Acceleration and localization of matter in a ring trap

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(Received 26 October 2006; revised manuscript received 24 January 2007; published 29 May 2007)

A toroidal trap combined with external time-dependent electric field can be used for implementing different dynamical regimes of matter waves. In particular, we show that dynamical and stochastic acceleration, localization, and implementation of the Kapitza pendulum can be originated by means of proper choice of the external force.

DOI: 10.1103/PhysRevA.75.053614 PACS number(s): 03.75.Lm, 03.75.Kk, 03.75.Ss

I. INTRODUCTION

Exploring different geometries of potentials trapping cold condensed atoms is of both fundamental and practical importance. Toroidal traps play a special role allowing for "infinite" atomic trajectories and for realization of quasi-one-dimensional (quasi-1D) regimes. These advantages are relevant for designing highly precise sensors based on matter wave interferometry [1,2] as well as for accurate study of such phenomena as superfluid currents, stability of sound waves, solitons, and vortices in Bose-Einstein condensates (BECs) [3,4]. Traps with circular geometry are also believed to be conceptually important for implementation of the main ideas of the accelerator physics at ultralow temperatures [2], and, in particular, for acceleration of ultracold atoms [5]. In this last context, existence of well localized wave packets and thus attenuation of the dispersion, the latter being the intrinsic property of a quantum systems, is of primary importance. In the first experimental studies [2] it was shown that the dispersive spreading out [1] can be compensated by using betatron resonances in a storage ring. An alternative way of contrabalancing dispersion is also well known—it is nonlinearity, leading in quasi-1D regime to existence of bright and dark matter solitons (see, e.g. [6,7] and [8–10], respectively). This issue has already been explored [11] from the point of view of acceleration of matter waves in a toroidal trap with help of a modulated optical lattice, which is known to be an efficient tool for acceleration of matter waves [12].

In this paper we propose two alternative ways of accelerating matter wave solitons—either by time varying or by stochastic external electric field. These new ways of soliton acceleration are especially relevant in view of radiative losses [13] and distortions [12] of solitons moving in optical lattices (the effects acquiring significance for long trajectories). At the same time, it turns out that the toroidal geometry of a trap confining a BEC allows one to realize a number of other dynamical regimes, like dynamical localization of solitons and solitonic implementation of the celebrated Kapitza pendulum. Theoretical description of all mentioned phenomena can be observed by using this framework, based on perturbation theory for solitons. This is done in the present paper. More specifically, in Sec. II we formulate the model and the main physical constraints determining its validity. In Secs. III and IV we describe how by applying external time-dependent electric field matter solitons can be accelerated in the usual sense and in the sense of the time increase of the velocity variance (the stochastic acceleration), respectively. In Sec. V we describe localized states of the matter in circular trap subject to external field, and in Sec. VI we show that a matter soliton affected by rapidly varying force represents an example of the Kapitza pendulum [14]. Summary and discussion of the results are given in the Conclusion.

II. SCALING AND THE EVOLUTION EQUATION

We assume that a BEC is loaded in a circular trap, which in cylindrical coordinates $\mathbf{r}=(\rho,\varphi,z)$ is described by $V=V_c(\rho)+m\omega_z^2z^2/2$, where ω_z is the frequency of the magnetic trap in the z direction, $V_c(\rho)$ is the potential in the radial direction, forming the trap circular in the (x,y) plane, and m is the mass of an atom. We also suppose that the BEC is subject to external electric field with amplitude E_0, which produces an additional potential $V_{\rm ext}=-\alpha' E_0^2/4$, where α' is the polarizability of the atoms (see, e.g. [15]). If the amplitude E_0 or direction of the field vary along some direction, say, along the x-axis, smoothly on the scale of the trap radius R, the potential energy $V_{\rm ext}$ can be expanded in the Taylor series; after neglecting the nonessential constant, it can be rewritten in the form $V_{\rm ext}=-\alpha x$, where $\alpha=(\alpha'/4)\,\partial(E_0)^2/\partial x|_{x=0}$ and consideration is restricted only to the first term of the expansion. In order to realize one-dimensional geometry we require torus radius to be much larger than the core radius r_c, which allows us to define a small parameter $\epsilon=r_c/R\ll 1$. In order to introduce quantitative characteristics, we consider the normalized ground state ϕ of the eigenvalue problem

