# Scattering of polarized laser light by an atomic gas in free space: A quantum stochastic differential equation approach

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We propose a model, based on a quantum stochastic differential equation (QSDE), to describe the scattering of polarized laser light by an atomic gas. The gauge terms in the QSDE account for the direct scattering of the laser light into different field channels. Once the model has been set, we can rigorously derive quantum filtering equations for balanced polarimetry and homodyne detection experiments, study the statistics of output processes, and investigate a strong driving, weak coupling limit.

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## I. INTRODUCTION

Many recent experimental [1–4] and theoretical [5–10] works have been based on a simple experimental scenario, in which a polarized atomic gas is continuously probed with a polarized off-resonant optical beam (Fig. 1). By measuring the Faraday rotation of the optical polarization resulting from the interaction, one can in principle prepare conditionally spin-squeezed states or perform quantum metrology tasks, e.g., estimating a magnetic field that rotates the spins.

Central to the description of these experiments is the *quantum filtering equation*, which propagates the expectation value of the atomic gas observables conditioned on prior measurement results. The conditional expectation is the mean least squares estimate of an atomic gas observable given the observations thus far. The conditional expectations of "all" atomic observables can be summarized in an information state $\pi_t$. The filtering equation propagates this information state in real time.

In quantum optics the filtering equation is often referred to as the *stochastic master equation* [11]. For the polarimetry example considered here, previous modelling efforts have either produced an unconditional description [8] or arrived at a conditional description by heuristically "adding the usual measurement terms" [4] in analogy with a physicallly different homodyne measurement scheme with only a single polarization mode [6,7]. In this paper we treat the conditional evolution of the state (due to detection of Faraday rotation with a polarimeter) in a rigorous manner, allowing the atomic system to mediate exchange between two orthogonal optical polarization modes. In particular, we derive the quantum filtering equation from an underlying quantum stochastic model, i.e., the quantum stochastic differential equation (QSDE) governing the interaction of the atomic gas with the laser light.

Formal quantum filtering theory was pioneered by Belavkin in [12,13] using martingale techniques (see also [14]). We here employ the reference probability method, based on the quantum Bayes formula [15,16], to obtain the quantum filter from the QSDE (see also [17]).

The QSDE model we use here is based on a simple Faraday Hamiltonian, $H=\kappa F_z S_z$, where $\kappa$ is a small interaction strength prefactor, $S_z$ is a Stokes operator measuring the circularity of optical polarization, and $F_z$ is the $z$ component of the collective atomic spin. Under this Hamiltonian, photons with a right circular polarization rotate the collective atomic spin over a positive angle $\kappa$ along the $z$ axis, while photons with a left circular polarization rotate the collective spin over a negative angle $-\kappa$. With linearly polarized light, the angle of linear polarization will Faraday rotate by a degree proportional to the $z$ component of the spin. Note that we entirely neglect "tensor" terms of the interaction Hamiltonian (non-

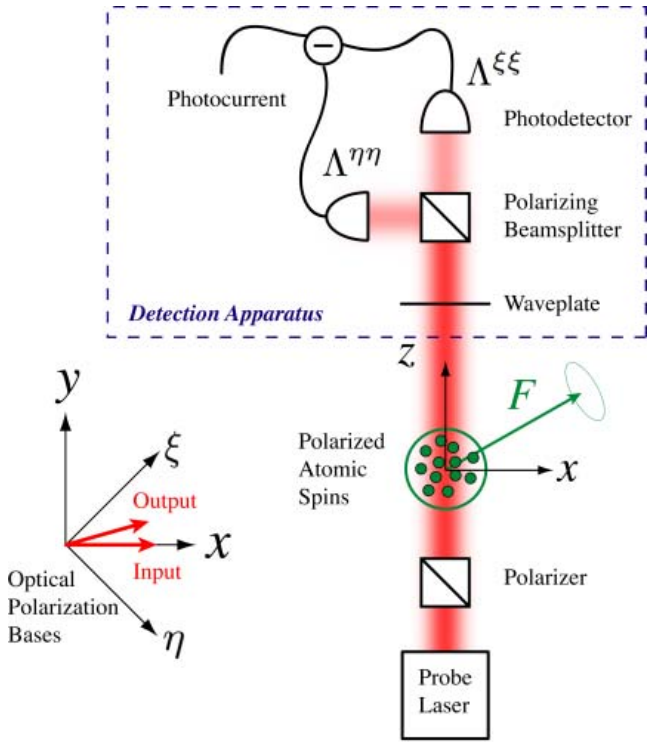


FIG. 1. (Color online) Schematic depicting balanced polarimetric detection of laser light after interacting with a polarized cloud of atomic spins via the Faraday Hamiltonian. The light is initially linearly polarized along the $x$ direction. After the interaction, the light carries off information about the atomic gas encoded in a small optical polarization rotation. The light is measured in the $\xi$-$\eta$ basis rotated 45° from the $x$-$y$ basis, such that without the atomic gas the mean output of the polarimeter is balanced to zero. The change of measurement basis is achieved with the waveplate located just before the polarizing beamsplitter.

