which cannot be directly extended to implement an *N*-qubit gate. More recently, several cavity QED proposals have been presented for implementation of multiqubit controlled-phase gates [15–17]. In Refs. [15], schemes were proposed for implementing *N*-qubit phase gates which required a single-photon source and its injection into an optical cavity. Based on a dispersive interaction, Gabris *et al.* proposed a scheme for realizing a three-qubit phase gate, which cannot be generalized to the *N*-qubit case [16]. Based on resonant interaction, several schemes have been proposed for implementing multiqubit phase gates [17]. However, these schemes require that one can tune the atom-cavity coherent coupling strength, and it is difficult to realize experimentally since the atom-cavity coupling depends on the atomic position.

In this paper, we propose a scheme to implement an *N*-qubit phase gate with *N* three-level atoms trapped in a high-*Q* optical cavity. Similar to other schemes for implementing two- and multiqubit phase gates, the present scheme also requires the individual addressing of the trapped atoms. However, the motivation of our scheme is at least twofold. First, the present scheme no longer needs to control the atom-cavity coupling between different atoms and cavity modes [17] nor the single-photon injection into an optical cavity [15], so that it is easier to realize in experiment. Second, the present scheme realizes an *N*-qubit phase gate in a one-step operation, and the interaction time required to implement the scheme does not change with increasing number *N* of qubits.

To build the basic model, we consider that *N* identical atoms are trapped in a high-finesse optical cavity as shown in Fig. 1(a). The atoms are separated by at least one optical wavelength so that single laser pulses can address each atom individually. This requirement has also been met by other

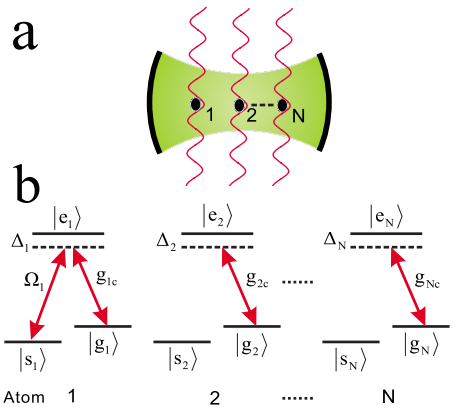

FIG. 1. (Color online) (a) Schematics of *N* identical atoms interacting with a single high-*Q* cavity mode. Single neutral atoms are delivered into the optical cavity and controlled by using *N* corresponding translating one-dimensional optical lattices [19]. (b) Relevant energy level structure of the atoms. The atomic transition $|e_i\rangle \leftrightarrow |g_i\rangle$ is nonresonantly coupled to the cavity mode *a* with the coupling strength g_{ic}, while the atomic transition $|e_1\rangle \leftrightarrow |s_1\rangle$ is driven by a classical laser field with Rabi frequency Ω_1.

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schemes for implementing two- and multiqubit phase gates [11–14,16,17]. The energy level configuration of the atoms is depicted in Fig. 1(b). It includes two stable ground states $|g\rangle$, $|s\rangle$, and an excited state $|e\rangle$. The atomic transition from the ground state $|g\rangle$ to the excited state $|e\rangle$ is coupled to a single cavity mode with frequency ω_c. To implement an N-qubit phase gate, a classical laser pulse with the Rabi frequency Ω_1 is used to individually drive the atomic transition $|s_1\rangle \leftrightarrow |e_1\rangle$ of the atom 1. In the interaction picture, the Hamiltonian describing the system is given by

$$H = \sum_{j=1}^{N} H_j \tag{1}$$

with $H_1 = \Delta_1 |e_1\rangle\langle e_1| + (g_{1c}|e_1\rangle\langle g_1|a + \Omega_1|e_1\rangle\langle s_1| + \text{H.c.})$ and $H_{j\neq 1} = \Delta_j |e_j\rangle\langle e_j| + (g_{jc}|e_j\rangle\langle g_j|a + \text{H.c.})$, where the subscript j represents the jth atom, and a is the annihilation operator associated with the quantized cavity mode; g_{ic} describes the coherent coupling strength between the atom i and the cavity mode; $\Delta_i \equiv \omega_{ei} - \omega_{gi} - \omega_c$ represents the atom-field detuning associated with the corresponding atomic transition $|e\rangle_i \leftrightarrow |g\rangle_i$, which can be controlled by introducing an auxiliary classical field to the individually addressed atom 1; H.c. stands for the Hermitian conjugate.