$$-\frac{\hbar^2}{2m}\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}\phi+V_c(\rho)\phi=\varepsilon_r\phi,\quad \int_0^\infty\phi^2\rho\,d\rho=1 \qquad (1)$$

and define $R_1=\int_0^\infty\phi^2\rho^2d\rho$, $R_2=(\int_0^\infty\phi^2d\rho/\rho)^{-1/2}$, and $\lambda=(\int_0^\infty\phi^4\rho\,d\rho)^{-1/2}$. In the case at hand, $\lambda\sim\sqrt{Rr_c}\sim\epsilon^{1/2}R$, and thus, $\lambda\ll R_1\sim R_2\sim R$.

In the present paper we are interested in the dynamics of matter waves in which spatial extension is much less than the trap perimeter, allowing us to treat them similarly to the matter solitons in an infinite one-dimensional trap. This, in particular is the case where the spatial size of the BEC excitations along the trap are of order of λ, which is the well defined parameter and thus convenient for formulating the constraints of the theory. Indeed, now we can estimate the kinetic energy of the longitudinal excitations as $\varepsilon_\parallel = \hbar^2/(2m\lambda^2)$ and require it to be much less than the kinetic energy of the transverse excitations, $\varepsilon_r \sim \hbar^2/(2mr_c^2)$ (for the sake of simplicity, here, we assume that the size of the trap in z direction is of order of the core radius, $a_z = \sqrt{\hbar/m\omega_z} \sim r_c$). Adding the requirement for the energy of the two-body interactions, which is estimated as $|g|n$ (where $g = 4\pi\hbar^2 a_s/m$, a_s is the scattering length, $n \sim N/\mathcal{V}$ is a mean density, N is the total number of atoms and $\mathcal{V}$ is the effective volume occupied by the atoms and estimated as $\mathcal{V} \sim \pi\lambda r_c a_z$), to be of order of $\varepsilon_\parallel$ and to be much less than ε_r (or more precisely, requiring $|g|n/\varepsilon_r \sim \epsilon$), we can neglect in the leading order the transitions between the transverse energy levels [9,10], and employ the multiple scale expansion [6,10] for description of the quasi-one-dimensional evolution of the BEC. We also notice that subject to the assumptions introduced, one has the estimate $N \sim \epsilon^{3/2} R/(8|a_s|)$.

In order to get an insight on practical numbers, let us consider ^{7}Li atoms ($a_s = -2$ nm) in a trap with $R = 100$ μm, $r_c = 5$ μm, and $a_z = 10$ μm. Then $\epsilon = 0.05$, the characteristic size of solitonic excitations is $\lambda \approx 22$ μm and the number of particles is estimated as $N \approx 140$. We emphasize that these estimates indicate only an order of the parameters. Thus, for example, a condensate of $10^2 - 10^3$ lithium atoms satisfy the conditions of the theory.

We will be interested in managing soliton dynamics by means of weak (i.e., not destroying solitons) electromagnetic field varying in time. Respectively, we consider α time-dependent and characterized by the estimate $\alpha \sim \hbar^2/(mR_1\lambda^2)$. Then, starting with the Gross-Pitaevskii equation, in which the external potential in cylindrical coordinates has the form $V_{\rm ext} = -\alpha\rho\cos(\varphi) \approx -\alpha R_1 \cos(\varphi)$, and using the multiple-scale expansion, one ensures that the BEC macroscopic wave function in the leading order allows factorization

$$\Psi = \pi^{-1/4} a_z^{-1/2} e^{-i(\omega_r+\omega_z)t/2} e^{-z^2/2a_z^2} \phi(r)\psi(t,\varphi), \qquad (2)$$

where $\omega_r = 2\varepsilon_r/\hbar$ and $\psi(t,\varphi)$ solves the nonlinear Schrödinger equation, which we write in terms of $A = \sqrt{|g|m/\sqrt{2\pi}\hbar^2 a_z}\psi$, $\zeta = R_2\varphi/\lambda$, and $\tau = \hbar t/m\lambda^2$

$$i\frac{\partial A}{\partial \tau} = -\frac{1}{2}\frac{\partial^2 A}{\partial \zeta^2} - \cos(\kappa\zeta) f(\tau) A + \sigma |A|^2 A, \qquad (3)$$

