Collective modes in a uniform Fermi gas with Feshbach resonances

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The collective modes in a uniform fermionic atomic gas with Feshbach resonance are investigated with the path integral method in the frame of a fermion-boson model Hamiltonian. We mainly concentrated on the long-wavelength and low-frequency limits at $T=0$ K and got an analytical expression for the collective modes across the whole BCS-Bose-Einstein condensate (BEC) crossover. We completely recover the Anderson-Bogoliubov modes in the BCS limit and the Bogoliubov modes of the bosonic systems in the BEC limit. The numerical results show that there exists a continuous interpolation for sound velocity between BCS and BEC limits.

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I. INTRODUCTION

Since Eagles [1] and Leggett [2] independently extended the BCS theory to get a continuous picture of evolution from weak to strong coupling, the crossover from BCS to Bose-Einstein condensate (BEC) has attracted considerable attention [3–13]. This crossover leads to a particular interpretation of a fascinating, but not well-understood phase, known as pseudogap state and the short coherence length. In 1995, Randeria *et al.* [14] raised the interesting possibility that the crossover physics might be relevant to a high temperature superconductor. Importantly, a number of experimental results [15,16] have claimed evidence in support of the BCS-BEC crossover picture for high temperature superconductor materials.

In the system of the condensed matter, it is difficult to strongly modify the strength of the pairing interaction, which gives a challenge to realize this process in the experiment. Fortunately we can use gas consisting of ultracold fermionic atoms to complete this process. To make all fermionic atoms to a high degree of quantum degeneracy, low enough temperature must be satisfied. The advent of the laser opened the way to the development of powerful new methods for manipulating and cooling atoms, but only laser cooling is not enough, which must be followed by the so-called evaporative cooling [17,18]. In such a fermionic system, it should be possible to adjust the interaction strength to tune the system continuously between BCS-type superfluidity and Bose-Einstein condensation. This phenomenon, which is referred to as Feshbach resonance, was first investigated in the context of nuclear physics [19]. Feshbach resonance has become an important tool in investigations of the basic physics of cold atoms. Experimentally, long-lived molecules have been produced [20–22] from ultracold fermionic atoms near the Feshbach resonance, where a molecule state is resonant with the atomic state and molecules can form without heat release. In most of the experiments, molecules formed by sweeping an external magnetic field through Feshbach resonance [21,22]. In the same year, BEC of long-lived molecules was observed by using the standard techniques such as time-of-flight expansion images [4,23,24] and three-body recombination [6]; but demonstrating condensation of fermionic pairs on the BCS side of resonance presents a significant challenge. Observation of fermionic pairs is insufficient to demonstrate condensation and rather a probe of the momentum distribution is required. One year later, condensation of fermionic pairs [7] was discovered by introducing a technique that takes advantage of the Feshbach resonance to pairwise project the fermionic atoms onto molecules. To a certain extent, most of the fundamental predictions of the crossover between BCS and BEC have been realized in the experiment.

In this paper, we calculate the Goldstone modes in the fermionic atomic gas. The existence of Goldstone modes are a universal result, which is caused by the spontaneous breakdown of gauge symmetry associated with the superfluid phase transition. One issue under current investigation is how to track the evolution of the system in superfluid phase. One possibility is to measure low-lying collective modes of the gas [25]. In fact, in the symmetry-broken state, the velocity of the collective modes is dependent on the adjusting parameter. So we can track the evolution of the system by observing collective modes. In Ref. [26], the collective modes have been calculated by the way of the Feynman diagram, in which we must be careful to pick up the corresponding diagram to get the Goldstone modes. Functional integrals overcome this difficulty. Keeping the Gaussian fluctuation, we can naturally get the same result as in [26].

This paper is organized as follows. In Sec. II, we use the path integral to obtain the effective action of the system. In Sec. III, we start from the effective action and study the collective modes across the whole crossover regime to acquire an analytical expression for the sound velocity of Goldstone modes. In Sec. IV, we deal with analytical results in BCS and BEC limits and numerical results. In Sec. V, a conclusion is given.

II. THE EFFECTIVE ACTION

The Hamiltonian for the dilute gas of Fermi atoms can be written as ($\hbar=K_B=1$)

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$$H = \int d^3\vec{x}\Bigg\{\sum_\sigma \psi_\sigma^\dagger(\vec{x})\left[-\frac{\nabla^2}{2m}-\mu\right]\psi_\sigma(\vec{x}) + \phi^\dagger(\vec{x})\left[-\frac{\nabla^2}{2M}+2\nu - 2\mu\right]\phi(\vec{x}) - U\psi_\uparrow^\dagger(\vec{x})\psi_\downarrow^\dagger(\vec{x})\psi_\downarrow(\vec{x})\psi_\uparrow(\vec{x}) + g_r[\phi^\dagger(\vec{x})\psi_\downarrow(\vec{x})\psi_\uparrow(\vec{x}) + \phi(\vec{x})\psi_\uparrow^\dagger(\vec{x})\psi_\downarrow^\dagger(\vec{x})]\Bigg\}. \quad (1)$$

