PHYSICAL REVIEW A **75**, 052509 (2007)

Fermionic density functional at a Feshbach resonance

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(Received 5 December 2006; published 29 May 2007)

We consider a dilute gas of neutral unpolarized fermionic atoms at zero temperature. The atoms interact via a short-range (tunable) attractive interaction. We demonstrate analytically a curious property of the gas at unitarity. Namely, the correlation energy of the gas, evaluated by second-order perturbation theory, has the same density dependence as the first-order exchange energy, and the two almost exactly cancel each other at a Feshbach resonance irrespective of the shape of the potential, provided $(\mu r_s) \gg 1$. Here $(\mu)^{-1}$ is the range of the two-body potential, and r_s is defined through the number density, $n = 3/(4\pi r_s^3)$. The implications of this result for universality are discussed.

DOI: 10.1103/PhysRevA.75.052509 PACS number(s): 31.15.Ew, 34.20.Cf, 34.50.−s, 71.15.Mb

I. INTRODUCTION

There has been much interest amongst theorists to calculate the properties of a dilute Fermi gas in the so-called unitary regime. In general, the properties of the dilute gas are determined by the number density n and the scattering length a. At unitarity, the short-range interaction between two atoms (with mass M) may be adjusted to give rise to an infinite scattering length a. As a consequence, no length scale is left from the interaction. The only length scale available is the inverse of the Fermi wave number $k_F = (3\pi^2 n)^{1/3}$, and the total energy E of the gas must have the same density dependence as the noninteracting kinetic energy, $N\frac{3}{5}\hbar^2 k_F^2/2M$. In particular, at $T=0$, the energy per particle is calculated to be

$$\frac{E}{N} = \xi \frac{3}{5}\frac{\hbar^2 k_F^2}{2M} \propto n^{2/3}, \tag{1}$$

where $\xi \simeq 0.44$ [1]. The experimental value of ξ is about 0.5, but with large error bars [2]. Recently, there have been two Monte Carlo (MC) finite temperature calculations [3,4] of an untrapped gas at unitarity, where various thermodynamic properties as a function of temperature have been computed. It is clear that at unitarity, the kinetic and potential energies should have the same density dependence. This has been assumed *a priori* in a previous density functional treatment of a unitary gas [5]. However, such scaling behavior is not evident from the usual density functionals for the direct, exchange and correlation energies [6]. The aim of the present paper is to examine this point in some detail.

To this end, we consider a dilute gas of $N \gg 1$ neutral fermionic atoms (mass M) at $T=0$, interacting with a short-range attractive potential which is assumed to have the two-parameter form

$$v(r) = -v_0 f(\mu r). \tag{2}$$

Here, $v_0 > 0$ is the strength of the interaction, $R_0 = \frac{1}{\mu}$ is its range, and $f(x)$ is a dimensionless shape function. The Hamiltonian of this N-particle system reads

$$\hat{H} = \frac{\hbar^2\mu^2}{M}\left(-\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial \mathbf{x}_i^2} - \lambda \sum_{i<j} f(|\mathbf{x}_i - \mathbf{x}_j|)\right). \tag{3}$$

Here, $\mathbf{x}_i = \mu \mathbf{r}_i$ are dimensionless coordinates and

$$\lambda = \frac{M v_0}{\hbar^2 \mu^2} \tag{4}$$

is a dimensionless interaction strength. Not written explicitly here, there is also an external potential $v_{\mathrm{ext}}(\mathbf{r})$ that forces the N atoms to stay within a large box with volume $\Omega = N/n$ [with $v_{\mathrm{ext}}(\mathbf{r}) \equiv 0$ inside]. Consequently, in addition to λ and μ, there is a third independent parameter r_s, defined by $n = \frac{N}{\Omega} = \frac{3}{4\pi r_s^3}$.