Here $\sigma = \operatorname{sgn} a_s$, $f(\tau) \equiv mR_1\lambda^2\alpha(t)/\hbar^2$, and $\kappa = \lambda/R_2 \sim \sqrt{\epsilon}$. We choose the scaling in such a way that all terms in Eq. (3) are of the unity order, and in particular $A = \mathcal{O}(1)$. This can be done, taking into account the normalization

$$\int_0^L |A|^2 d\zeta = 2\sqrt{2\pi}\frac{|a_s|N}{\kappa a_z}, \qquad (4)$$

$L = 2\pi/\kappa$, which follows from the normalization condition for the order parameter $\int |\Psi|^2 d^3\mathbf{r} = N$, and considering $N \sim a_z/|a_s|$, which is of order of 10^3, in a typical experimental setting. Equation (3) is subject to periodic boundary conditions $A(\zeta,\tau) = A(\zeta + L, \tau)$.

III. ACCELERATION OF BRIGHT MATTER SOLITONS BY TIME-DEPENDENT EXTERNAL FORCE

First we consider the acceleration, γ, which can be achieved due to the potential V_e properly dependent on time. An order of magnitude of γ can be estimated by taking into account that Eq. (3) makes sense provided that all terms are of the unity order. In the physical units this gives $\gamma \sim \hbar^2/(m^2\lambda^3)$. Then, recalling the above example of the lithium condensate we estimate $\gamma \sim 7$ mm/s^2, which is of order of the acceleration announced in [11]. This, however, does not provide the best estimate in our case because it is based on the 1D model, while lowering dimensionality imposes constraints on the atomic density and, consequently, on the amplitude of the applied force.

To describe the physics of the phenomenon we consider a BEC with a negative scattering length ($\sigma = -1$). Then a "bright soliton" solution of Eq. (3) at $f(\tau) \equiv 0$ (or, more precisely, a periodic solution mimicking a bright soliton in an infinite 1D system) which moves with a constant velocity v_n, can be written down as follows [8]:

$$A_s = e^{-i(\omega(k)+v_n^2/2)\tau + iv_n\zeta}\eta(k)\,\mathrm{dn}[\eta(k)(\zeta - v_n\tau),k]. \qquad (5)$$

Here $\mathrm{dn}(x,k)$ is the Jacobi elliptic function [16], k is the elliptic modulus parametrizing the solution. The frequency and the amplitude are given by $\omega(k) = (k^2/2 - 1)\eta^2(k)$ and $\eta(k) = 2K(k)/L$ [$K(k)$ is the complete elliptic integral of the first kind]. The velocity of the soliton is quantized $v_n = 2\pi n/L$ with n being the integer.

To ensure that the solution A_s satisfies the scaling relations imposed above, we notice that the size of the soliton can be estimated as $\pi/K(k)$ and its smallness implies that k is close to unity. In that case we obtain the estimates

$$1 - k^2 \sim 16\exp(-2\pi/\sqrt{\epsilon})$$

and

$$\mathrm{dn}[\eta(k)(\zeta - v_n\tau),k] \approx 1/\cosh[\eta(k)(\zeta - v_n\tau)].$$

In the limit $k \to 1$, quantization of the velocity does not play a significant role. This was verified numerically. For example, $L = 10$ deviation of the initial velocity from the quantized one produces appreciable effect on dynamics during intervals $\tau \lesssim 100$ only if $k \lesssim 0.99$.