This is the so-called fermion-boson model [27–29], where a fermionic atom and a quasimolecular boson associated with the Feshbach resonance are described by the field operators $\psi(\vec{x})$ and $\phi(\vec{x})$, respectively. 2ν represents the lowest excitation energy of the Feshbach molecule, also referred to as the threshold energy of the Feshbach resonance. m and $M=2m$ are the masses of the Fermi atom and the molecule, respectively. μ is the chemical potential. The last term in Eq. (1) describes the Feshbach resonance with a coupling constant g_r, which describes how a molecular boson can dissociate into two Fermi atoms and vice versa. The Hamiltonian also includes an attractive interaction $-U(<0)$ between atoms, arising from the nonresonant process. The factor of 2 in 2μ and $M=2m$ reflects the fact that one boson consists of two fermionic atoms. In the imaginary-time functional integration formalism, the partition function is written as

$$Z = \int D[\psi_\sigma^\dagger, \psi_\sigma, \phi^\dagger, \phi]\exp(-S), \quad (2)$$

where

$$S = \int_0^\beta d\tau \int d^3\vec{x}\Bigg(\sum_\sigma \psi_\sigma^\dagger(x)\left[\partial_\tau - \frac{\nabla^2}{2m}-\mu\right]\psi_\sigma(x) + \phi^\dagger(x)\left[\partial_\tau - \frac{\nabla^2}{2M}+2\nu-2\mu\right]\phi(x) - U\psi_\uparrow^\dagger(x)\psi_\downarrow^\dagger(x)\psi_\downarrow(x)\psi_\uparrow(x) + g_r[\phi^\dagger(x)\psi_\downarrow(x)\psi_\uparrow(x) + \phi(x)\psi_\uparrow^\dagger(x)\psi_\downarrow^\dagger(x)]\Bigg), \quad (3)$$

with $\beta=1/T$ and τ imaginary time. $D[\cdots]$ represents the functional integral for field operators. $x=(\tau,\vec{x})$ is a four-dimensional vector.

In order to integrate out the Fermi field, bosonic-like variables $B(\vec{x},\tau)$ and $\bar{B}(\vec{x},\tau)$ are introduced by the Hubbard-Strotonovich transformation

$$\exp\{U\psi_\uparrow^\dagger\psi_\downarrow^\dagger\psi_\downarrow\psi_\uparrow\} = \int D[\bar{B},B]\exp\left\{-\frac{1}{U}\bar{B}B + \psi_\uparrow^\dagger\psi_\downarrow^\dagger B + \bar{B}\psi_\downarrow\psi_\uparrow\right\}. \quad (4)$$

It is further convenient to introduce the space-time Fourier transform for Grassman, bosonic, and Feshbach molecules variables.

$$\psi_\sigma(\vec{x},\tau) = \frac{1}{\beta V}\sum_{\vec{k},\omega_n}\exp[i(\vec{k}\cdot\vec{x}-\omega_n\tau)]C_\sigma(\vec{k},\omega_n),$$

$$\phi(\vec{x},\tau) = \frac{1}{\beta V}\sum_{\vec{q},\Omega_n}\exp[i(\vec{q}\cdot\vec{x}-\Omega_n\tau)]b(\vec{q},\Omega_n), \quad (5)$$

where $\omega_n=\frac{(2n+1)\pi}{\beta}$ and $\Omega_n=\frac{2n\pi}{\beta}$ are the Matsubara frequency for the Fermi and boson field, respectively. Then, the action S reads

$$S = \frac{1}{\beta V}\Bigg\{\sum_{k,\sigma}\left[-i\omega_n + \frac{\vec{k}^2}{2m}-\mu\right]\bar{C}_\sigma(k)C_\sigma(k) + \sum_q\left[-i\Omega_n + \frac{\vec{q}^2}{2M}+2\nu-2\mu\right]\bar{b}(q)b(q) + \frac{1}{U}\sum_q \bar{B}(q)B(q) - \frac{1}{\beta V}\sum_{k,k'}\left[\bar{C}_\uparrow(k)\bar{C}_\downarrow(-k')B(k-k') + C_\downarrow(-k')C_\uparrow(k)\bar{B}(k-k')\right] + \frac{1}{\beta V}\sum_{k,k'}g_r\left[\bar{b}(k-k')\times C_\downarrow(-k')C_\uparrow(k) + b(k-k')\bar{C}_\uparrow(k)\bar{C}_\downarrow(-k')\right]\Bigg\}, \quad (6)$$

where $k=(\omega_n,\vec{k})$ and $q=(\Omega_n,\vec{q})$ are used to denote both momentum and Matsubara frequency.

