Controllability of the coupled spin-$\frac{1}{2}$ harmonic oscillator system

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We present a control-theoretic analysis of a system consisting of a two-level atom coupled with a quantum harmonic oscillator. We show that, by applying external fields with just two resonant frequencies, any desired unitary operator can be generated.

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I. INTRODUCTION

In this paper, we apply theoretical concepts of quantum control to the joint system consisting of a two-state system coupled with a quantum harmonic oscillator. Such systems are ubiquitous in Nature. For example, coupled atom-oscillator systems form the basis for the ion trap quantum computer [1]. Other examples include a single atom in a cavity [2], a superconducting qubit in a cavity [3], and control of single-atom lasers [4,5]. In [6], Law and Eberly showed that arbitrary states can be synthesized by using just two resonant frequencies, a result experimentally verified in [7], and [8] showed that the two-level atom-oscillator system could be controlled by fine tuning the Lamb-Dicke parameter. Here we prove that the dynamics of such systems is controllable without any fine tuning or special state preparation: with the proper sequence of pulses, it is possible to perform any desired unitary transformation on the Hilbert space spanned by the atomic states together with the lowest n energy levels of the oscillator.

In this paper, we will use the ion trap as our model system. An ion trap quantum computer can be modeled as a collection of N particles with spin $\frac{1}{2}$ in a one-dimensional harmonic potential. Laser pulses incident on the ions can be tuned to simultaneously cause internal spin transitions and vibrational (phonon) excitations, thus allowing local internal states to be mapped into shared phonon states. The computational qubits are encoded by two internal states of each ion and the collective vibration of the trapped ions acts as the information bus. In this manner, quantum information can be communicated between any pair of ions and logic gates can be performed. Several key features of the original proposal in [1], including the production of entangled states and the implementation of quantum controlled operations between a pair of trapped ions, have already been experimentally demonstrated (see, e.g., [9–12]). Meanwhile, several alternative theoretical schemes (see, e.g., [8,13–17]) have also been developed for overcoming various difficulties in realizing a practical ion-trap quantum information processor. All these proposals require either fine-tuning of the Lamb-Dicke parameter or an initial eigenstate of the vibration motion. Here we present a control theoretical analysis and show that, in the Lamb-Dicke regime by using two resonant frequencies, any unitary transformation within a finite level of the harmonic oscillator can be generated. Unlike in, e.g., [8], no fine tuning of the Lamb-Dicke parameter is required to obtain complete control. While the proof of controllability is somewhat involved, because of the fundamental nature of the system to be controlled and because of the wide range of potential application, we present this proof in detail. As will be seen below, the difficulty of the proof arises because, in the absence of controllability of the Lamb-Dicke parameter, one must combine discrete and continuous control-theoretic techniques. The current proof can be regarded as extending the techniques of the papers [17,18] from controlling four states to controlling m states, where m can be arbitrarily large. We begin in Sec. II by presenting the usual Jaynes-Cumming model for spin boson interactions. We then make the controllability analysis of the system in Sec. III.

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II. LASER-ION INTERACTION MODEL

The physical situation we consider is a two-state atom (frequency ω_c) coupled to a harmonic oscillator (transition frequency ω_z), driven additionally by an external field (frequency ω) as shown in Fig. 1. We will follow the ion trap model [19,17]. The free Hamiltonian of this system is $H_0 = \hbar\omega_c\sigma_z/2 + \hbar\omega_z a^\dagger a$, where σ_z is a Pauli spin operator and a annihilates a phonon. Turning on the electromagnetic field of a laser gives an interaction Hamiltonian

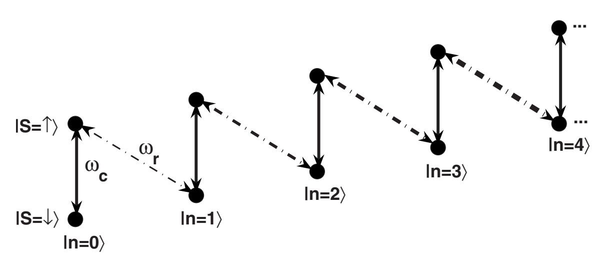

FIG. 1. Graphical representation of the quantum harmonic oscillator driven by sinusoidal resonant field fields ω_c and $\omega_r = \omega_c - \omega_z$ as shown. The strengths of the ω_c transition couplings are independent of the harmonic oscillator quantum number n, whereas the strength of the ω_r transition couplings increases as the square root of the quantum number n.