When external force is applied, $f(\tau) \neq 0$, the velocity is no longer preserved, which manifests itself in evolution of the momentum $P = (1/2i)\int_0^L (A_\zeta \bar{A} - A\bar{A}_\zeta)$ (here $\bar{A}$ stands for complex conjugation of A) according to the law:

$$\frac{dP}{d\tau} = - f(\tau)\int_0^L \cos(\kappa\zeta)\frac{\partial|A|^2}{\partial\zeta}d\zeta. \qquad (6)$$

The external field, however, does not affect the norm: $\mathcal{N} = \int_0^L |A|^2\, d\zeta = \text{const}$. It follows from Eq. (5) that in the adiabatic approximation the solution of the perturbed equation (3) can be searched in the form

$$A = e^{-i[\omega(k)+V(\tau)^2/2]\tau + iV(\tau)\zeta}\eta\,\mathrm{dn}(\eta[\zeta - X(\tau)],k), \qquad (7)$$

where $V(\tau)=dX(\tau)/d\tau$ is the time-dependent velocity of the soliton and $X(\tau)$ is the coordinate of the soliton center. Substituting Eq. (7) in Eq. (6) and taking into account the parity of the functions in the integrand as well as the fact that all of them are periodic with the same period L, we obtain the equation for the soliton coordinate

$$\frac{d^2X}{d\tau^2} = - \kappa C(k) f(\tau)\sin(\kappa X). \qquad (8)$$

Here

$$C(k) = \frac{K(k)}{2\pi E(k)}\int_0^{2\pi} \cos(\theta)\mathrm{dn}^2\left(\frac{K(k)}{\pi}\theta,k\right)d\theta \qquad (9)$$

and it is taken into account that $\mathcal{N}=2\eta E(k)$, where $E(k)$ is the complete elliptic integral of the second kind.

Depending on the choice of the function $f(\tau)$, Eq. (8) describes different dynamical regimes. Now we are interested in acceleration which occurs during the rotational movement of the soliton in the trap (i.e., X is a growing function). We illustrate this acceleration using an example of the simplest steplike dependence $f(\tau)$. To this end we assume that initially the soliton is centered at $X(0)=0$ and require $f(\tau)$ to be a constant f_0 for time intervals such that the soliton coordinates $X(\tau) \in I_p$ and to be zero for $X(\tau) \notin I_p$ where the intervals I_p are given by $I_p=[(p+\frac{1}{2})L,(p+1)L]$ with $p=0,1,\ldots$. Then, as it is clear, the acceleration of the soliton, which is given by the right-hand side of Eq. (8), is positive for all times. The above requirement introduces natural splitting of the temporal axis in the set of intervals $T_l=[\tau_l,\tau_{l+1}]$ ($l=0,1,\ldots$), with $\tau_0=0$, such that $f(\tau)=0$ for $\tau \in T_{2p}$ and $f(\tau)=f_0$ for $\tau \in T_{2p+1}$ [here $X(\tau_l)=lL/2$]. Thus our task is to find τ_l. This can be done by taking into account that during each of the "odd" intervals T_{2p+1}, Eq. (8) describes conservative nonlinear oscillator, the solution for which is well-known. During "even" intervals T_{2p}, the motion is free (with a constant velocity) which means that the time T_{2p} necessary for the soliton to cross an interval $[pL,(p+\frac{1}{2})L]$ is

$$T_{2p} = \tau_{2p+1} - \tau_{2p} = L/(2v_{2p}), \qquad (10)$$

where v_{2p} is the velocity in the point pL. During the time interval T_{2p+1}, the soliton has to cross the interval I_p. From this condition we obtain

$$T_{2p+1} = \tau_{2p+2} - \tau_{2p+1} = \frac{\sqrt{2}K[\sqrt{2E_0/(H_{2p+1}+E_0)}]}{\kappa\sqrt{H_{2p+1}+E_0}}, \qquad (11)$$

where $H_{2p+1}=v_{2p+1}^2/2+E_0$ is the energy of the soliton in the point $(p+1/2)L$, $E_0=C(k)f_0$, and $v_{2p+1}=v_{2p}$ is the soliton velocity in the same point. At the end of the interval T_{2p+1} the soliton velocity is given by $v_{2(p+1)}=\sqrt{2(H_{2p+1}+E_0)}$. Thus, one computes that after p rotations, the soliton acquires the velocity $v_{2(p+1)}$, which can be obtained from the recurrent relation $v_{2(p+1)}=\sqrt{v_{2p}^2+4C(k)f_0}$.