In this paper, we consider the large-detuning limit of the atom 1, i.e., $|\Delta_1| \gg |\Omega_1|, |\Delta_2|, \ldots, |\Delta_N|, |g_{1c}|, \ldots, |g_{Nc}|$. In this case, neglecting the effect of rapidly oscillating terms, we can eliminate adiabatically the level $|e_1\rangle$ and then obtain an effective Hamiltonian [20]

$$\begin{aligned} H_{\text{eff}} = & -\frac{g_{1c}^2}{\Delta_1} a^\dagger a |g_1\rangle\langle g_1| - \frac{\Omega_1^2}{\Delta_1} |s_1\rangle\langle s_1| \\ & - \left(\frac{g_{1c}\Omega_1}{\Delta_1} a |s_1\rangle\langle g_1| + \text{H.c.} \right) + \sum_{j=2}^{N} [\Delta_j |e_j\rangle\langle e_j| + (g_{jc} a |e_j\rangle \\ & \times \langle g_j| + \text{H.c.})]. \end{aligned} \tag{2}$$

Here the first two terms represent the dynamical energy shifts of the atomic levels $|g_1\rangle$ and $|s_1\rangle$. The third term describes the effective atomic transition between the states $|s_1\rangle$ and $|g_1\rangle$. The last term depicts the dynamics of the atom j $(j=2,3,\ldots,N)$.

Based on the effective Hamiltonian, we now turn to the problem of implementation of the N-qubit conditional phase gate U_p of the form $U_p |\otimes_{j=1}^N |\psi\rangle_j = e^{i\pi \otimes_{j=1}^N |s_j\rangle\langle s_j| \otimes_{j=1}^N} \otimes_{j=1}^N |\psi_j$ where $|\psi\rangle_j = \alpha_j |s_j\rangle + \beta_j |g_j\rangle$ denotes the arbitrary initial state of the atom j. It demonstrates that the phase changes by π if all atoms are in the state $|s\rangle$ and does not change otherwise.

Initially, we assume that the cavity field is in the vacuum state $|0\rangle_c$. It is obvious that, if the atom 1 is initially in the state $|g_1\rangle$ and all other atoms are in the states $|g_j\rangle$ or $|s_j\rangle$, the atom-cavity system does not experience any dynamical evolution, i.e., the system state $|g_1\rangle \otimes_{j=2}^N |i_j\rangle |0\rangle_c$ does not evolve with the time since $H_{\text{eff}} |g_1\rangle \otimes_{j=2}^N |i_j\rangle |0\rangle_c = 0$ where $i_j = g_j, s_j$. Therefore, we only need consider the temporal evolution of the initial system state with the form of $|s_1\rangle \otimes_{j=2}^N |i_j\rangle |0\rangle_c$.

First we consider the temporal evolution of the initial system state with the form $|a\rangle = |s_1\rangle |\cdots g_{i_1} \cdots g_{i_2} \cdots g_{i_m} \cdots\rangle |0\rangle_c$, where the qubits $i_1, i_2, \ldots, i_m$ $(1 \leq m \leq N-1)$ are initially in the ground state $|g\rangle$, and other states are in the ground state $|s\rangle$, i.e., at least one qubit initially occupies the ground state $|g\rangle$. For simplicity, we choose that the coupling strengths satisfy the condition $g_{2c} = g_{3c} = \cdots = g_{Nc}$. Then, the system state $|a\rangle$ interacts with a system state $|b\rangle = |g_1\rangle |\cdots g_{i_1} \cdots g_{i_2} \cdots g_{i_m} \cdots\rangle |1\rangle_c$ with an effective coupling strength $\Omega_{\text{eff}} = -g_{1c}\Omega_1/\Delta_1$. Then the system state $|b\rangle$ can be coupled to another system state $|c\rangle$ which has the form

$$\begin{aligned} |c\rangle = (1/\sqrt{m}) |g_1\rangle (& |\cdots e_{i_1} \cdots g_{i_2} \cdots g_{i_m} \cdots\rangle \\ & + |\cdots g_{i_1} \cdots e_{i_2} \cdots g_{i_m} \cdots\rangle + \cdots \\ & + |\cdots g_{i_1} \cdots g_{i_2} \cdots e_{i_m} \cdots\rangle) |0\rangle_c, \end{aligned}$$