The Grassman variables can be integrated out, yielding

$$S_{eff} = -\operatorname{tr}\ln G^{-1} + \frac{1}{\beta V}\left\{\sum_q\left[-i\Omega_n + \frac{\vec{q}^2}{2M}+2\nu-2\mu\right]\bar{b}(q)b(q) + \frac{1}{U}\sum_q \bar{B}(q)B(q)\right\} \quad (7)$$

where

$$G^{-1}(k,k') = \begin{pmatrix} \epsilon(k)\delta_{k,k'} & \frac{1}{\beta V}[\bar{B}(k-k') - g_r\bar{b}(k-k')] \\ \frac{1}{\beta V}[B(k'-k) - g_r b(k'-k)] & -\epsilon(-k)\delta_{k,k'} \end{pmatrix}, \quad (8)$$

with $\epsilon(k)=i\omega_n-\xi_k$, $\xi_k=\frac{\vec{k}^2}{2m}-\mu$.

To proceed further, we consider a quadratic expansion of the effective action in terms of fluctuation near the mean-field value. Choosing mean-field value Δ and ϕ_m for $B(k)$ and $b(k)$, respectively, that is

$$\bar{B}(q)\to\beta V[\bar{\Delta}\delta_{q=0}+\bar{B}(q)],$$

$$\bar{b}(q)\to\beta V[\bar{\phi}_m\delta_{q=0}+\bar{b}(q)], \tag{9}$$

we can express the matrix $G^{-1}(k,k')$ as

$$G^{-1}(k,k')=G_0^{-1}(k,k')+G_1(k,k'), \tag{10}$$

where

$$G_0^{-1}(k,k')=\begin{pmatrix}\epsilon(k)\delta_{k,k'} & \bar{\Delta}-g_r\bar{\phi}_m\\ \Delta-g_r\phi_m & -\epsilon(-k)\delta_{k,k'}\end{pmatrix}, \tag{11}$$

$$G_1=\begin{pmatrix}0 & \bar{B}(k-k')-g_r\bar{b}(k-k')\\ B(k'-k)-g_rb(k'-k) & 0\end{pmatrix}.$$

Correspondingly, the effective action takes the form

$$S_{eff}=-\operatorname{tr}\ln G_0^{-1}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\operatorname{tr}(G_0G_1)^n+\beta V\Bigg[\frac{1}{U}[\bar{\Delta}B(0)+\Delta\bar{B}(0)]+2(\nu-\mu)[\bar{\phi}_mb(0)+\phi_m\bar{b}(0)]+\frac{1}{U}\Delta^2+2(\nu-\mu)\phi_m^2+\sum_q\left[-i\Omega_n+\frac{\vec{q}^2}{2M}+2\nu-2\mu\right]\bar{b}(q)b(q)+\frac{1}{U}\sum_q\bar{B}(q)B(q)\Bigg]. \tag{12}$$

The constant Δ and ϕ_m are determined by requiring the coefficient of the linear terms in $B(0)$, $\bar{B}(0)$, $b(0)$, and $\bar{b}(0)$ to vanish, leading to the gap equation

$$1=U_{eff}\int\frac{d^3k}{(2\pi)^3}\frac{\tanh\left(\frac{\beta}{2}E_k\right)}{2E_k}, \tag{13}$$

where

$$U_{eff}=U+\frac{g_r^2}{2(\nu-\mu)},$$

$$E_k=\sqrt{\xi_k^2+\Delta_r^2},$$

$$\Delta_r=\Delta-g_r\phi_m, \tag{14}$$

Eq. (12) is still exact, the approximation depends on the number of power considered. As usual, we will only consider until the Gaussian approximation. In this approximation

$$S_{eff}=-\operatorname{tr}\ln G_0^{-1}+\beta V\left[\frac{1}{U}\Delta^2+2(\nu-\mu)\phi_m^2\right]+S_{eff}^{(2)}, \tag{15}$$

where

$$S_{eff}^{(2)}=\frac{\beta V}{2}\sum_q[\bar{b}(q)b(-q)B(-q)\bar{B}(q)]M(q)\begin{pmatrix}b(q)\\ \bar{b}(-q)\\ \bar{B}(-q)\\ B(q)\end{pmatrix} \tag{16}$$

and

$$M(q)=\begin{pmatrix}M_{11}(q) & g_r^2Q(q) & -g_rQ(q) & g_rP(q)\\ g_r^2Q(q) & M_{22}(q) & g_rP^*(q) & -g_rQ(q)\\ -g_rQ(q) & g_rP^*(q) & M_{33}(q) & Q(q)\\ g_rP(q) & -g_rQ(q) & Q(q) & M_{44}(q)\end{pmatrix}, \tag{17}$$