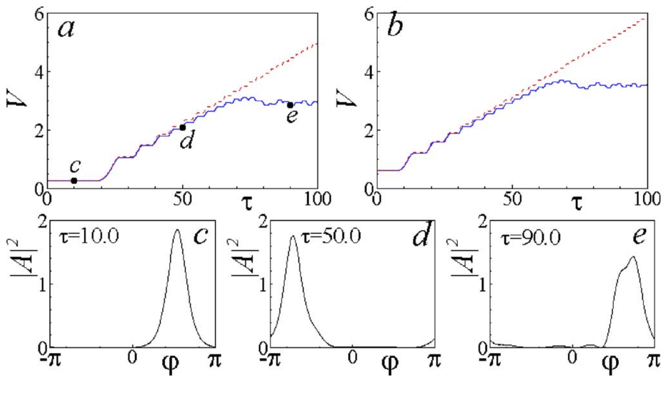

FIG. 1. (Color online) The soliton velocity vs time [(a) and (b)] for the parameters $k=0.99999$, $L=10.0$, $f_0=0.3$, $n=0.43$ [(a), the nonquantized velocity], $n=1.0$ [(b), the quantized velocity], and the forms of the soliton [(c), (d), and (e)] in the instants of time indicated in (a). In (a) and (b) solid and dashed lines represent the velocity numerically computed from Eqs. (3) and (8), respectively.

In Figs. 1(a) and 1(b) we compare the solution obtained from the perturbation theory Eq. (8), with numerical simulation of Eq. (3) for $f_0=0.3$. Nevertheless, during the numerical simulation we used the values for T_{2p} and T_{2p+1} [Eqs. (10) and (11)], obtained for the case of adiabatic approximation. It follows from the results presented that the dashed and solid lines perfectly match until $\tau \approx 50.0$. At larger times appreciable discrepancy appears. It occurs due to failure of the adiabatic approximation and can be removed by introducing temporal corrections to T_{2p} and T_{2p+1}. This naturally leads to an optimization problem, which requires numerical approach and goes beyond the scope of the present work. Finally we notice, that for the above example of ^{7}Li condensate the obtained acceleration is 0.36 mm/s^2.

Comparison of (a) and (b) in Fig. 1 shows that for $k \approx 1$, quantization of the velocity is not important, which is also confirmed by the evolution of the solitonic forms depicted in Figs. 1(c)–1(e).

IV. STOCHASTIC ACCELERATION OF MATTER SOLITONS

Now we concentrate on another dynamical regime—on the *stochastic acceleration*—where increase of the velocity of a matter soliton in a toroidal trap is achieved by applying a fluctuating external field. To this end, holding all conditions of the applicability of the model (3), we consider the case of a stochastic force $f(\tau)$, which is a delta-correlated Gaussian process with characteristics $\langle f(\tau)\rangle=0$ and $\langle f(\tau)f(\tau')\rangle=D\delta(\tau-\tau')$ (the angular brackets stand for the stochastic averaging and D is the dispersion). Now the dynamics can be described in terms of the distribution function

$$\mathcal{P}(V,\Phi,\tau) = \langle \delta[\Phi - \Phi(\tau)]\delta[V - V(\tau)]\rangle, \qquad (12)$$

where $\Phi(\tau) \equiv \kappa X$ is the angular coordinate of the soliton, $\Phi(\tau)$ and $V(\tau)$ with explicit time dependence stand for the soliton coordinates obtained from the dynamical equations, while Φ and V are considered as independent variables. The distribution function solves the Fokker-Planck equation, which is obtained by the standard procedure (see e.g., [17])

$$\frac{\partial \mathcal{P}}{\partial \tau} = - V\frac{\partial \mathcal{P}}{\partial \Phi} + \tilde{D}\sin^2(\Phi)\frac{\partial^2 \mathcal{P}}{\partial V^2}. \tag{13}$$

Here $\tilde{D}=\kappa^4 C^2(k) D$ is the diffusion coefficient. Due to the circular geometry of the trap, Eq. (13) is considered on the interval $-\pi<\Phi<\pi$ with the periodic boundary conditions $\mathcal{P}(V,\Phi-\pi,\tau)=\mathcal{P}(V,\Phi+\pi,\tau)$, with respect to Φ, and zero boundary conditions with respect to V:$\mathcal{P}\to 0$ as $V\to\pm\infty$. The normalization condition for the probability density reads $\int_{-\infty}^{\infty} dV \int_{-\pi}^{\pi} d\Phi\, \mathcal{P}=1$.