with a coupling strength $\sqrt{m} g_{2c}$. In addition, we choose the tunable parameters Ω_1 and $\Delta_2, \ldots, \Delta_N$ to satisfy the following conditions

$$\Omega_1 = g_{1c}, \tag{3a}$$

$$\Delta_2 = \Delta_3 = \cdots = \Delta_N = -g_{1c}^2/\Delta_1. \tag{3b}$$

Equations (3a) and (3b) result in the resonant transitions $|a\rangle \leftrightarrow |b\rangle$ and $|b\rangle \leftrightarrow |c\rangle$, respectively. Therefore, the temporal evolution of the system state is then obtained, within the state subspace $\{|a\rangle, |b\rangle, |c\rangle\}$,

$$\begin{aligned} |\psi(t)\rangle_1 = & \frac{\Omega_{\text{eff}}^2 \cos(\Lambda t) + m g_{2c}^2}{\Lambda^2} |a\rangle - i \frac{\Omega_{\text{eff}}}{\Lambda} \sin(\Lambda t) |b\rangle \\ & + \frac{\Omega_{\text{eff}} \sqrt{m} g_{2c} [\cos(\Lambda t) - 1]}{\Lambda^2} |c\rangle, \end{aligned} \tag{4}$$

where $\Lambda = \sqrt{\Omega_{\text{eff}}^2 + m g_{2c}^2}$.

Second, as a special case, if the system is initially prepared in the state $\otimes_{j=1}^N |s_j\rangle |0\rangle_c \equiv |s_1, s_2, \ldots, s_N\rangle |0\rangle_c$, i.e., all atoms are initially prepared in the ground state $|s\rangle$, it interacts only with the state $|g_1\rangle \otimes_{j=2}^N |s_j\rangle |1\rangle_c$ with the coupling rate Ω_{eff}. The temporal evolution of this state is then obtained as

$$|\psi(t)\rangle_2 = \cos(\Omega_{\text{eff}} t) \otimes_{j=1}^N |s_j\rangle |0\rangle_c - i \sin(\Omega_{\text{eff}} t) |g_1\rangle \otimes_{j=2}^N |s_j\rangle |1\rangle_c. \tag{5}$$

With the choice of the interaction time $t = \tau = \pi/|\Omega_{\text{eff}}|$, and in the weak-excitation limit $(|g_{2c}| \gg |\Omega_{\text{eff}}|)$, $|\psi(\tau)\rangle_1$ can be approximately reduced to the initial state $|a\rangle$, and the system does not acquire any phase shift, while the state $|\psi(\tau)\rangle_2$ acquires a phase shift $e^{i\pi}$. Thus we can implement the N-qubit conditional z gate U_p. In the present scheme, only the state when all atoms are in the state $|s\rangle$ experiences a dynamic evolution in the weak-excitation limit, and other states do not evolve with time. This mechanics is different from that proposed in [18], in which a highly controlled and selective atom-field interaction is realized by adjusting the photon-number-dependent Stark shift. Although such selective interaction can be used to create nontrivial Dicke states, it is not clear how to extend such scheme to realize a quantum phase gate.

In order to validate the feasibility of the above theoretical analysis, we perform a direct numerical simulation of the

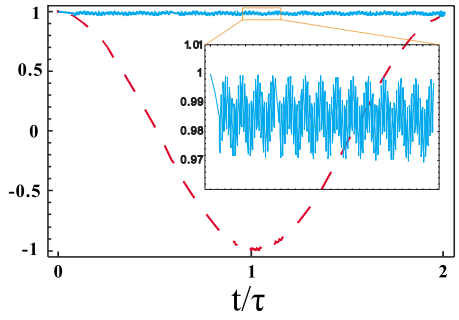

FIG. 2. (Color online) Real parts of the coefficients of the system states $|a\rangle$ (solid line) and $\otimes_{j=1}^{N}|s_j\rangle|0\rangle_c$ (dashed line). The inset shows the expanded drawing of a small part of the solid line. Other parameters: $N=3$, $m=2$, $\Omega_1=g_{1c}=g_{2c}$, $\Delta_1=10g_{1c}$, $\Delta_2=-g_{1c}^2/\Delta_1$, $\tau=\pi/|\Omega_{\text{eff}}|$, $\Omega_{\text{eff}}=-g_{1c}\Omega_1/\Delta_1$.