In the true ground state of the Hamiltonian (3) the attractive atoms may form dimers or even clusters. We are, however, looking for a metastable state where there is a dilute gas of separated atoms with uniform density n, satisfying the condition $\mu r_s \gg 1$. Even then, for a weak v_0, there will be BCS-type pairing, followed by dimer formation as the strength of the interaction increases. This was predicted long ago by Leggett [7], and has been observed experimentally [8]. For the density-functional analysis of the uniform gas at a Feshbach resonance, we shall disregard the BCS condensed pairs in this paper.

To determine the value λ_{uty} of the parameter $\lambda \propto v_0$ at unitarity, we consider the atom-atom scattering problem with the attractive interaction $v(r)$ in the relative s state. Separating the center-of-mass motion (and using $x = \mu r$), we are left with the relative Hamiltonian

$$\hat{H}_{\mathrm{rel}} = \frac{\hbar^2\mu^2}{M}\left(-\frac{d^2}{dx^2} - \frac{2}{x}\frac{d}{dx} - \lambda f(x)\right). \tag{5}$$

(Note that $M = 2M_{\mathrm{red}}$ where M_{red} is the reduced mass.) Keeping the range of the potential small enough such that $\mu r_s \gg 1$, the strength v_0 (or λ) must be adjusted such that the potential can support a single bound state at zero energy. Then, the scattering length diverges, $a \to \infty$, leaving no length scale from the interaction. The gas is said to be at unitarity. [The corresponding value λ_{uty} is calculated in Sec. III for different shape functions $f(x)$ of the potential.] Such a tuning of the interaction is possible experimentally, and gives rise to a Feshbach resonance [9]. The scattering cross section in the given partial wave (s wave in our case) reaches the unitary limit. The gas is then expected to display universal behavior [10].

In the present paper (Sec. III), we are able to show analytically that, at unitarity, the first-order exchange energy [which does not have the expected density dependence of Eq. (1)] is almost exactly canceled by the leading contribution to the correlation energy (calculated in second-order perturbation theory in Sec. II). This happens irrespective of the shape of the potential as specified by the function $f(x)$ in Eq. (2), provided that the condition $\mu r_s \gg 1$ is satisfied. We show that our general results ensure this necessary cancellation at unitarity for a variety of two-parameter potentials, including the square well and the delta shell, as well as the smoothly varying $\cosh^{-2}(\mu r)$ and Gaussian potentials (Table I). This is the main result of the present work. The implications of this result for universality as well as their limitations are discussed in Sec. IV.

II. PERTURBATION EXPANSION

Treating the interaction (2) as a weak perturbation in the Hamiltonian (3) (with λ as the perturbation parameter), the unperturbed energy $E^{(0)}$ is the kinetic energy of a noninteracting Fermi gas, given by Eq. (1) with $\xi = 1$. The unperturbed ground state $|\Phi_0\rangle$ is a Slater determinant of plane waves.

Note that, at unitarity, the perturbation parameter is not really small. For example, for the square-well potential, the zero-energy single bound state occurs when $\lambda = \frac{\pi^2}{4}$ (see Sec. III). Nevertheless the low-order terms can point to important information, even when the expansion is divergent [11].

In our problem, there are three parameters, μ, v_0, and r_s. The unitarity condition $\lambda = \lambda_{\text{uty}}$ relates μ and v_0 via Eq. (4) [where λ_{uty} is fixed by the respective shape $f(x)$ of the potential as explained after Eq. (5)], so two independent parameters are left. This freedom is constrained farther by the low-density condition $(\mu r_s)^{-1} \ll 1$.

A. First order

Formally, the first-order correction,

$$E^{(1)} = \langle \Phi_0 | \hat{V}_{\text{int}} | \Phi_0 \rangle, \tag{6}$$

where $\hat{V}_{\text{int}}$ is the operator for the interaction (2), has a direct contribution $U(r_s,\mu) = N u(r_s,\mu)$ with

$$u(r_s,\mu) = \frac{n^2}{2N} \int_\Omega d^3r \int_\Omega d^3r' v(|\mathbf{r} - \mathbf{r}'|) = -\frac{3}{2} \frac{v_0}{(\mu r_s)^3} f_2. \tag{7}$$

Here, $f_2 = \int_0^\infty dx x^2 f(x)$.