Multiplying Eq. (13) by V and Φ, and integrating over V and Φ, one readily obtains that the average velocity and angular position of the soliton are constants, which for the sake of simplicity will be considered zeros, i.e., $\langle V\rangle=0$ and $\langle\Phi\rangle=0$. Next, multiplying Eq. (13) by V^2, Φ^2, and $V\Phi$, and performing the integration, one obtains the equations of the second momenta. They are not closed and can be written down as follows

$$\frac{d}{d\tau}\langle V^2\rangle = 2\tilde{D}\langle \sin^2\Phi\rangle, \tag{14}$$

$$\frac{d}{d\tau}\langle \Phi^2\rangle = 2\langle V\Phi\rangle, \tag{15}$$

$$\frac{d}{d\tau}\langle V\Phi\rangle = -2\pi\int_{-\infty}^{\infty} P(\pi,V,\tau)V^2\, dV + \langle V^2\rangle. \tag{16}$$

Equation (14) means that the average square velocity is growing with time, i.e., the soliton undergoes the stochastic acceleration. The law of the growth of the velocity invariance deviates form the linear, as it would happen for the Brownian diffusion in the momentum space, which happens because the diffusion coefficient in the Fokker-Planck equation (13) is not a constant, but depends on the angular variable. However, due to the diffusion one can expect that the phase distribution will tend to be homogeneous, i.e., that $\mathcal{P}\to 1/(2\pi)$ as $\tau\to\infty$. In this formal limit one obtains that $\langle V\Phi\rangle\to 0$, $\langle\sin^2\Phi\rangle\to 1/2$, and hence $\langle V^2\rangle\to\tilde{D}\tau$. In other words, the system (14)–(16), describes a random walk, which in the limit of large time, approaches the Brownian diffusion in the velocity space. In that limit the stochastic acceleration, which can be defined as $\tilde{\gamma}=d\sqrt{\langle V^2\rangle}/d\tau$, would tend to zero according to the law $\tilde{\gamma}\propto\tau^{-1/2}$.

In order to check the above predictions and reveal other features of the stochastic dynamics of a soliton in a ring trap, we solved Eq. (13) numerically, subject to the initial condition $\mathcal{P}(V,\Phi,0)=\langle\delta(\Phi)\delta(V)\rangle$. The results is summarized in Fig. 2. In Fig. 2(a) one observes the predicted monotonic growth of the mean velocity with time, which is slightly different from the linear law. In Fig. 2(b) one can see that the stochastic acceleration $\tilde{\gamma}$ is a monotonically decreasing function, which at sufficiently large times tends to zero. In particular, at $\tau\gtrsim 15$, decreasing of the acceleration with time can be well approximated by the predicted law $\tilde{\gamma}\propto\tau^{-1/2}$, shown by dashed curves in Fig. 2(b) (it was verified that $\langle\sin^2\Phi\rangle\approx 1/2$, which is in agreement with the analytical predictions). Also, Fig. 2(b) shows that the stochastic acceleration is larger for larger D. The physical explanation of this last fact is simple: The acceleration is generated by the stochastic force, whose intensity is determined by the dispersion D.

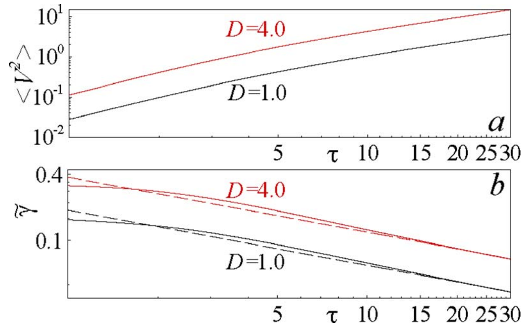

FIG. 2. (Color online) (a) The mean square velocity and (b) the stochastic acceleration $\tilde{\gamma}$ of the soliton vs time for different values of the dispersion, obtained by numerical integration of Eq. (13) with parameters $L=10.0$ and $k=0.99999$. In (b) dashed lines depict the approximation of numerical data by the law $\tilde{\gamma}\propto\tau^{-1/2}$. All axes in (a) and (b) are represented in logarithmic scale.