where

$$M_{11}(q)=-i\Omega_n+\frac{\vec{q}^2}{2M}+2\nu-2\mu-g_r^2P(q),$$

$$M_{22}(q)=M_{11}^*(q),$$

$$M_{33}(q)=\frac{1}{U}-P^*(q),$$

$$M_{44}(q)=M_{33}^*(q), \tag{18}$$

$$P(q)=\frac{1}{\beta V}\sum_k g(k)g(q-k),$$

$$Q(q)=\frac{1}{\beta V}\sum_k f(k)f(q-k), \tag{19}$$

and $g(k)$ and $f(k)$ are the ordinary Gorkov function, that is

$$g(k)=\frac{\epsilon(-k)}{\epsilon(k)\epsilon(-k)+\Delta_r^2},$$

$$f(k)=\frac{\Delta_r}{\epsilon(k)\epsilon(-k)+\Delta_r^2}. \tag{20}$$

III. COLLECTIVE MODES AT $T=0$ K

The collective modes are determined by the poles of the propagator matrix $M^{-1}(q)$. The poles of $M^{-1}(q)$ are determined by the condition $\det M(q)=0$ and lead to a dispersion relation for the collective modes $\omega=\omega(q)$, when the usual analytical continuation $\Omega_n\to\omega+i\delta$ is performed. In order to obtain the collective mode spectrum, we introduce the transformation [30,31]

$$b(x) = \frac{1}{\sqrt{2}}[\lambda_1(x) + i\theta_1(x)],$$

$$B(x) = \frac{1}{\sqrt{2}}[\lambda_2(x) + i\theta_2(x)], \quad (21)$$

where $\lambda_i(x)$ and $\theta_i(x)$ are real and may be identified with amplitude and phase fields. This transformation corresponds to a representation transformation which leads to a subsequent transformation matrix:

$$\begin{pmatrix} b(q) \\ \bar{b}(-q) \\ \bar{B}(-q) \\ B(q) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ 1 & -i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & i & 1 \end{pmatrix} \begin{pmatrix} \lambda_1(q) \\ \theta_1(q) \\ \theta_2(q) \\ \lambda_2(q) \end{pmatrix}. \quad (22)$$

So that matrix $M(q)$ corresponds to

$$\tilde{M}(q) = \begin{pmatrix} \tilde{M}_{11}(q) & -i[\omega + g_r^2 P^{(O)}(q)] & i g_r P^{(O)}(q) & g_r[P^{(E)}(q) - Q(q)] \\ i[\omega + g_r^2 P^{(O)}(q)] & \tilde{M}_{22}(q) & g_r[P^{(E)}(q) + Q(q)] & -i g_r P^{(O)}(q) \\ -i g_r P^{(O)}(q) & g_r[P^{(E)}(q) + Q(q)] & \tilde{M}_{33}(q) & i P^{(O)}(q) \\ g_r[P^{(E)}(q) - Q(q)] & i g_r P^{(O)}(q) & -i P^{(O)}(q) & \tilde{M}_{44}(q) \end{pmatrix}. \quad (23)$$

where

$$\tilde{M}_{11}(q) = \frac{\vec{q}^2}{2M} + 2\nu - 2\mu - g_r^2[P^{(E)}(q) - Q(q)],$$

$$\tilde{M}_{22}(q) = \frac{\vec{q}^2}{2M} + 2\nu - 2\mu - g_r^2[P^{(E)}(q) + Q(q)],$$

$$\tilde{M}_{33}(q) = \frac{1}{U} - [P^{(E)}(q) + Q(q)],$$

$$\tilde{M}_{44}(q) = \frac{1}{U} - [P^{(E)}(q) - Q(q)]. \quad (24)$$

Function $P(q)$ can be expanded into the addition of two parts even $P^{(E)}(q)$ and odd $P^{(O)}(q)$. $Q(q)$ is an even function.

In this paper, we only think about the long-wavelength and low-frequency collective modes. Thus we need to expand the function $P^{(E)}(q)+Q(q)$, $P^{(E)}(q)-Q(q)$, and $P^{(O)}(q)$ as the series of $\vec{q}$ and ω. Before doing this, let us notice that low-frequency expansion must conform to two conditions [30,31]. (i) Landau damping of the collective modes do not exist. So we must keep the imaginary part of the propagator to vanish. This condition can be satisfied by setting $T=0$ K. (ii) The collective modes cannot damp into two quasiparticles. Owing to the minimum energy of single quasiparticle is Δ_r, so the condition $\omega \ll 2\Delta_r$ must be carried out. Under satisfying these conditions above, these functions can be expanded as

$$P^{(O)}(q) = B\omega,$$

$$P^{(E)}(q) + Q(q) = A + Cq^2 + D\omega^2,$$

$$P^{(E)}(q) - Q(q) = R + Sq^2 + T\omega^2, \quad (25)$$

where

$$B = \sum_k \frac{\xi_k}{4E_k^3}, \quad A = \sum_k \frac{1}{2E_k},$$

$$C = -\sum_k \frac{1}{8E_k^3}\left[\frac{\xi_k}{m} - \left(1 - \frac{3\Delta_r^2}{E_k^2}\right)\frac{k^2\cos^2\theta}{m^2}\right],$$

$$D = \sum_k \frac{1}{8E_k^3}, \quad R = \sum_k \frac{\xi_k^2}{2E_k^3},$$

$$T = \sum_k \frac{1}{8E_k^3}\left(1 - \frac{\Delta_r^2}{E_k^2}\right),$$

$$S = \sum_k \frac{1}{8E_k^3}\left\{-\frac{\xi_k}{m}\left(1 - \frac{3\Delta_r^2}{E_k^2}\right) + \left[1 - \frac{10\Delta_r^2}{E_k^2}\left(1 - \frac{\Delta_r^2}{E_k^2}\right)\frac{k^2\cos^2\theta}{m^2}\right]\right\}. \quad (26)$$