The other first-order contribution is the exchange energy $E_{\text{x}}(r_s,\mu) = N e_{\text{x}}(r_s,\mu)$ [10],

$$e_{\text{x}}(r_s,\mu) = -\frac{3k_F}{\pi} \int_0^\infty dr j_1(k_F r)^2 v(r). \tag{8}$$

Here, $j_1(z)$ is a spherical Bessel function. Since $v(r)$ is short range and k_F is small in a dilute gas, we can use the small-z expansion $j_1(z) = \frac{z}{3} + O(z^3)$ to find

$$e_x(r_s,\mu) = \frac{3}{4} \frac{v_0}{(\mu r_s)^3} f_2 + O(\mu r_s)^{-5}. \tag{9}$$

B. Second order

Just as $E^{(1)}$, also the second-order correction,

$$E^{(2)} = -\sum_{n \neq 0} \frac{|\langle \Phi_n | \hat{V}_{\text{int}} | \Phi_0 \rangle|^2}{E_n - E_0} = N(e_{\text{dir}}^{(2)} + e_{\text{ex}}^{(2)}), \tag{10}$$

has direct and exchange contributions [12],

$$e_{\text{dir}}^{(2)}(r_s,\mu) = -\frac{3}{32\pi^5} \left(\frac{2M}{\hbar^2 \mu^2} \right) v_0^2 \frac{k_F^4}{\mu^4} \int d^3q \tilde{f}\left(\frac{k_F}{\mu} q \right)^2 \times \int_D \frac{d^3k_1 d^3k_2}{\mathbf{q} \cdot (\mathbf{q} + \mathbf{k}_1 - \mathbf{k}_2)}, \tag{11}$$

$$e_{\text{ex}}^{(2)}(r_s,\mu) = +\frac{3}{64\pi^5} \left(\frac{2M}{\hbar^2 \mu^2} \right) v_0^2 \frac{k_F^4}{\mu^4} \int d^3q \tilde{f}\left(\frac{k_F}{\mu} q \right) \times \int_D d^3k_1 d^3k_2 \frac{\tilde{f}\left(\frac{k_F}{\mu} |\mathbf{q} + \mathbf{k}_1 - \mathbf{k}_2| \right)}{\mathbf{q} \cdot (\mathbf{q} + \mathbf{k}_1 - \mathbf{k}_2)}. \tag{12}$$

While $v_0^2 (2M/\hbar^2\mu^2)$ has dimensions of energy, the integration variables $\mathbf{q}$, $\mathbf{k}_1$, and $\mathbf{k}_2$ are dimensionless. The domain of the integral over $d^3k_1 d^3k_2$ depends on $\mathbf{q}$,

$$D: \quad |\mathbf{k}_1|, |\mathbf{k}_2| < 1; \quad |\mathbf{k}_1 + \mathbf{q}|, |\mathbf{k}_2 - \mathbf{q}| > 1. \tag{13}$$

Furthermore, $\tilde{f}(y)$ is a dimensionless transform of $f(x)$,

$$\tilde{f}(y) = \int_0^\infty dx x^2 f(x) j_0(yx) \equiv \frac{1}{y} \int_0^\infty dx x f(x) \sin(yx). \tag{14}$$

To recover Eqs. (8) and (9) of Ref. [12], set $M = m_e$, $v_0 = -e^2\mu$, and $f(x) = \frac{1}{x}$ or $\tilde{f}(y) = \frac{1}{y^2}$, such that $v(r) = \frac{e^2}{r}$ becomes the electronic Coulomb repulsion. (Note that Ref. [12] uses Rydberg units, $m_e e^4 / 2\hbar^2 = e^2 / 2a_B = 1$.)