 

linear in individual spin operators) which are important near resonance with realistic atoms of spin greater than 1/2 [4]. We have also omitted the evolution due to any driving magnetic field, e.g., $H=\gamma BF_y$, purely for reasons of simplicity, it can easily be added at the end.

In our QSDE description, the Faraday interaction is described as a "direct" scattering process, without coherent absorption and reemission. This is a consequence of the fact that the interaction Hamiltonian is derived from an approximation in which the excited states are adiabatically eliminated [4]. At present, however, no mathematically rigorous treatment of this elimination is available in the literature (see [18] for rigorous results on the adiabatic elimination of a leaky cavity mode). Therefore, we have chosen to directly base our QSDE model on the Faraday Hamiltonian without proceeding through a rigorous Markov limit [19,20] followed by adiabatic elimination of the excited states. Mathematically, the direct scattering is represented by gauge terms in the QSDE [21].

Having set the underlying model, i.e., the QSDE, we rigorously derive the quantum filtering equation for the balanced polarimetry setup and for homodyne detection of the $y$-polarized channel. We investigate the statistics of the output processes for these two experiments and take a limit where the driving laser power $\alpha^2$ goes to infinity but where the product $M=\kappa^2\alpha^2$ is kept constant ($\kappa$ is the parameter that couples the field to the atomic gas). We show that in this strong driving, weak coupling limit the statistics of the output processes for the balanced polarimetry experiment and the homodyne detection experiment are equivalent. Furthermore, we show that in the strong driving, weak coupling limit we obtain the quantum filter that has already been intuitively assumed in the literature [4].

The remainder of this paper is organized as follows. Section II introduces the fundamental noises and the quantum stochastic calculus, and Sec. III sets our QSDE model. Section 4 derives the filter when counting in the 45°-rotated $xy$ basis (balanced polarimetry), and Sec. V derives the quantum filter for the homodyne detection experiment. In Secs. VI and VII we study the statistics of the observation processes, and investigate the strong driving, weak coupling limit. We close the paper with a discussion of the results obtained.

## II. THE QUANTUM CALCULUS

One polarized photon in a beam of light can be described by the one particle space

$$\mathcal{H}=\mathbb{C}^2\otimes L^2(\mathbb{R})\cong L^2(\mathbb{R};\mathbb{C}^2),$$

of $\mathbb{C}^2$-valued quadratically integrable functions on the real line. The polarized light field is described by the *bosonic Fock space* $\mathcal{F}(\mathcal{H})$ over $\mathcal{H}$

$$\mathcal{F}(\mathcal{H})=\mathbb{C}\oplus\bigoplus_{n=1}^{\infty}\mathcal{H}^{\otimes_s n},$$

which enables arbitrary superpositions between states with a different number of photons. Note that photons are bosons and therefore need to be described by symmetric wave functions. For an $f\in\mathcal{H}$ we can define the *exponential vector* $e(f)$ in $\mathcal{F}(\mathcal{H})$ by

$$e(f)=1\oplus\bigoplus_{n=1}^{\infty}\frac{1}{\sqrt{n!}}f^{\otimes n}.$$

We call the span of the exponential vectors the *exponential domain*. The exponential domain is a dense set in $\mathcal{F}(\mathcal{H})$ and we allow ourselves the freedom to only provide the definition of the fundamental noises (a little further below) on this domain. If we normalize the exponential vectors then we obtain the coherent vectors $\psi(f)=\exp(-\frac{1}{2}\|f\|^2)e(f)$. An important vector is the *vacuum vector*, given by $\Phi=\psi(0)=e(0)=1\oplus 0\oplus 0\cdots$. The *vacuum state* $\phi=\langle\Phi,\cdot\,\Phi\rangle$ is obtained by taking inner products with the vacuum vector.