V. LOCALIZATION OF MATTER INDUCED BY THE EXTERNAL FIELD

Let us now turn to localized states of matter in a toroidal trap and concentrate on the states generated by the constant external electric field, i.e., by $f(\tau)\equiv f_0$. Respectively, we look for stationary solutions of Eq. (3) in the form $A=e^{-i\omega\tau}\mathcal{A}(\zeta)$ and obtain for $\mathcal{A}(\zeta)$ the equation

$$-\frac{1}{2}\frac{d^2\mathcal{A}}{d\zeta^2} - f_0\cos(\kappa\zeta)\mathcal{A} + \sigma|\mathcal{A}|^2\mathcal{A} = \omega\mathcal{A}, \tag{17}$$

which is subject to periodic boundary conditions $\mathcal{A}(\zeta,\tau)=\mathcal{A}(\zeta+L,\tau)$.

Several lowest branches of the numerically obtained solutions of Eq. (17) are shown in Fig. 3. The lowest branch approaches zero at the frequency $\omega_0\approx -0.143$ [it is interesting to mention that this frequency coincides with the lowest

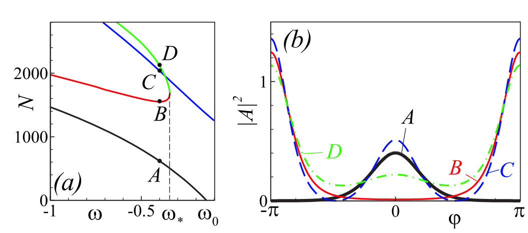

FIG. 3. (Color online) The number of bosons N (for the example of lithium condensate described in the text) vs (a) frequency ω and (b) shapes of the localized modes at $\omega=-0.4$ for the case where $L=10.0$, $f_0=0.3$, and $\sigma=-1$.

gap edge of the spectrum of the Mathieu equation (17) considered on the whole axis], where the amplitude of the nonlinear periodic mode is small and it transforms into the linear periodic Bloch mode at the lowest gap edge. Such behavior of the branch is similar to that of the strongly localized modes in a BEC in the optical lattice [18]. The lowest mode A is localized in the vicinity of $\varphi=0$ [Fig. 3(b)], i.e., around the minimum of the effective potential, and that is why such a mode is stable and can exist even in the linear case, where the two-body interactions are negligible (here it is important that we are dealing with periodic boundary conditions). The modes of the upper branches—B and D [their typical examples are shown in Fig. 3(b)] —bifurcate at $\omega_*\approx -0.345$. They are localized at $\varphi=\pm\pi$, i.e., near the points where the potential has maxima. As is clear, this is a pure nonlinear effect and occurs due to delicate balance between the attractive interactions and repulsive forces of the external potential. Such balance can easily be destroyed even by an infinitesimal perturbation, which allows us to expect instability of the modes. The mode C represents two local maxima of the atomic distributions at $\varphi=0,\pi$. Similar to the modes B and D, one can expect it to be unstable, which can be explained by the existence of local atomic maxima at the maxima of the potential. By direct numerical solution of Eq. (3) [more specifically, by perturbing the mode profiles by the factor $1+0.1\cos(21\zeta)$ and computing the dynamics until $\tau=1000$] we have verified that, indeed, only the mode A on Fig. 3 is dynamically stable, while the modes B, C, and D are unstable.