C. Second order in the limit $\mu r_s \gg 1$

For a dilute gas (small k_F) with short-range interaction (large μ), Eqs. (11) and (12) can be evaluated in the limit $\mu / k_F \equiv \alpha \mu r_s \gg 1$ where $\alpha^3 = \frac{4}{9\pi}$. Following Ref. [13], we choose a number q_1 such that $1 \ll q_1 \ll \mu / k_F$ and split the integrals over d^3q into two parts,

$$\int d^3q = \int_{q<q_1} d^3q + \int_{q>q_1} d^3q. \tag{15}$$

In the first part with $q < q_1$, we have $q \ll \mu / k_F$ and $|\mathbf{q} + \mathbf{k}_1 - \mathbf{k}_2| \ll \mu / k_F$ (note that $|\mathbf{k}_1|, |\mathbf{k}_2| < 1 \ll q_1$). Therefore, we may expand $\tilde{f}(y)=f_2+O(y^2)$ in Eqs. (11) and (12) and keep the leading term f_2 only. The sum of the two resulting $q<q_1$ contributions reads

$$e^{(2)}_{q<q_1}(r_s,\mu)=-\frac{3}{64\pi^5}\left(\frac{2M}{\hbar^2\mu^2}\right)v_0^2\frac{k_F^4}{\mu^4}f_2^2 \times\int_{q<q_1}d^3q\int_D\frac{d^3k_1d^3k_2}{\mathbf{q}\cdot(\mathbf{q}+\mathbf{k}_1-\mathbf{k}_2)}. \qquad (16)$$

The number q_1 can be chosen independently of $\mu/k_F\gg 1$, despite the condition $1\ll q_1\ll\mu/k_F$. Then, the integral in Eq. (16) is a finite constant and we conclude [13]

$$e^{(2)}_{q<q_1}(r_s,\mu)=O(\mu r_s)^{-4}. \qquad (17)$$

In the second part $q>q_1\gg 1$ of the integral (15), we can set $\mathbf{q}+\mathbf{k}_1-\mathbf{k}_2\approx\mathbf{q}$, since $|\mathbf{k}_1|,|\mathbf{k}_2|<1$. The resulting contributions to Eqs. (11) and (12) add up to

$$e^{(2)}_{q>q_1}(r_s,\mu)=-\frac{3}{64\pi^5}\left(\frac{4\pi}{3}\right)^2\left(\frac{2M}{\hbar^2\mu^2}\right)v_0^2\frac{k_F^4}{\mu^4} \times\int_{q>q_1}\frac{d^3q}{q^2}\tilde{f}\left(\frac{k_F}{\mu}q\right)^2, \qquad (18)$$

where $\int_D d^3k_1\,d^3k_2=\left(\frac{4\pi}{3}\right)^2$ has been used. Now,

$$\int_{q>q_1}\frac{d^3q}{q^2}\tilde{f}\left(\frac{k_F}{\mu}q\right)^2=\frac{\mu}{k_F}4\pi\int_{y_1}^{\infty}dy\tilde{f}(y)^2, \qquad (19)$$

where $y_1=k_Fq_1/\mu\ll 1$. If $\int_{y_1}^{\infty}dy\tilde{f}(y)^2$ in Eq. (19) did not depend on y_1, expression (18) did rigorously have the order $O(\mu r_s)^{-3}$. However, using the small-y expansion $\tilde{f}(y)=f_2+O(y^2)$, we have $\int_0^{y_1}dy\tilde{f}(y)^2=f_2^2y_1+O(y_1^3)$. Consequently, shifting the lower limit y_1 of the integral (19) to zero does not affect the leading-order contribution to expression (18),

$$e^{(2)}_{q>q_1}(r_s,\mu)=O(\mu r_s)^{-3}. \qquad (20)$$

Therefore, the quantity (17) does not contribute to the leading order of $e^{(2)}_{\mathrm{c}}=e^{(2)}_{\mathrm{dir}}+e^{(2)}_{\mathrm{ex}}$ which is purely due to expression (18),

$$e^{(2)}_{\mathrm{c}}(r_s,\mu)=-\frac{3}{4\pi}\left(\frac{2M}{\hbar^2\mu^2}\right)\frac{v_0^2}{(\mu r_s)^3}F+O(\mu r_s)^{-4}, \qquad (21)$$

where $F=\int_0^{\infty}dy\tilde{f}(y)^2$.