If we choose an orthonormal basis $\{e_1,e_2\}$ in $\mathbb{C}^2$ then we can decompose every $f\in L^2(\mathbb{R};\mathbb{C}^2)$ along this basis, i.e., $f=f_1e_1+f_2e_2$ with $f_1$ and $f_2$ in $L^2(\mathbb{R})$. We now introduce the fundamental noises $A_t^i$, $A_t^{i^*}$, and $\Lambda_t^{ij}$ on the exponential domain by (see also [21–23])

$$A_t^ie(f)=\left(\int_0^t f_i(s)ds\right)e(f),$$

$$\langle e(g),A_t^{i^*}e(f)\rangle=\left(\int_0^t\bar{g}_i(s)ds\right)\langle e(g),e(f)\rangle,$$

$$\langle e(g),\Lambda_t^{ij}e(f)\rangle=\left(\int_0^t\bar{g}_i(s)f_j(s)ds\right)\langle e(g),e(f)\rangle. \qquad (1)$$

$A_t^i$ and $A_t^{i^*}$ are called the *annihilation* and *creation* processes, respectively. The processes $\Lambda_t^{ij}$ are called *gauge* processes. Formally, we can write the noises as $A_t^i=\int_0^t a_s^i ds$, $A_t^{i^*}=\int_0^t a_s^{i^*}ds$, and $\Lambda_t^{ij}=\int_0^t a_s^{i^*}a_s^j ds$ where $a_t^i$ and $a_s^j$ are the usual Bose fields. Mathematically, the objects $a_t^i$ and $a_s^j$ are ill-defined and therefore we resort to the definition of Eq. (1). The formal expressions do show very explicitly though, that the operator $\Lambda_t^{ii}$ counts the number of photons with a polarization in the $e_i$ direction up to time $t$ and that the operator $\Lambda_t^{ij}$ scatters the polarization of a photon from the $e_j$ direction to the $e_i$ direction.

We will usually work in the basis $\{\vec{e}_x,\vec{e}_y\}$ which physically corresponds to an orthonormal basis in the plane orthogonal to the direction of propagation of the light. Apart from this basis, we also use the *circular basis* given by $\{e_+=-(\vec{e}_x+i\vec{e}_y)/\sqrt{2},\ e_-=(\vec{e}_x-i\vec{e}_y)/\sqrt{2}\}$, and the 45 degrees rotated $xy$ basis given by $\{\{\vec{e}_\xi=(\vec{e}_x+\vec{e}_y)/\sqrt{2},\ \vec{e}_\eta=(\vec{e}_x-\vec{e}_y)/\sqrt{2}\}$. Given the definitions in Eq. (1) it is easy to work out how the noises transform under basis transformations. For example, we have

$$\Lambda_t^{++}=\tfrac{1}{2}(\Lambda_t^{xx}+\Lambda_t^{yy}-i\Lambda_t^{yx}+i\Lambda_t^{xy}).$$

Denote by $\mathfrak{h}$ the Hilbert space of the atomic gas. The space of the combined system of atomic gas and field together is then given by $\mathfrak{h}\otimes\mathcal{F}(\mathcal{H})$. Define $\mathcal{H}_{t]}=\mathbb{C}^2\otimes L^2(-\infty,t]$ and $\mathcal{H}_{[t}=\mathbb{C}^2\otimes L^2[t,\infty)$. For all $t$ the bosonic Fock

space splits in a natural way as a tensor product $\mathcal{F}(\mathcal{H}) = \mathcal{F}(\mathcal{H}_{[t]}) \otimes \mathcal{F}(\mathcal{H}_{[t}$. A process $L_s$, $(s \geqslant 0)$ on $\mathfrak{h} \otimes \mathcal{F}(\mathcal{H})$ is called *adapted* if $L_s$ acts nontrivially only on $\mathfrak{h} \otimes \mathcal{F}(\mathcal{H}_{s]})$ and is the identity on $\mathcal{F}(\mathcal{H}_{[s})$ for all $s \geqslant 0$.

Hudson and Parthasarathy [22] defined stochastic integrals of adapted processes $L_s$ against the fundamental noises, i.e., they gave meaning to the expression $X_t = X_0 + \int_0^t L_s dM_s$, where $M_s$ is one of the fundamental noises $A_s^i$, $A_s^{i*}$ or $\Lambda_s^{ij}$. The expression can be written in shorthand as $dX_t = L_t dM_t$. More importantly, Hudson and Parthasarathy [22] provided the calculus with which these stochastic integrals can be manipulated in calculations. The calculus consists of the following. Suppose $X_t$ and $Y_t$ are stochastic integrals, i.e., $dX_t = L_t^1 dM_t^1$ and $dY_t = L_t^2 dM_t^2$ where $L^1$ and $L^2$ are adapted processes and $M^1$ and $M^2$ are fundamental noises, then the product $X_t Y_t$ is itself a stochastic integral. Moreover, the product $X_t Y_t$ satisfies the following quantum Itô rule (partial integration rule):