Schrödinger equation with the nonapproximate Hamiltonian H in the absence of system decoherence mechanisms. For this, without loss of generality we consider that there are three atoms trapped in the cavity ($N=3$) and choose the parameters $\Omega_1=g_{1c}=g_{2c}$, $\Delta_1=10g_{1c}$, $\Delta_2=-g_{1c}^2/\Delta_1$, so $\Omega_{\text{eff}}=-g_{1c}/10\ll g_{2c}$. In the following simulation, we calculate the temporal evolution of the system with two distinct initial states $|s_1,g_2,g_3\rangle|0\rangle_c$ and $|s_1,s_2,s_3\rangle|0\rangle_c$. As shown in Fig. 2, the solid and dashed lines describe the real parts of the coefficients of the basic states $|s_1,g_2,g_3\rangle|0\rangle_c$ and $|s_1,s_2,s_3\rangle|0\rangle_c$, respectively. It is obvious that the system returns to its initial state but obtains a global phase shift π at the time $t=\tau=\pi/|\Omega_{\text{eff}}|$ when all atoms initially occupy the ground state $|s\rangle$, while it is almost unchanged for the initial state $|s_1,g_2,g_3\rangle|0\rangle_c$.

We now discuss the effect of decoherence and the mismatch of the parameters on the scheme. The dissipation path of the N-qubit operations includes the atomic spontaneous emission (at the rate γ_s) and the cavity decay (at the rate κ). First, the spontaneous decay of the atom 1 can be strongly suppressed due to the large atom-field detuning $|\Delta_1|\gg|\Omega_1|,|g_{1c}|$, so that it does not play a remarkable role in the dynamical evolution. Second, with the appropriate parameter choice, the weak-excitation limit $|g_{2c}|\gg|\Omega_{\text{eff}}|$ is satisfied, and the system is approximately restricted in its initial state $|a\rangle$. Thus, both the atomic spontaneous emission and the cavity decay play negligible roles since neither the atoms nor the cavity field can be excited effectively. Third, when the system is initially prepared in the state $\otimes_{j=1}^{N}|s_j\rangle|0\rangle_c$, it evolves in a two-dimensional subspace $\{\otimes_{j=1}^{N}|s_j\rangle|0\rangle_c,|g_1\rangle\otimes_{j=2}^{N}|s_j\rangle|1\rangle_c\}$ which can be simplified to $\{|s_1\rangle|0\rangle_c,|g_1\rangle|1\rangle_c\}$ since the atoms $2,\ldots,N$ do not take part in the dynamical evolution. With this understanding, we easily know that the spontaneous emission of the atoms still plays a minor role, but the cavity decay should play a dominant role in the whole dissipation. We confirm this point by numerically calculating the evolution of the system density matrix governed by the Hamiltonian H. The quality of an N-qubit controlled-phase gate can be described by a fidelity $F=\langle\varphi(0)|U_p^{\dagger}\rho U_p(t=\tau)|\varphi(0)\rangle$ where $\rho(t)$ represents the temporal reduced density matrix obtained by tracing out the cavity mode part, and $|\varphi(0)\rangle$ denotes the initial atomic state. Here, for simplicity, we choose

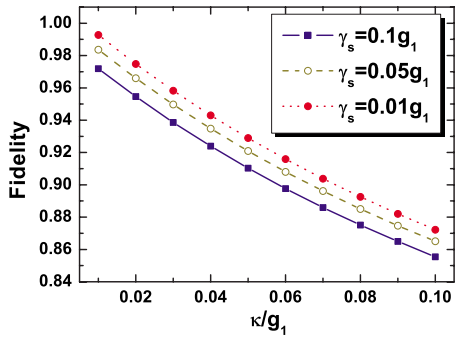

FIG. 3. (Color online) (a) Fidelity (F) of a three-qubit phase gate vs the cavity decay rate κ for the initial atomic state $|\varphi(0)\rangle=\otimes_{j=1}^{3}(|s_j\rangle+|g_j\rangle)/\sqrt{2}$. The solid, dashed, and dotted lines describe the cases of the atomic spontaneous rates $\gamma_s=0.1g_{1c},0.05g_{1c},0.01g_{1c}$, respectively. (b) Fidelity vs δ under $\gamma_s=\kappa=0.01g_{1c}$. Other common parameters: $N=3$, $\Omega_1=g_{1c}=g_{2c}$, $\Delta_1=10g_{1c}$, $\Delta_2=-g_{1c}^2/\Delta_1$, $\tau=\pi/|\Omega_{\text{eff}}|$, $\Omega_{\text{eff}}=-g_{1c}\Omega_1/\Delta_1$.