III. DENSITY SCALING AT UNITARITY

If the perturbation expansion is convergent [12], the total energy $E(r_s,\mu)=Ne(r_s,\mu)$ of the gas can be expressed in the form

$$e(r_s,\mu)=\frac{3}{5}E_F+e_x(r_s,\mu)+\sum_{n=2}^{\infty}e^{(n)}_{\mathrm{c}}(r_s,\mu). \qquad (22)$$

TABLE I. The moments f_2 and F of four different profiles $f(x)$ for the potential (2). λ_{uty} is the value at unitarity of the parameter λ in Eq. (24). At unitarity, the ratio Q of the LHS of Eq. (23) to the RHS is always close to 1.

$f(x)$	f_2	F	λ_{uty}	Q
$\Theta(1-x)$	$\frac{1}{3}$	$\frac{\pi}{15}$	$\frac{\pi^2}{4}$	0.987
$\mathrm{sech}(x)^2$	$\frac{\pi^2}{12}$	0.596	2	0.922
$\delta(1-x)$	1	$\frac{\pi}{2}$	1	1.000
$\exp(-x^2)$	$\frac{1}{4}\sqrt{\pi}$	$\frac{1}{8}\left(\frac{\pi}{2}\right)^{3/2}$	2.684	0.949

At unitarity, when the relative Hamiltonian (5) has a single bound state at zero energy, the exchange plus correlation energy $e_x+\Sigma_{n=2}^{\infty}e^{(n)}_{\mathrm{c}}$ should display the same density scaling as the kinetic energy, $\frac{3}{5}E_F\propto r_s^{-2}\propto n^{2/3}$. This is obviously not the case with any one of the leading-order results, Eqs. (9) and (21). However, since the exchange energy Eq. (9) and the second-order correlation energy Eq. (21) have opposite signs, they can cancel each other at some value of μ. This happens when

$$\lambda\equiv\frac{Mv_0}{\hbar^2\mu^2}=\frac{\pi f_2}{2F}, \qquad (23)$$

where $f_2=\int_0^{\infty}dx\,x^2f(x)$ and $F=\int_0^{\infty}dy\tilde{f}(y)^2$. This is the main result of our paper, and we check it by considering four different potentials. The results of this analysis, summarized in Table I, are discussed in detail in Sec. IV.

Generally, we need an eigenfunction $\psi(x)=\frac{u(x)}{x}$ of the relative Hamiltonian (5) with eigenvalue zero. The corresponding dimensionless Schrödinger equation reads

$$u''(x)=-\lambda f(x)u(x). \qquad (24)$$

Precisely, we wish to determine that particular value λ_{uty} of λ for which this zero-energy solution is the only bound state. Then, $u(x)$ must obey $u(0)=0$, $u'(x)>0$ for $x\geqslant 0$, and $u(x)\to$ const for $x\to\infty$. In the following examples (A)–(D), the solution $u(x)$ can be found analytically or numerically.

(A) Square-well potential of radius $R_0=1/\mu$,

$$v(r)=-v_0\Theta(R_0-r), \qquad (25)$$

where $\Theta(z)$ denotes the Heaviside step function, $\Theta(z)=1$ for $z>0$ and $\Theta(z)=0$ for $z\leqslant 0$. By setting the dimensionless variable $\mu r=x$, we see that $f(x)=\Theta(1-x)$. The square-well potential (25) supports a single zero energy bound state when the left-hand side (LHS) of Eq. (23) is $\lambda_{\mathrm{uty}}=\pi^2/4$. It may be easily checked analytically that for the square-well potential (25), $f_2=\frac{1}{3}$ and $F=\frac{\pi}{15}$ so that the right-hand side (RHS) of Eq. (23) is $\frac{5}{2}$, very close to its LHS, $\pi^2/4=2.47$.