$$d(X_t Y_t) = X_t dY_t + (dX_t) Y_t + dX_t dY_t,$$

where to evaluate $dX_t dY_t$ we use that the increment $dM_t$ of a fundamental noise commutes with all adapted processes, and products $dM_t^1 dM_t^2$ are given by the following quantum Itô table [22]:

| $dM^1$ \ $dM^2$ | $dA_t^{i*}$ | $d\Lambda_t^{ij}$ | $dA_t^i$ |
|---|---|---|---|
| $dA_t^{k*}$ | 0 | 0 | 0 |
| $d\Lambda_t^{kl}$ | $\delta_{li} dA_t^{k*}$ | $\delta_{li} d\Lambda_t^{kj}$ | 0 |
| $dA_t^k$ | $\delta_{ki} dt$ | $\delta_{ki} dA_t^j$ | 0 |

and all products $dM_t dt$ and $dt dM_t$ are zero. As an example, suppose $dX_t = L_t^1 dA_t^i$ and $dY_t = L_t^2 dA_t^{i*}$, then $d(X_t Y_t) = X_t L_t^2 dA_t^{i*} + L_t^1 Y_t dA_t^i + L_t^1 L_t^2 dt$.

It can be shown [19,20] that in the weak coupling limit (a Markov limit) QED models converge to quantum stochastic models, i.e., in the limit the unitary time evolution $U_t$ satisfies a quantum stochastic differential equation in the sense of Hudson and Parthasarathy. Usually, the QSDE obtained via the weak coupling limit can be simplified further by adiabatic elimination of degrees of freedom of the initial system. See for instance [18] for rigorous results on the adiabatic elimination of a leaky cavity. We, however, are interested in the adiabatic elimination of the excited states of the atoms in the atomic gas. Unfortunately, at present, no rigorous results on this kind of adiabatic elimination are available. Therefore we choose not to go through a weak coupling limit and/or adiabatic elimination procedure here, but rather write down a phenomenological QSDE based on the Faraday interaction, see Eqs. (3) and (6) below. See [4] for a derivation of the Faraday Hamiltonian of Eq. (2) via usual nonrigorous adiabatic elimination methods.

## III. THE MODEL

The interaction between the laser light and the spin polarized atomic gas is governed by the Faraday interaction given by

$$H dt = 2\kappa F_z S_z dt = \kappa F_z (d\Lambda_t^{++} - d\Lambda_t^{--}). \tag{2}$$

Here $\kappa$ is a coupling parameter, $F_z$ is the $z$ component of the collective spin vector of the atoms, and $2S_z = a_t^{+*} a_t^+ - a_t^{-*} a_t^-$ is the $z$ component of the Stokes vector $\mathbf{S}$ of the polarized light. The time evolution of the coupled system of light and atomic gas together is given by the exponential (the superscript 0 distinguishes $U_t^0$ from $U_t$ to be introduced later)

$$U_t^0 = \exp\left( i \int_0^t \kappa F_z (d\Lambda_s^{++} - d\Lambda_s^{--}) \right).$$

Since $\Lambda_t^{++}$ and $\Lambda_t^{--}$ are jump processes, the Itô rule leads to a quantum stochastic differential equation [22] that contains the following difference terms $(U_0^0 = I)$:

$$dU_t^0 = [(e^{i\kappa F_z} - 1) d\Lambda_t^{++} + (e^{-i\kappa F_z} - 1) d\Lambda_t^{--}] U_t^0. \tag{3}$$

That is, right circular polarized photons rotate the collective spin of the atoms over an angle $\kappa$ along the $z$ axis, whereas left circular polarized photons rotate the collective spin of the atoms over an angle $-\kappa$ along the $z$ axis. If we express the gauge processes in the linearly polarized $xy$ basis, then Eq. (3) reads $(U_0^0 = I)$,

$$dU_t^0 = \{[\cos(\kappa F_z) - 1](d\Lambda_t^{xx} + d\Lambda_t^{yy}) - \sin(\kappa F_z)(d\Lambda_t^{xy} - d\Lambda_t^{yx})\} U_t^0. \tag{4}$$

The second term shows that the interaction can scatter $x$-polarized photons to $y$-polarized photons and vice versa.