$$H_I = -\vec{\mu}\cdot\vec{B}, \tag{1}$$

where $\vec{\mu}=\mu\vec{\sigma}/2$ is the magnetic moment of the ion and $\vec{B}=B\hat{x}\cos(kz-\omega t+\Phi)$ is the magnetic field produced by the laser. Here $z=z_0(a+a^\dagger)$, where $z_0=\sqrt{\hbar/2Nm\omega_z}$ is a characteristic length scale for the motional wave functions and m is the mass of an ion.

We consider the regime in which $\eta\equiv kz_0\ll 1$. In this regime, we may determine the effect of a laser pulse at a specific frequency ω by expanding Eq. (1) in powers of η and neglecting rapidly rotating terms. Then pulsing on resonance ($\omega=\omega_c$) allows one to perform the transformation

$$R(\theta,\phi) = \exp[i\theta(e^{i\phi}\sigma^+ + e^{-i\phi}\sigma^-)], \tag{2}$$

and pulsing at the red sideband frequency ($\omega=\omega_c-\omega_z$) gives

$$R^-(\theta,\phi) = \exp[i\theta(e^{i\phi}\sigma^+ a^\dagger + e^{-i\phi}\sigma^- a)]. \tag{3}$$

In each case, the parameter θ depends on the strength and duration of the pulse and ϕ depends on its phase. In the next section, we show that by just using these two frequencies any unitary operator can be generated. The basic idea in proving controllability is an extension of [17,18]. Using the feature that the transition frequencies increase as the square root of the quantum number, we apply only pulses that leave the system confined within the Hilbert space spanned by the first n oscillator levels. This requirement means that only a discrete set of pulses can be applied at the red sideband frequency. Meanwhile, a continuous set of pulses can be applied at the resonance frequency. As a result of the use of both discrete and continuous controls, the resulting control problem is technically somewhat involved. Nonetheless, it can be solved completely, as we now show.

III. CONTROLLABILITY ANALYSIS

We denote as E_{pq} the matrix that has all the entries equal to zero except the pq entry, which equals 1. It is easy to check that $E_{pq}E_{rs}=\delta_q^r E_{ps}$.

The Hamiltonian, after absorbing the imaginary number i, can be represented as skew-Hermitian matrices. If we take the eigenstate of the free Hamiltonian as the basis, then, after rescaling the time unit, the various Hamiltonians in the interaction frame can be represented as

(1) $\omega=\omega_c$, $\phi=0$,

$$H_1 = i\sum_{k=0}^{\infty} E_{(2k+1)(2k+2)} + E_{(2k+2)(2k+1)} = i\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \tag{4}$$

(2) $\omega=\omega_c$, $\phi=\pi/2$,

$$H_2 = \sum_{k=0}^{\infty} E_{(2k+1)(2k+2)} - E_{(2k+2)(2k+1)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \tag{5}$$

(3) $\omega=\omega_c-\omega_z$, $\phi=0$,

$$H_3 = i\sum_{k=1}^{\infty} \sqrt{k}(E_{(2k)(2k+1)} + E_{(2k+1)(2k)}) = i\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \tag{6}$$

and (4) $\omega=\omega_c-\omega_z$, $\phi=\pi/2$,

$$H_4 = \sum_{k=1}^{\infty} \sqrt{k}(E_{(2k)(2k+1)} - E_{(2k+1)(2k)}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{7}$$

When the Hamiltonian H_3 or H_4 is applied, $|\uparrow\rangle|m\rangle$ is connected to $|\downarrow\rangle|m+1\rangle$. We restrict the evolution time T under these two Hamiltonians to satisfy $T\sqrt{m}=k\pi$, while k is integer, so the subspace of states spanned by $\{|\downarrow,\uparrow\rangle|j\rangle|j \leq m\}$ is preserved. We show that under these restrictions any unitary matrix within any finite harmonic level can still be generated.