Similarly, we expand $\tilde{M}_{11}(q)$, $\tilde{M}_{22}(q)$, $\tilde{M}_{33}(q)$, and $\tilde{M}_{44}(q)$ in the same form:

$$\tilde{M}_{11}(q) = Z_1 + Z_2 q^2 - g_r^2 T\omega^2,$$

$$\tilde{M}_{22}(q) = Z_3 + Z_4 q^2 - g_r^2 D\omega^2,$$

$$\tilde{M}_{33}(q) = Z_5 - Cq^2 - D\omega^2,$$

$$\tilde{M}_{44}(q) = Z_6 - Sq^2 - T\omega^2. \quad (27)$$

As is known to all [32], low energy collective modes are related to phase-phase fluctuation. In order to extract the information about the phase-phase fluctuation, we integrate out the amplitude field and only keep the phase field, then the effective action of the second order takes the form

$$S_{eff}^{(2)} = \frac{\beta V}{2}\sum_{q}[\theta_1^*(q),\theta_2^*(q)]\widetilde{M}_p(q)\begin{pmatrix}\theta_1(q)\\ \theta_2(q)\end{pmatrix}, \quad (28)$$

with

$$\widetilde{M}_p(q) = \begin{pmatrix} \widetilde{M}_{22}(q) + \frac{\omega^2}{W_a}G_1 & \Theta - \frac{\omega^2}{W_a}G_2 \\ \Theta - \frac{\omega^2}{W_a}G_2 & \widetilde{M}_{33}(q) + \frac{\omega^2}{W_a}G_3 \end{pmatrix} \quad (29)$$

where

$$\Theta = g_r[P^{(E)}(q) + Q(q)] = g_r(A + Cq^2 + D\omega^2)$$

$$= y_1 + y_2 q^2 + y_3\omega^2,$$

$$\Xi = g_r[P^{(E)}(q) - Q(q)] = g_r(R + Sq^2 + T\omega^2),$$

$$G_1 = 2\Xi g_r B(1 + g_r^2 B) + \widetilde{M}_{11}g_r^2B^2 + \widetilde{M}_{44}(1 + g_r^2 B)^2,$$

$$G_2 = \Xi B(1 + g_r^2 B) + \Xi g_r^2 B^2 + \widetilde{M}_{11}g_r B^2 + \widetilde{M}_{44}g_r B(1 + g_r^2 B),$$

$$G_3 = 2\Xi g_r B^2 + \widetilde{M}_{11}B^2 + \widetilde{M}_{44}g_r^2 B^2,$$

$$W_a = \Xi^2 - \widetilde{M}_{11}\widetilde{M}_{44} = Z_7 + Z_8 q^2 + Z_9\omega^2. \quad (30)$$

Again, the dispersion relation for the phase-phase collective modes is obtained from the condition $\det \widetilde{M}_p(q)=0$. Let $\det \widetilde{M}_p(q)=0$ and only retain the second order of $\vec{q}$ and ω, we obtain

$$Z_7(Z_3Z_5 - y_1^2) + (-CZ_3Z_7 + Z_4Z_5Z_7 + Z_3Z_5Z_8 - 2y_1y_2Z_7 - y_1^2Z_8)q^2 + (-Z_3Z_7D + Z_3Z_5Z_9 - Z_5Z_7Dg_r^2 + G_3Z_3 + G_1Z_5 - 2y_1y_3Z_7 - y_1^2Z_9 + 2y_1G_2)\omega^2 = 0. \quad (31)$$

For Goldstone modes, the constant term must vanish. As proven below, this requirement is automatically satisfied. Owing to $Z_3=2(\nu-\mu)-g_r^2A$, $Z_5=\frac{1}{U}-A$, $y_1^2=g_r^2A^2$, and gap equation (13), there is an identity $Z_3Z_5-y_1^2=0$. This result is very important for us, which makes us assure that the low energy phase-phase mode is no energy gap. No gap is completely related to spontaneous breakdown of the gauge symmetry associated with the superfluid phase transition. The method of path integral directly obtained the gapless result. When we use the quantum field theory method, we must be careful in choosing a Feynman diagram to ensure the behavior of no gap, or the gapless result is not possible. So that the sound velocity

$$v_s^2 = \frac{CZ_3Z_7 - Z_4Z_5Z_7 + 2y_1y_2Z_7}{G_3Z_3 + G_1Z_5 + 2y_1G_2 - Z_3Z_7D - Z_5Z_7Dg_r^2 - 2y_1y_3Z_7}. \quad (32)$$

For the crossover, to completely obtain an analytical result is not possible, a numerical accession is essential; but in the limit of BCS and BEC, we can obtain the analytical result.