Initially, the atomic gas is in an $x$-spin polarized state, denoted $\rho$, and the field is in an $x$-polarized coherent state $\psi^x(f)$ which represents the driving laser. The function $f \in L^2(\mathbb{R}^+)$ gives the phase and amplitude of the driving laser field at every time $t \in \mathbb{R}^+$. In computations it is often convenient to work with respect to the vacuum state $\phi = \langle \Phi, \cdot \Phi \rangle$ for the field. We can obtain a coherent state by acting with a *displacement* or *Weyl operator* $W^x(f)$ on the vacuum vector

$$\psi^x(f) = W^x(f) \Phi.$$

If we work with respect to the vacuum state $\phi$, then we must sandwich all operators with the Weyl operator $W^x(f)$. An observable $S$ of the combined system of atomic gas and field up to time $t$ is therefore at time $t$ given by

$$j_t(S) = W^x(f)^* U_t^{0*} S U_t^0 W^x(f) = W^x(f_t)^* U_t^{0*} S U_t^0 W^x(f_t). \tag{5}$$

Here, $f_t$ denotes the function $f$ truncated at time $t$, i.e., $f_t(s) = f(s)$ for all $s \leqslant t$ and $f_t(s) = 0$ for all $s > t$. The relation Eq. (5) follows since all the operators split as a tensor product at time $t$ and $U_t^0$ and $S$ act as the identity operator after time $t$. Since $W^x(f)$ is unitary, it then cancels against its adjoint for the part that is after time $t$.

It can be shown [23] that $W^x(f_t)$ satisfies the following QSDE $[W^x(f_0) = I]$:

$$dW^x(f_t) = \left[ f(t) dA_t^{x*} - \bar{f}(t) dA_t^x - \tfrac{1}{2} |f(t)| dt \right] W^x(f_t).$$

Defining $U_t = U_t^0 W^x(f_t)$ and using the quantum Itô rule [22], we obtain $(U_0 = I)$

$$dU_t = \Big\{[\cos(\kappa F_z) - 1](d\Lambda_t^{xx} + d\Lambda_t^{yy}) - \sin(\kappa F_z)(d\Lambda_t^{xy} - d\Lambda_t^{yx}) + f(t)\cos(\kappa F_z)dA_t^{x^*} - \bar{f}(t)dA_t^x + f(t)\sin(\kappa F_z)dA_t^{y^*} - \tfrac{1}{2}|f(t)|^2 dt\Big\}U_t. \quad (6)$$

Summarizing, we work in the state $\mathbb{P} := \rho \otimes \phi$, the time evolution of (adapted) observables $S$ is given by $j_t(S) = U_t^* S U_t$, with $U_t$ given by Eq. (6).

## IV. THE QUANTUM FILTER

After the interaction, the light carries off information about the atomic gas. Therefore, measuring the field will enable us to make inference about the atomic gas observables. Let us suppose that we are counting the photons with a polarization along the $\vec{e}_\xi = 1/\sqrt{2}(\vec{e}_x + \vec{e}_y)$ axis, and that we are separately counting the photons with a polarization along the $\vec{e}_\eta = 1/\sqrt{2}(\vec{e}_x - \vec{e}_y)$ axis, see Fig. 1. That is, our observations are given by

$$Y_t^\xi = U_t^* \Lambda_t^{\xi\xi} U_t = \tfrac{1}{2} U_t^* (\Lambda_t^{xx} + \Lambda_t^{yy} + \Lambda_t^{xy} + \Lambda_t^{yx}) U_t,$$

$$Y_t^\eta = U_t^* \Lambda_t^{\eta\eta} U_t = \tfrac{1}{2} U_t^* (\Lambda_t^{xx} + \Lambda_t^{yy} - \Lambda_t^{xy} - \Lambda_t^{yx}) U_t. \quad (7)$$

Let $X$ be an atomic gas operator, its time evolution is given by

$$j_t(X) = U_t^* X U_t. \quad (8)$$

Equation (8) is called *the system*. Together Eqs. (8) and (7) form a *system-observations pair*.

It is easily checked that $[Y_t^\alpha, Y_s^\beta] = 0$ for all $\alpha, \beta \in \{\xi, \eta\}$ and for all $t, s \geqslant 0$. This is called the *self-nondemolition* property and ensures that our observations are simultaneously observable classical processes. Furthermore, it can be shown that $[j_t(X), Y_s^\alpha] = 0$ for all $t \geqslant s \geqslant 0$ and $\alpha \in \{\xi, \eta\}$. This is called the *nondemolition* property. Together the self-nondemolition and the nondemolition property ensure the existence of the conditional expectation $\mathbb{P}(j_t(X)|\mathcal{Y}_t)$ of a system operator at time $t$ on the observations up to time $t$. Since the conditional expectation is linear in the atomic gas operators $X$, we can define an *information state* $\pi_t$ on the atomic gas system by

$$\pi_t(X) = \mathbb{P}(j_t(X)|\mathcal{Y}_t).$$

Note that $\pi_t$ is a stochastic state since it depends on the observations $Y^\xi$ and $Y^\eta$ up to time $t$.