(B) Rosen-Morse hyperbolic potential [1]. This potential is given by

$$v(r)=-v_0\,\mathrm{sech}^2(\mu r), \qquad (26)$$

which supports a single zero energy bound state when the LHS of Eq. (23) is $\lambda_{\mathrm{uty}}=2$ instead of $\pi^2/4$. For this potential, it is easy to check that $f_2=\pi^2/12$. The quantity F, however, must be calculated numerically, and is given by $F=0.596$. Again, Eq. (23) is approximately satisfied, since its RHS for this potential is 2.17.

(C) Delta-shell potential [14]. Consider the potential

$$v(r)=-\eta\frac{\hbar^2}{M}\delta(r-R_0)=-\eta\frac{\hbar^2}{M}\frac{1}{R_0}\delta\left(\frac{r}{R_0}-1\right)=-v_0f(\mu r). \tag{27}$$

Thus, we have $v_0=\eta\frac{\hbar^2}{MR_0}$, $\mu=\frac{1}{R_0}$, and $f(x)=\delta(x-1)$. So we get $f_2=1$, $\tilde{f}(y)=\frac{\sin y}{y}$, and $F=\frac{\pi}{2}$. Hence the RHS of Eq. (23) is unity. The LHS is (ηR_0), which is exactly unity when the s-state scattering length goes to infinity [14]. Thus Eq. (23) is exactly obeyed in this case.

(D) Gaussian potential,

$$v(r)=-v_0\exp(-\mu^2r^2). \tag{28}$$

For this example, $f(x)=\exp(-x^2)$ in Eq. (2). We find $f_2=\frac{1}{4}\sqrt{\pi}$ and $F=\frac{1}{8}(\frac{\pi}{2})^{3/2}$ so that the RHS of Eq. (23) becomes $\pi f_2/2F=2^{3/2}$. Solving Eq. (24) numerically for this $f(x)$, we obtain a single bound state at zero energy when the LHS of Eq. (23) is $\lambda_{\mathrm{uty}}=0.949\times2^{3/2}$, close to $2^{3/2}$.

IV. DISCUSSION

While the total energy per particle should, at unitarity, have the same density dependence as the noninteracting kinetic energy, our first-order direct and exchange (potential) energy terms given by Eqs. (7) and (9) do not. However, Table I shows that, at unitarity, the exchange energy Eq. (9) is cancelled almost exactly by the second-order correlation term, Eq. (21). Irrespective of the direct term, Eq. (7), which stays intact, this cancellation helps achieve the required density dependence.

To be more specific, note that the ground-state energy per particle of the Hamiltonian (3) is a function of the three independent parameters r_s, μ, and $\lambda\sim v_0$. Its perturbation expansion can be written in the form

$$\frac{E(r_s,\mu,\lambda)}{N}=\frac{\hbar^2\mu^2}{M}\sum_{\alpha=0}^{\infty}\varepsilon_\alpha(\mu r_s)\lambda^\alpha. \tag{29}$$

While α denotes the order in the perturbation expansion, we may further expand for low densities ($\mu r_s\gg1$),

$$\varepsilon_\alpha(\mu r_s)=\sum_{\beta=0}^{\infty}\frac{\varepsilon_{\alpha\beta}}{(\mu r_s)^\beta}. \tag{30}$$

The coefficients can be read off from our energy expressions. From the noninteracting kinetic energy [given by Eq. (1) with $\xi=1$], we find $\varepsilon_{02}=\frac{3}{10}(\frac{9\pi}{4})^{2/3}$ while $\varepsilon_{0\beta}=0$ for $\beta\neq2$. Equations (7) and (9) imply that $\varepsilon_{1\beta}=0$ for $\beta<3$ and $\varepsilon_{13}=(-\frac{3}{2}+\frac{3}{4})f_2$. Eventually, due to Eq. (21), $\varepsilon_{2\beta}=0$ for $\beta<3$ and $\varepsilon_{23}=(-\frac{3}{4\pi})2F$. Our result is, that, at unitarity, ε_{23} is almost exactly the negative of the contribution $\frac{3}{4}f_2$ to ε_{13} (Table I). Consequently, an essential part of the $r_s^{-3}\propto n$ contribution to the low-density expansion of E/N nearly cancels out at unitarity.