IV. ANALYTICAL AND NUMERICAL RESULTS

A. Weak coupling limit

In the weak coupling limit, the threshold energy 2ν is very large. $\nu \gg \mu$ can be realized because the chemical potential is at most the order of ϵ_F, so the effective interaction is dominated by nonresonant interaction U. The gap equation is reduced to

$$\frac{1}{U} = \sum_k \frac{1}{2E_k}. \quad (33)$$

At the same time, fermionic atoms are dominated over Feshbach molecules, so Feshbach resonance can be neglected and Feshbach molecules are negligible because of $\nu \gg \mu$. Then the Hamiltonian of the fermion-boson model is reduced to that of the continuum model. Thus in this limit, a complete BCS theoretic result is recovered. This discussion corresponds to matrix $\widetilde{M}(q)$ to be reduced to

$$M_{BCS}(q) = \begin{pmatrix} \widetilde{M}_{33}(q) & iP^{(O)}(q) \\ -iP^{(O)}(q) & \widetilde{M}_{44}(q) \end{pmatrix}. \quad (34)$$

Thus

$$v_{BCS}^2 = -\frac{C}{D + \frac{B^2}{A-R}}. \quad (35)$$

This analytical result is consistent with the result in [30], which shows that our results in the BCS limit are qualitatively correct. The weak coupling limit is particularly simple with all integrals peaked near the Fermi surface. We can take the approximation

$$\sum_k F(\xi_k) = N(\epsilon_F)\int_{-\infty}^{\infty} d\xi\, F(\xi), \quad (36)$$

where the $N(\epsilon_F)$ is the density of state on the Fermi surface. By means of Eq. (36), $B=0$, $D=\frac{N(\epsilon_F)}{4\Delta^2}$, $C=-\frac{N(\epsilon_F)\epsilon_F}{6m\Delta^2}$, and $v_{BCS} = \frac{1}{\sqrt{3}}v_F$, which recovered the Anderson-Bogoliubov result [34].

B. Strong coupling limit

In the BEC limit, $\nu<0$, $\nu \ll -\epsilon_F$. Since the Feshbach molecules have energy lower than the energy of the two fermionic atoms, most fermionic atoms will combine to form molecules. We can ignore the fermionic atoms and identify the system with a bosonic system. These correspond to $\widetilde{M}(q)$ equal to

$$M_{BEC}(q) = \begin{pmatrix} \widetilde{M}_{11}(q) & -i\omega(1 + g_r^2 B) \\ i\omega(1 + g_r^2 B) & \widetilde{M}_{22}(q) \end{pmatrix}. \quad (37)$$

Meanwhile, since the chemical potential μ approaches ν, which leads to $U_{eff}=\frac{g_r^2}{2(\nu-\mu)}$ and the gap equation is

$$1 = \frac{g_r^2}{4(\nu-\mu)}\sum_k \frac{1}{E_k}, \quad (38)$$

using the gap equation,

$$v_{BEC}^2 = \frac{g_r^2\left(\frac{1}{2M} - g_r^2 C\right)\sum_k \frac{\Delta_r^2}{2E_k^3}}{(1+g_r^2 B)^2 + g_r^4 D \sum_k \frac{\Delta_r^2}{2E_k^3}}. \quad (39)$$

In the case of weak Feshbach resonance, we can ignore the high order of g_r. This leads to

$$v_{BEC}^2 = \frac{g_r^2}{4M}\sum_k \frac{\Delta_r^2}{E_k^3}. \quad (40)$$

We can approximate $E_k=\sqrt{\left(\frac{k^2}{2m}-\mu\right)^2+\Delta_r^2} \to E_k=\frac{k^2}{2m}-\mu$, then

$$v_{BEC} = \frac{g_r \Delta_r}{16}\sqrt{\frac{3\pi N}{m(-\nu\epsilon_F)^{3/2}}}. \quad (41)$$

This result is consistent with that of [26] quantitatively. As in the discussion in [26], in the BEC limit, Cooper pairs order parameter is negligible, and Δ_r is reduced to ϕ_m and v_{BEC} is proportional to $\sqrt{N_B^C}$. This dependence on N_B^C is characteristic of the Bogoliubov phonon mode in a BEC gas. Thus Eq. (41) may be regarded as the velocity of the Bogoliubov phonon associated with a condensate of Feshbach molecules.