It is the goal of quantum filtering theory to obtain a recursive stochastic differential equation that propagates the information state $\pi_t$ in time. Our approach here is based on the reference probability method [15,16]. In these references, the interested reader can find further details on the exposition below.

Our first step is one of mere convenience. It is a change of picture that will simplify subsequent calculations. Let $W_t$ be given by $W_0 = I$ and

$$dW_t = \Big[\bar{f}(t)dA_t^x - f(t)dA_t^{x^*} - \tfrac{1}{2}|f(t)|dt\Big]W_t.$$

Note that $W_t$ is the adjoint of $W^x(f_t)$. Now define $U_t' = W_t U_t$, where $U_t$ is given by Eq. (6). It easily follows from the quantum Itô rule [22] that

$$dU_t' = \{[\cos(\kappa F_z) - 1](d\Lambda_t^{xx} + d\Lambda_t^{yy}) - \sin(\kappa F_z)(d\Lambda_t^{xy} - d\Lambda_t^{yx}) + f(t)[\cos(\kappa F_z) - 1]dA_t^{x^*} - \bar{f}(t)[\cos(\kappa F_z) - 1]dA_t^x + f(t)\sin(\kappa F_z)dA_t^{y^*} - \bar{f}(t)\sin(\kappa F_z)dA_t^y + |f(t)|^2[\cos(\kappa F_z) - 1]dt\}U_t'. \quad (9)$$

Define a new state on the combined system of atomic gas and field by $\mathbb{Q}^t(S) = \mathbb{P}(U_t'^* S U_t')$. To complete our change of picture we need to sandwich the observables with the opposite rotation. That means that the system Eq. (8) is now simply given by $U_t' U_t^* X U_t U_t'^* = W_t X W_t^* = X$. In the last step we used that $X$ acts on the atoms and is the identity on the field and vice versa for $W_t$. In the new picture the observations read

$$Z_t^\xi = U_t' Y_t^\xi U_t'^* = W_t \Lambda_t^{\xi\xi} W_t^*,$$

$$Z_t^\eta = U_t' Y_t^\eta U_t'^* = W_t \Lambda_t^{\eta\eta} W_t^*.$$

Using the quantum Itô rule, it easily follows that

$$dZ_t^\xi = d\Lambda_t^{\xi\xi} + \tfrac{1}{2}[\bar{f}(t)(dA_t^x + dA_t^y) + f(t)(dA_t^{x^*} + dA_t^{y^*}) + |f(t)|^2 dt],$$

$$dZ_t^\eta = d\Lambda_t^{\eta\eta} + \tfrac{1}{2}[\bar{f}(t)(dA_t^x - dA_t^y) + f(t)(dA_t^{x^*} - dA_t^{y^*}) + |f(t)|^2 dt]. \quad (10)$$

Denote $\mathcal{C}_t = U_t' \mathcal{Y}_t U_t'^*$, i.e., $\mathcal{C}_t$ consists of the processes $Z^\xi$ and $Z^\eta$ up to time $t$. It can easily be shown that the conditional expectations in the two different pictures are related by $\mathbb{P}(j_t(X)|\mathcal{Y}_t) = U_t'^* \mathbb{Q}^t(X|\mathcal{C}_t) U_t'$. That completes our discussion of the change of picture. We will now focus on deriving an equation that propagates $\mathbb{Q}^t(X|\mathcal{C}_t)$.

At the heart of the reference probability method is the following *quantum Bayes formula* [15,16]. Let $V$ be an operator that commutes with $Z_s^\alpha$ for all $\alpha \in \{\xi, \eta\}$ and for all $0 \leqslant s \leqslant t$. Moreover, suppose that $V^* V > 0$ and that $\mathbb{P}(V^* V) = 1$. Then we can define a state $\mathbb{Q}$ by $\mathbb{Q}(S) = \mathbb{P}(V^* S V)$, and for all operators $X$ that commute with $Z_s^\alpha$ ($\alpha \in \{\xi, \eta\}, 0 \leqslant s \leqslant t$), we have (see [15,16] for a proof)

$$\mathbb{Q}(X|\mathcal{C}_t) = \frac{\mathbb{P}(V^* X V|\mathcal{C}_t)}{\mathbb{P}(V^* V|\mathcal{C}_t)}.$$

We would like to apply the quantum Bayes formula to $\mathbb{Q}^t$, i.e., with $V = U_t'$. However, Eq. (9) shows that $U_t'$ is driven by noises that do not commute with $Z_s^\alpha$ ($\alpha \in \{\xi, \eta\}, 0 \leqslant s \leqslant t$), i.e., $U_t'$ itself does not commute with the $Z_s^\alpha$'s.