Note that small contributions $O(r_s^{-3})$ (with coefficients $\varepsilon_{\alpha3}$, $\alpha\geqslant3$) may also come from higher-order terms of the perturbation expansion, since it is carried out with respect to the parameter λ, but not $1/\mu r_s$. For a dilute gas with $\mu r_s\gg1$, the expansions (30) should converge rapidly. The parameter λ of the perturbation expansion (29), however, is not small at unitarity, see Table I. Due to the short range of the interaction, however, the coefficients $\varepsilon_{\alpha3}$ with $\alpha\geqslant3$ are expected to be small [13].

In passing, we note that our first-order direct and exchange (potential) energy terms given by Eqs. (7) and (9) are the same as those obtained in the Hartree-Fock calculation [see, for example, Eq. (10) of Heiselberg [10]].

Two limitations of our result must be pointed out. First, the direct first-order term, Eq. (7) (which corresponds to the contribution $-\frac{3}{2}f_2$ to ε_{13}), is not canceled. In the electron gas, the direct term is repulsive and gets canceled by the interaction of the electrons with the positive ionic background. There is no such mechanism of cancellation here, unless we assume, rather arbitrarily, that the short-range interatomic repulsion cancels this direct (attractive) contribution. Even without any such assumptions, however, our main result (Table I), applicable at a Feshbach resonance, is interesting from the perspective of potential theory.

Second, at extremely low densities $\mu r_s\gg1$, our exchange term (9) which for the square-well potential is given by

$$e_x(r_s,\mu)=\frac{\pi}{18}\left(\frac{9\pi}{4}\right)^{1/3}\frac{E_F}{\mu r_s}, \tag{31}$$

becomes negligible against the noninteracting kinetic energy $\frac{3}{5}E_F$. In this case, the cancellation of the leading exchange and second-order correlation terms at unitarity would be of academic interest only. However, taking a modestly large value, $\mu r_s=3$, we obtain the ratio of e_x to $\frac{3}{5}E_F$ to be about 0.56. This is a considerable proportion and its cancellation by the second-order correlation term is quite significant.

So far as the unitary point is concerned, we are interested in a situation where $k_F|a|\gg1\gg k_FR_0\sim(\mu r_s)^{-1}$. For the example (A) of the square-well potential in Sec. III, we have

$$(k_Fa)=\left(\frac{9\pi}{4}\right)^{1/3}\frac{1}{(\mu r_s)}\left(1-\frac{\tan\sqrt{\lambda}}{\sqrt{\lambda}}\right). \tag{32}$$

At unitarity ($\lambda=\pi^2/4$), the RHS diverges for any finite value of μr_s, however large. Even in the neighborhood of unitarity, it is possible to have $k_F|a|\gg1$ for $\mu r_s\gg1$.

In summary, we have demonstrated that important potential-energy contributions that do not have the same density dependence as the noninteracting kinetic energy $\frac{3}{5}E_F$, cancel out each other almost exactly at unitarity. This, however, does not include the direct first-order term, Eq. (7), which seems to need some extra counterpiece to get canceled.

We conclude by emphasizing that the result in this paper is displayed in Table I, and should be of interest from the point of view of potential theory.

ACKNOWLEDGMENTS

The authors would like to thank Brandon van Zyl for discussions. One of the authors (M. S.) gratefully acknowledges the warm hospitality at McMaster University where part of this work was done. This research was financed by NSERC of Canada.

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