A. SU(4)

Let us first work out the case of $m=1$, showing that we can generate SU(4) on the subspace spanned by states

$$|\downarrow\rangle|0\rangle, \quad |\uparrow\rangle|0\rangle, \quad |\downarrow\rangle|1\rangle, \quad |\uparrow\rangle|1\rangle.$$

Restricted to this subspace,

$$H_1 = i\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

and the unitary operators we can generate using H_3 and H_4 are

$$R^{-}\left(\frac{k\pi}{\sqrt{2}},0\right) = \exp\left(\frac{k\pi}{\sqrt{2}}H_3\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(k\pi/\sqrt{2}) & i\sin(k\pi/\sqrt{2}) & 0 \\ 0 & i\sin(k\pi/\sqrt{2}) & \cos(k\pi/\sqrt{2})0 & 0 \\ 0 & 0 & 0 & (-1)^k \end{bmatrix}, \tag{8}$$

$$R^{-}\left(\frac{k\pi}{\sqrt{2}},\frac{\pi}{2}\right) = \exp\left(\frac{k\pi}{\sqrt{2}}H_4\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(k\pi/\sqrt{2}) & \sin(k\pi/\sqrt{2}) & 0 \\ 0 & -\sin(k\pi/\sqrt{2}) & \cos(k\pi/\sqrt{2}) & 0 \\ 0 & 0 & 0 & (-1)^k \end{bmatrix}. \tag{9}$$

Choosing $k=2p$ and varying p, $R^{-}(k\pi/\sqrt{2},\pi/2)$ forms a dense subset of the one-parameter group

$$\exp\left(t\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right).$$

Thus we have the generator

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Adding it to H_2, we get

$$H_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Choosing $k=2p+1$, we have

$$U_1 = R^{-}\left(\frac{k\pi}{\sqrt{2}},\frac{\pi}{2}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(k\pi/\sqrt{2}) & \sin(k\pi/\sqrt{2}) & 0 \\ 0 & -\sin(k\pi/\sqrt{2}) & \cos(k\pi/\sqrt{2}) & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{10}$$

Since U(4) is compact, the infinite sequence $\{U_1, U_1^2, U_1^3, U_1^4, \ldots\}$ has a convergent subsequence, i.e., there exists $p_1 > p_2 \in N$ such that $U_1^{p_1} - U_1^{p_2}$ is arbitrarily close to zero. When this is true then $U_1^{p_1-p_2-1}$ is arbitrarily close to U_1^{-1}, i.e., U_1^{-1} can be approximately generated to arbitrary accuracy. But

$$U_1^{-1}H_1U_1 = i\begin{bmatrix} 0 & \cos(k\pi/\sqrt{2}) & \sin(k\pi/\sqrt{2}) & 0 \\ \cos(k\pi/\sqrt{2}) & 0 & 0 & \sin(k\pi/\sqrt{2}) \\ \sin(k\pi/\sqrt{2}) & 0 & 0 & -\cos(k\pi/\sqrt{2}) \\ 0 & \sin(k\pi/\sqrt{2}) & -\cos(k\pi/\sqrt{2}) & 0 \end{bmatrix}. \tag{11}$$

Choosing k such that $k/\sqrt{2}$ is arbitrarily close to an integer, we can get the transformation

$$i\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Subtracting this from H_1 and dividing by a factor 2 yields

$$H_6 = i\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Similarly, by using H_2 and U_1, we can get

$$H_7 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

We now show that $\{H_5, H_6, H_7\}$ generate all the skew-Hermitian matrices on the subspace.

First,

$$H_8 = H_5 - H_7 = \sum_{k=1}^{N-2} E_{k(k+1)} - E_{(k+1)k}.$$

Here $N=4$. We will do the following computation using the general N, as this will be used for the proof of the general case. Now,

$$H_7 = E_{(N-1)N} - E_{N(N-1)}.$$

We first show that H_7 and H_8 generate all the real skew-symmetric matrices of size $N\times N$ [20,21]. Let

$$M_{N-1} = H_7 = E_{(N-1)N} - E_{N(N-1)},$$

$$M_{N-2} = [H_8, M_{N-1}] = E_{(N-2)N} - E_{N(N-2)},$$

$$M_{N-3} = [H_8, M_{N-2}] + M_{N-1} = E_{(N-3)N} - E_{N(N-3)},$$

$$M_{N-4} = [H_8, M_{N-3}] + M_{N-2} = E_{(N-4)N} - E_{N(N-4)},$$

$$\vdots$$

$$M_1 = [H_8, M_2] + M_3 = E_{1N} - E_{N1},$$

and $[M_p, M_q] = E_{qp} - E_{pq}$, $\forall\, p \neq q \in \{1,2,\ldots,N-1\}$. Thus we can generate the complete basis for skew-symmetric matrices. Similarly,