$|\varphi(0)\rangle=\otimes_{j=1}^{N}(|s_j\rangle+|g_j\rangle)/\sqrt{2}$. In Fig. 3(a), we plot the fidelity (F) of a three-qubit phase gate vs κ and γ. It can be seen that the cavity decay rate κ is the dominant noise source in the gate operation, and the fidelity F rapidly decreases to 0.87 from 0.995 when κ increases from $0.01g_1$ to $0.1g_1$, while there is not much change when the atomic spontaneous emission is taken into an account for a given κ. This is in excellent agreement with the above-mentioned theoretical analysis.

On the other hand, the effective interaction is difficult to set precisely, since atom-cavity coupling Ω_{eff} is dependent on atomic position. If we take the deviation of the coupling into account by the value $\delta\Omega_{\text{eff}}$, in Fig. 3(b), we show the fidelity vs the comparative deviation δ. It can be seen that there is not much change when the deviation is taken into account. It should be noted that we need a very small cavity decay (smaller than the atom-field coherent coupling strength by approximately two orders of magnitude) for the sake of a high gate fidelity, which still cannot fully match the current experimental technique. As a possible implementation using ^{87}Rb, the parameters $(g,\kappa,\gamma_s)/2\pi=(16,1.4,3)$ MHz have been reported [21]. Further improvement of the cavity characteristic for a bigger g/κ can be accomplished, in principle, by adjusting the cavity length L and finesse $\mathcal{F}$ since $g\sim L^{-3/4}$ and $\kappa\sim(\mathcal{F}L)^{-1}$.

Finally, technical difficulties with the present scheme should be pointed out. Our scheme requires the large-detuning limit to the atom 1 ($|\Delta_1|\gg|\Omega_1|,|\Delta_2|,\ldots,|\Delta_N|,|g_{1c}|,\ldots,|g_{Nc}|$) and the weak-excitation limit ($|g_{2c}|\gg|\Omega_{\text{eff}}|=|g_{1c}\Omega_1|/|\Delta_1|$). Both limits are relatively easy to reach by adjusting the cavity mode and the classical laser field. However, it is noted that we have assumed that the atoms $2,\ldots,N$ have the same coherent coupling rate with the single cavity mode in the above analysis. In fact, this assumption is only chosen for a simple and clear deduction, and our scheme can work robustly without the condition $g_2=g_3=\cdots=g_N$, but we also require $|g_{jc}|\gg|\Omega_{\text{eff}}|$ for $j=2,\ldots,N$. The main difficulty of our scheme with respect to an experimental demonstration consists in the requirement for the regime of strong coupling, i.e., the frequency scale

Ω_{eff} associated with reversible evolution of the atom-cavity system exceeds the dissipative rates, $\Omega_{eff} \gg \kappa$. This might be achieved by integrating the techniques of laser cooling and trapping with those of cavity quantum electrodynamics. Single cesium atoms have been trapped within the mode of a small, high-finesse optical cavity in the regime of strong coupling [10]. Another difficulty is the technical capability to individually address multiple atoms that are strongly coupled to a cavity. Although significant experimental advances have been reported in trapping single atoms in high-finesse cavities, no experiment has yet achieved a well-defined number of atoms, each of which is strongly coupled to the cavity mode and individually addressed. To satisfy the requirements of the presented scheme will be experimentally challenging.

In summary, we have considered a simple cavity QED scheme to directly implement an N-qubit controlled-phase gate. The interaction time that is required to implement the gate does not increase with increasing number of qubits. The present scheme does not require a single-photon source or single-photon injection into an optical cavity. Numerical simulations show that the scheme is insensitive to atomic spontaneous emission, and some experimental difficulties are pointed out.

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