VI. MATTER SOLITON AS A KAPITZA PENDULUM

As the final example of nontrivial dynamics of a matter soliton in a toroidal trap we consider dynamical localization induced by a rapidly oscillating force $f(\tau)=f_0[\nu+\cos(\Omega\tau)]$. In this case the solitonic motion mimics the famous Kapitza pendulum, which acquires an additional stable point due to rapid oscillation of the pivot [14]. Assuming that the physical conditions of the validity of the quasi-1D approximation (3) hold and that the frequency Ω is large enough, i.e., $\Omega^2 \gg \kappa^2 C(k) f_0$, one can perform the standard analysis (see, e.g. [14]), i.e., look for a solution of Eq. (3) in a form $X(\tau)+\xi(\tau)$ where ξ is small, $|\xi|\ll|X|$, and rapidly varying, and provide averaging over rapid oscillations. One then arrives at equation $d^2X/d\tau^2=-\partial U/\partial X$ with the effective potential

$$U=-C(k)f_0\left[\nu\cos(\kappa X)+\kappa^2 C(k)\frac{f_0}{8\Omega^2}\cos(2\kappa X)\right]. \tag{18}$$

If condition $\kappa^2 C(k) f_0/(2\Omega^2\nu)>1$ is met, the effective potential U possesses two stable points: $X=0$ ($\Phi=0$) and $X=L/2$ ($\Phi=\pi$). Thus, it opens the possibility for the new type of soliton moving around the new stable point. Two typical trajectories of the soliton, obtained by numerical integration of Eq. (3), are presented in Fig. 4. One of the trajectories displays oscillations around the new equilibrium point, while the other one shows the large oscillations around the equilibrium point $\Phi=2\pi$ started with the same initial data, but in the case where $\Phi=\pi$, is no longer in equilibrium. The amplitude of large oscillations decay with time because of energy losses of the soliton in the nonconservative system.

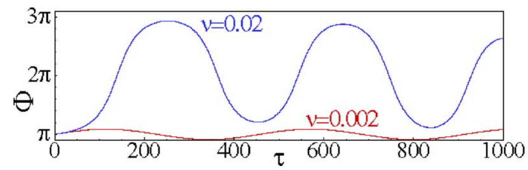

FIG. 4. (Color online) The angular coordinate of the soliton center vs time for the soliton motion affected by the rapidly oscillating external force, obtained numerically from Eq. (3) with parameters $L=10.0$, $n=0.01$, $\sigma=-1$, $f_0=0.15$, $\Omega=2.0$, and $k=0.99999$.

VII. CONCLUSIONS

In the present paper we have shown that dynamics of a matter soliton in a toroidal trap, reproducing one-dimensional geometry, can be efficiently governed by time varying external electric field. In particular, such regimes like dynamical acceleration, stochastic acceleration, localization and implementation of the Kapitza pendulum can be realized by proper choices of the time dependence of the external force.

Experimental detection of the acceleration can be implemented either by direct imaging of the atomic cloud, which is well localized in space and has well specified trajectory, or by measurement of the atomic distribution in the momentum space displaying shift of the maximum towards higher kinetic energies. Alternatively, one can study the evolution of the atomic cloud releasing from the trap (by switching the trap off) after some period of accelerating motion. The respective dynamics will be a spreading out cloud whose center of mass is moving with the acquired velocity.

The obtained results were based on the one-dimensional model, although deduced using the multiple-scale method and thus mathematically controllable. This means that a number of problems are still left unsolved. One problem is the limitation on the soliton velocity, and thus acceleration, introduced by lowering the space dimension, which appears when the solitonic kinetic energy becomes comparable with the transverse kinetic energy. Another limitation on the soliton acceleration emerges from velocity quantization when the radius of the ring trap is not large enough. We also left for further studies the diversity of localized stationary atomic distributions supported by the external filed, indicating only the lowest modes. We thus believe that the richness of the phenomena which can be observed by simple combination of the trap geometry and varying external field will stimulate new experimental and theoretical studies.

ACKNOWLEDGMENTS

The authors are indebted to H. Michinel for providing additional data on Ref. [11]. Y.V.B. was supported by the FCT grant SFRH/PD/20292/2004. V.V.K. was supported by the Secretaria de Stado de Universidades e Investigacion (Spain) under the grant SAB2005-0195. The work was supported by the FCT (Portugal) and European program FEDER under the grant POCI/FIS/56237/2004.

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