$$J_{N-1} = H_6 = i(E_{(N-1)N} + E_{N(N-1)}),$$

$$J_{N-2} = [H_8, J_{N-1}] = i(E_{(N-2)N} + E_{N(N-2)}),$$

$$J_{N-3} = [H_8, J_{N-2}] + J_{N-1} = i(E_{(N-3)N} + E_{N(N-3)}),$$

$$\vdots$$

$$J_1 = [H_8, J_2] + J_3 = i(E_{1N} + E_{N1}),$$

and

$$[M_q, J_p] = i(E_{qp} + E_{pq}),$$

$$[i(E_{qp} + E_{pq}), E_{qp} - E_{pq}] = 2i(E_{pp} - E_{qq})$$

$\forall\, p \neq q \in \{1,2,\ldots,N-1\}$. So we can generate a full basis for all $N\times N$ skew-Hermitian matrices. This proves the SU(4) case.

B. General case

Now we generalize our proof to controllability on SU(n) for any n. It is not necessary to check the case for each n; as SU(n_1) is a subgroup of SU(n_2), for $n_1 < n_2$, the controllability on SU(n_2) implies controllability on SU(n_1). It is sufficient to prove the result for infinitely many n_i as $n_i \to \infty$.

Take the subspace up to harmonic level m, i.e.,

$$\{|\downarrow\rangle|0\rangle, |\uparrow\rangle|0\rangle, |\downarrow\rangle|1\rangle, |\uparrow\rangle|1\rangle, \ldots, |\downarrow\rangle|m\rangle, |\uparrow\rangle|m\rangle\},$$

where $(m-1, m+1)$ are both prime. We shall prove the controllability on this subspace. The twin-prime conjecture claims that there exist infinitely many such primes. If the twin-prime conjecture is false, then the following proof works only up to $n=2m+2$, where m is the largest known twin prime. As of 2006, the largest known twin prime is $100\,314\,512\,544\,015 \times 2^{171960} \pm 1$, which is large enough for most physical systems. Below, we generalize the twin-prime proof to show controllability for all n.

If we restrict the evolution time T for H_4 to satisfy $T\sqrt{m+1} = k\pi$, where k is an integer, then the angle rotated between $|\uparrow\rangle|p-1\rangle$ and $|\downarrow\rangle|p\rangle$ is $\sqrt{p}T = k\sqrt{p/(m+1)}\pi$. We divide the numbers $\{1,2,\ldots,m\}$ into groups G_i, $i=1,2,\ldots$, such that in the same group G_i the angles rotated under the above evolution are rationally related to each other, i.e.,

p_1, p_2 are in the same group if and only if $\sqrt{p_1/p_2}$ is a rational number. For example, $\{1, 1\times 2^2, 1\times 3^2, \ldots, 1\times p_1^2\}$ forms a group, where $p_1^2 \le m$, $(p_1+1)^2 > m$; similarly other groups are $\{2, 2\times 2^2, 2\times 3^2, \ldots, 2\times p_2^2\}$, $\{3, 3\times 2^2, 3\times 3^2, \ldots, 3\times p_2^2\}, \ldots$; in particular, $\{m-1\}$ itself forms a group.

As $m+1$ is a prime number, $k\sqrt{p/(m+1)}$ (mod 2) are irrational numbers for all $p \le m$. Accordingly, we can vary k such that, except for the angles related to one group G_i, all the other angles are arbitrarily close to zero. In this way we can construct the generator

$$\hat{H}_i = \sum_{j\in G_i} \sqrt{j}(E_{2j(2j+1)} - E_{(2j+1)2j}).$$

Adding all $\hat{H}_i$ to H_2, we get a matrix similar to H_5 in the SU(4) section, with only nonzero entries on the first off diagonal. Denote this matrix by $\tilde{H}_5$.