C. Numerical results

All of our numerical results are given in the canonical ensemble. To calculate sound velocity, we must self-consistently solve the energy gap equation (13) and the particle number equation at the level of the mean field

$$n = \int \frac{d^3\vec{k}}{(2\pi)^3}\left(1 - \frac{\xi_k}{E_k}\right) + 2\phi_m^2 \quad (42)$$

to get the values μ, Δ_r, and ϕ_m. In addition, owing to bare coupling parameters in fermion-boson model Hamiltonians U, g_r, and ν, which result in some ultraviolet divergence, renormalization to these parameters must be done [29]:

$$\frac{1}{U_{eff}^R} = \frac{1}{U_{eff}} - \int^{\Lambda} \frac{d^3\vec{k}}{(2\pi)^3}\frac{m}{\vec{k}^2},$$

$$\frac{1}{U^R} = \frac{1}{U} - \int^{\Lambda} \frac{d^3\vec{k}}{(2\pi)^3}\frac{m}{\vec{k}^2},$$

$$\frac{1}{g_r^R} = \frac{1}{g_r} - \frac{U}{g_r}\int^{\Lambda} \frac{d^3\vec{k}}{(2\pi)^3}\frac{m}{\vec{k}^2},$$

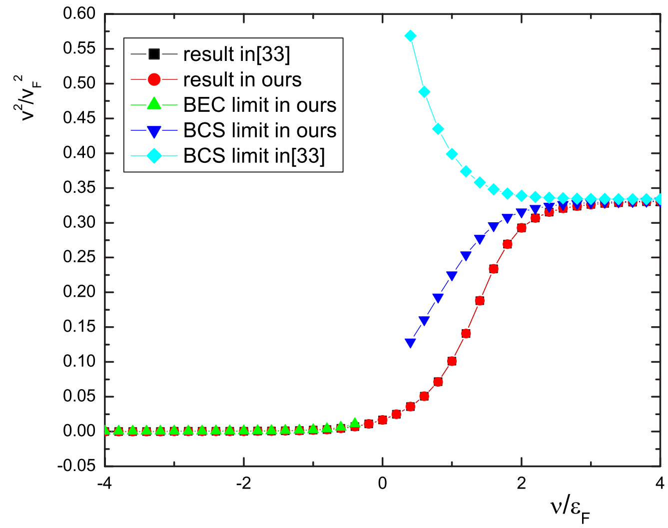

FIG. 1. (Color online) The behavior of the sound velocity v (circle line) as a function of the detuning ν across the BCS-BEC crossover. The results in the BCS and BEC (up-triangle line) limits are also included. Our results (down-triangle line) in the BCS limit are apparently different from that (diamond line) in [33]. Using the expression for the sound velocity in [33] we get the results represented by the square line. In order to make contrast, we have chosen $U=7.54\epsilon_F/K_F^3$ and $g=4.62\epsilon_F/K_F^{3/2}$.

$$\nu^R = \mu + \frac{(g_r^R)^2}{2(U_{eff}^R - U^R)}, \quad (43)$$

where Λ is the ultraviolet cutoff and all variables with superscript R are renormalized parameters. This procedure made our results convergent and significant.

All numerical results were insensitive to the ultraviolet cutoff Λ. In Fig. 1, we plot the behavior of the sound velocity as the function of the detuning parameter through the whole BCS-BEC crossover. In order to compare with [33], the same parameters have been chosen. The numerical results show that there exists a continuous interpolation for sound velocity between the BCS and BEC limits. We also make a plot for sound velocity using the expressions in [33] and find that our results are well-consistent with that in [33] except that there is an apparent difference for the behavior of the sound velocity in the BCS limit. In our paper, we neglected the contribution of the Feshbach molecules to the action in order to obtain the results of the BCS limit. Our results of the BCS limit directly came from the reduction to the action and agree with the results in [26,30] in the limit of BCS. In the process of calculating the sound velocity in [33], a gauge transformation which makes the action to be expressed in terms of the modulus of the order parameter and its phase was done but did not lead to any extra effect on the action. As it was pointed out [35–37], this gauge transformation has changed the measure of the functional integral and the ground state of the system. This transformation changed the local fermion concentration but kept the total fermion number remaining constant. So an extra term must be introduced to compensate for this change. We thought the statement above should be responsible for the difference between our results and the results in [33] in the BCS limit.

Hydrodynamic equations have also been used to calculate collective modes [38] across the BCS-BEC crossover. In the fermion-boson model Hamiltonian (1), the BEC limit is free gas consisting of Feshbach molecules, which leads the sound velocity to approach zero; but in the frame of hydrodynamic equations, interaction between Feshbach molecules is included and nonzero velocity is obtained in the BEC limit.

V. CONCLUSIONS

In this paper, we have investigated the BCS-BEC crossover in the superfluid phase of a uniform gas of Fermi atoms with Feshbach resonance and obtained an analytical expression for the collective modes. The results in BCS and BEC limits also have been recovered and there exists a continuous interpolation for sound velocity between BCS and BEC limits. Owing to the monotonic behavior of the sound velocity, we can track the evolution of the system by observing the phase-phase collective mode in the symmetry-broken phase in experiment.

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[1] D. M. Eagles, Phys. Rev. **186**, 456 (1969).
[2] A. J. Leggett, in *Modern Trends in the Theory of Condensed Matter*, edited by A. Pekalski and R. Przystawa (Springer-Verlag, Berlin, 1980).
[3] Q. J. Chen, J. Stajic, S. N. Tan, and K. Levin, Phys. Rep. **412**, 1 (2005).
[4] M. Greiner, C. A. Regal, and D. S. Jin, Nature (London) **426**, 537 (2003).
[5] S. Jochim, M. Bartenstein, A. Altmeyer, G. Hendl, S. Riedl, C. Chin, J. H. Denschlag, and R. Grimm, Science **302**, 2102 (2003).
[6] M. W. Zwierlein, C. A. Stan, C. H. Schunck, S. M. F. Raupach, S. Gupta, Z. Hadzibabic, and W. Ketterle, Phys. Rev. Lett. **91**, 250401 (2003).
[7] C. A. Regal, M. Greiner, and D. S. Jin, Phys. Rev. Lett. **92**, 040403 (2004).
[8] M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. H. Denschlag, and R. Grimm, Phys. Rev. Lett. **92**, 120401 (2004).
[9] M. W. Zwierlein, C. A. Stan, C. H. Schunck, S. M. F. Raupach, A. J. Kerman, and W. Ketterle, Phys. Rev. Lett. **92**, 120403 (2004).
[10] J. Kinast, S. L. Hemmer, M. E. Gehm, A. Turlapov, and J. E. Thomas, Phys. Rev. Lett. **92**, 150402 (2004).
[11] M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. H. Denschlag, and R. Grimm, Phys. Rev. Lett. **92**, 203201 (2004).
[12] C. Chin, M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, J. H. Denschlag, and R. Grimm, Science **305**, 1128 (2004).
[13] T. Bourdel, L. Khaykovich, J. Cubizolles, J. Zhang, F. Chevy, M. Teichmann, L. Tarruell, S. J. J. M. F. Kokkelmans, and C. Salomon, Phys. Rev. Lett. **93**, 050401 (2004).
[14] M. Randeria, in *Bose-Einstein Condensation*, edited by A. Griffin, D. W. Snoke, and S. Stringari (Cambridge University Press, Cambridge, 1995).
[15] Y. J. Uemura, Physica C **282-287**, 194 (1997).
[16] G. Deutscher, Nature (London) **397**, 410 (1999).
[17] W. Ketterle and N. J. Van Druten, Adv. At., Mol., Opt. Phys. **37**, 181 (1996).
[18] H. F. Hess, Phys. Rev. B **34**, 3476 (1986).
[19] H. Feshbach, Ann. Phys. (N.Y.) **19**, 287 (1962).
[20] K. E. Strecker, G. B. Partridge, and R. G. Hulet, Phys. Rev. Lett. **91**, 080406 (2003).
[21] C. A. Regal, C. Ticknor, J. L. Bohn, and D. S. Jin, Nature (London) **424**, 47 (2003).
[22] K. Xu, T. Mukaiyama, J. R. Abo-Shaeer, J. K. Chin, D. E. Miller, and W. Ketterle, Phys. Rev. Lett. **91**, 210402 (2003).
[23] M. H. Anderson *et al.*, Science **269**, 198 (1995).
[24] K. B. Davis, M. O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995).
[25] C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases* (Cambridge University Press, New York, 2002).
[26] Y. Ohashi and A. Griffin, Phys. Rev. A **67**, 063612 (2003).
[27] R. Friedberg and T. D. Lee, Phys. Rev. B **40**, 6745 (1989).
[28] T. Kostyrko and J. Ranninger, Phys. Rev. B **54**, 13105 (1996).
[29] Y. Ohashi and A. Griffin, Phys. Rev. A **67**, 033603 (2003).
[30] J. R. Engelbrecht, M. Randeria, and C. A. R. Sa de Melo, Phys. Rev. B **55**, 15153 (1997).
[31] M. Iskin and C. A. R. Sa de Melo, Phys. Rev. B **72**, 024512 (2005).
[32] S. De Palo, C. Castellani, C. Di Castro, and B. K. Chakraverty, Phys. Rev. B **60**, 564 (1999).
[33] Da-Shin Lee, Chi-Yong Lin, and Ray J. Rivers, Phys. Rev. Lett. **98**, 020603 (2007).
[34] P. W. Anderson, Phys. Rev. **112**, 1900 (1958).
[35] V. M. Yakovenko and Hsi-Sheng Goan, Phys. Rev. B **58**, 10648 (1998).
[36] K. Sengupta and N. Dupuis, Phys. Rev. B **61**, 13493 (1999).
[37] I. V. Krive and A. S. Rozhavsky, Phys. Lett. **113**, 313 (1985).
[38] T. K. Ghosh and K. Machida, Phys. Rev. A **73**, 013613 (2006).