To prove controllability, we just need to show that we can also generate matrices similar to H_6 and H_7, i.e.,

$$E_{(N-1)N} - E_{N(N-1)}$$

and

$$i(E_{(N-1)N} + E_{N(N-1)});$$

here $N=2m+2$.

As $\{m-1\}$ itself forms a group, say G_j, we can generate

$$S_1 = \frac{1}{\sqrt{m-1}}\hat{H}_j = E_{(2m-2)(2m-1)} - E_{(2m-1)(2m-2)}.$$

Bracketing it with $H_2 = \sum_{k=0}^{m} E_{(2k+1)(2k+2)} - E_{(2k+2)(2k+1)}$, we can get

$$S_2 = [H_2, S_1] = E_{(2m-3)(2m-1)} - E_{(2m-1)(2m-3)} + E_{2m(2m-2)} - E_{(2m-2)2m}.$$

Then bracketing S_2 with S_1,

$$S_3 = [S_1, S_2] = E_{(2m-3)(2m-2)} - E_{(2m-2)(2m-3)} + E_{(2m-1)2m} - E_{2m(2m-1)},$$

we see that S_3 is nothing but the restriction of H_2 on the subspace spanned by

$$|\downarrow\rangle|m-2\rangle, \quad |\uparrow\rangle|m-2\rangle, \quad |\downarrow\rangle|m-1\rangle, \quad |\uparrow\rangle|m-1\rangle.$$

Similarly, H_1 can also be restricted to this subspace. From the SU(4) case, we know we can generate any skew-Hermitian matrix on this subspace; specifically we can have $S_4 = E_{(2m-1)2m} - E_{2m(2m-1)}$.

Now pick the group G_p to which m belongs. We get

$$\hat{H}_p = \sum_{j\in G_p} \sqrt{j}(E_{2j(2j+1)} - E_{(2j+1)2j}) = E_{2m(2m+1)} - E_{(2m+1)2m} + \sum_{j\neq m\in G_p} \sqrt{j}(E_{2j(2j+1)} - E_{(2j+1)2j}). \quad (12)$$

Bracketing S_4 with $\hat{H}_p$, since all the numbers in G_p have the form m/q^2, the second term in the right side of the above equation commutes with S_4. Accordingly we obtain

$$S_5 = [S_4, \hat{H}_p] = E_{(2m-1)(2m+1)} - E_{(2m+1)(2m-1)}.$$

Now, bracketing S_5 with S_4,

$$S_6 = [S_5, S_4] = E_{2m(2m+1)} - E_{(2m+1)2m}.$$

Comparing with S_1, we see that we just moved one block down. Repeating what we did with S_1 to S_6, we can get

$$S_7 = E_{(2m+1)(2m+2)} - E_{(2m+2)(2m-1)} = E_{(N-1)N} - E_{N(N-1)}. \quad (13)$$

This is the matrix we need to generalize our proof of controllability on SU(4). Similarly, we can get

$$i(E_{(N-1)N} + E_{N(N-1)}).$$

Together with $\tilde{H}_5$, we are able to generate all the skew-Hermitian matrices of size $N\times N$, which proves the controllability on SU(N). This completes the proof: Driving the fundamental frequency and the red sideband suffices to control the two-level atom coupled to a harmonic oscillator.

Remark 1. From the proof we see that the only two properties of the pair $(m-1, m+1)$ we used are (1) $\sqrt{p/(m+1)}$ are irrational for all $p \le m$; and (2) there exists one group consisting of only one number. It is convenient to pick twin primes, but there exist other choices. For example, we can choose $m+1 = 2q$, where q is an odd prime. Under this choice condition 1 still holds, and q itself forms a group. So our proof, while expressed in terms of the twin-prime conjecture, actually holds for all n.

IV. DISCUSSION

We have proved the controllability of the dynamics of a coupled two-level system and harmonic oscillator. Because of the discrete nature of the controls, the proof was somewhat involved. In addition, the system is fully controllable only in the limit that the number of control pulses goes to infinity. In any realistic setting we will have only a finite time and a finite number of pulses that we can apply. The question of the rate of convergence of such discrete schemes is an important open question in control theory and in quantum information, and will be investigated elsewhere. For the moment, we note only that accurate generation of arbitary members of SU(4) and SU(n) for $n \le 10$ or so via the techniques described here is well within the reach of current experiment.

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