The following trick [17] solves this problem. Suppose $V_t'$ satisfies the QSDE,

$$dV'_t = \{[\cos(\kappa F_z) + \sin(\kappa F_z) - 1]dZ_t^{\xi} + [\cos(\kappa F_z) - \sin(\kappa F_z) - 1]dZ_t^{\eta}\}V'_t. \quad (11)$$

Then, the coefficients of $dA_t^{x*}$, $dA_t^{y*}$, and $dt$ are the same as in Eq. (19). Since $dA_t^{\alpha}$ and $d\Lambda_t^{\alpha\beta}$ $(\alpha,\beta \in \{x,y\})$ are zero when acting on the vacuum vector $\Phi$, we therefore have that for all operators $S$ [17],

$$\mathbb{Q}^t(S) = \mathbb{P}(U_t'^* S U_t') = \mathbb{P}(V_t'^* S V_t').$$

Moreover, since $V'_t$ is driven by $Z_t^{\xi}$ and $Z_t^{\eta}$, it commutes with $\mathcal{C}_t$, and we can therefore apply the Bayes formula with $V = V'_t$. That is, summarizing what we have achieved thus far

$$\mathbb{P}(j_t(X)|\mathcal{Y}_t) = U_t'^* \mathbb{Q}^t(X|\mathcal{C}_t) U'_t = \frac{U_t'^* \mathbb{P}(V_t'^* X V'_t|\mathcal{C}_t) U'_t}{U_t'^* \mathbb{P}(V_t'^* V'_t|\mathcal{C}_t) U'_t}. \quad (12)$$

The next step is to find the equation that propagates $\mathbb{P}(V_t'^* X V'_t|\mathcal{C}_t)$ in time. Using the quantum Itô rule we find

$$dV_t'^* X V'_t = V_t'^*(L^{\xi} X L^{\xi} - X)V'_t dZ_t^{\xi} + V_t'^*(L^{\eta} X L^{\eta} - X)V'_t dZ_t^{\eta}, \quad (13)$$

with

$$L^{\xi} = \cos(\kappa F_z) + \sin(\kappa F_z),$$

$$L^{\eta} = \cos(\kappa F_z) - \sin(\kappa F_z). \quad (14)$$

We can write Eq. (13) in integral form and approximate the stochastic integrals in the usual way with simple processes. If we proceed by taking the conditional expectation $\mathbb{P}(\cdot|\mathcal{C}_t)$, then we can pull the integrators which are elements of $\mathcal{C}_t$ out of the expectation. Furthermore, the conditional expectation $\mathbb{P}(L_s|\mathcal{C}_t)$ $(0 \leqslant s \leqslant t)$ of an adapted process $L$ equals $\mathbb{P}(L_s|\mathcal{C}_s)$. In this way we obtain

$$d\mathbb{P}(V_t'^* X V'_t|\mathcal{C}_t) = \mathbb{P}(V_t'^*(L^{\xi} X L^{\xi} - X)V'_t|\mathcal{C}_t)dZ_t^{\xi} + \mathbb{P}(V_t'^*(L^{\eta} X L^{\eta} - X)V'_t|\mathcal{C}_t)dZ_t^{\eta}.$$

Now define $\sigma_t(X) = U_t'^* \mathbb{P}(V_t'^* X V'_t|\mathcal{C}_t) U'_t$ for all atomic operators $X$. Using the quantum Itô rule, we obtain the linear version of the quantum filtering equation

$$d\sigma_t(X) = \sigma_t(\mathcal{L}(X))dt + \sigma_t(L^{\xi} X L^{\xi} - X)\left[dY_t^{\xi} - \tfrac{1}{2}|f(t)|^2 dt\right] + \sigma_t(L^{\eta} X L^{\eta} - X)\left[dY_t^{\eta} - \tfrac{1}{2}|f(t)|^2 dt\right], \quad (15)$$

where the Lindblad generator $\mathcal{L}$ is given by

$$\mathcal{L}(X) = |f(t)|^2[\sin(\kappa F_z)X\sin(\kappa F_z) + \cos(\kappa F_z)X\cos(\kappa F_z) - X], \quad (16)$$

for all atomic operators $X$. Now recall from Eq. (12) that $\pi_t(X) = \sigma_t(X)/\sigma_t(I)$, which is a quantum version of the classical *Kallianpur-Striebel* formula. Using the Itô rule once more, we obtain the following quantum filter:

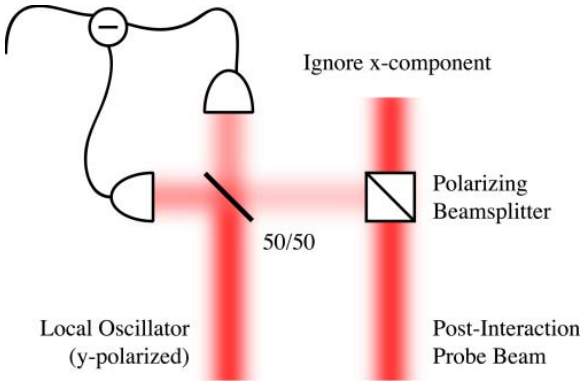


FIG. 2. (Color online) In the homodyne detection setup, the detection apparatus in the dashed box of Fig. 1 should be replaced with the apparatus depicted schematically here. As in Fig. 1, the light is initially polarized along $x$ and the polarization rotates slightly due to the interaction with the atoms. Here, however, the strong $x$ component is split off and ignored while the weak $y$ component is sent to a standard homodyne setup. The $y$ component is mixed at a 50/50 (nonpolarizing) beamsplitter with a strong local oscillator beam also polarized along the $y$ direction and derived from the same laser as the probe beam. The photocurrent representing the interference signal is then derived from the difference between the outputs of the photodetectors.

$$d\pi_t(X) = \pi_t(\mathcal{L}(X))dt + \left(\frac{\pi_t(L^{\xi} X L^{\xi})}{\pi_t(L^{\xi} L^{\xi})} - \pi_t(X)\right) \times \left(dY_t^{\xi} - \frac{1}{2}|f(t)|^2 \pi_t(L^{\xi} L^{\xi})dt\right) + \left(\frac{\pi_t(L^{\eta} X L^{\eta})}{\pi_t(L^{\eta} L^{\eta})} - \pi_t(X)\right) \times \left(dY_t^{\eta} - \frac{1}{2}|f(t)|^2 \pi_t(L^{\eta} L^{\eta})dt\right).$$

The processes $dY_t^{\xi} - \frac{1}{2}|f(t)|^2 \pi_t(L^{\xi} L^{\xi})dt$ and $dY_t^{\eta} - \frac{1}{2}|f(t)|^2 \pi_t(L^{\eta} L^{\eta})dt$ are called the *innovations* or *innovating martingales*. It can indeed be shown [13,14] that the innovations are martingales with respect to the filtration $\mathcal{Y}_t$ and the measure induced by $\mathbb{P}$.

## V. A DIFFERENT SETUP: HOMODYNE DETECTION

For the remainder of the paper we assume that $f(t) = \alpha e^{i\phi_t}$ with $\alpha$ real and $\phi_t$ in $[0, 2\pi)$. Now suppose that instead of the balanced polarimetry setup described in the preceding section, we use a homodyne detection setup to measure the $y$ component of the output light, see Fig. 2. It is well known [11,24] that for such a homodyne detection setup the observations are given by

$$Y_t = U_t^*(e^{-i\phi_t} A_t^y + e^{i\phi_t} A_t^{y*}) U_t. \quad (17)$$

That is, for homodyne detection of the $y$ channel the system-observations pair is given by Eqs. (8) and (17). It is easily checked that the homodyne system-observations pair satisfies the self-nondemolition and nondemolition properties,

meaning that the conditional expectation $\mathbb{P}(j_t(X)|\mathcal{Y}_t)$ is well defined. Here $\mathcal{Y}_t$ denotes the homodyne observations of Eq. (17) from time 0 up to time $t$. We will now derive the filter for the corresponding information state $\pi_t(X)=\mathbb{P}(j_t(X)|\mathcal{Y}_t)$.

Our first step is again one of convenience. We change to the Schrödinger picture by defining the following state on the combined system of atomic gas and field together $\mathbb{Q}^t(S)=\mathbb{P}(U_t^*SU_t)$. In the Schrödinger picture our system is simply given by $U_t j_t(X)U_t^*=X$ and the observations are given by

$$Z_t = U_t Y_t U_t^* = e^{-i\phi_t}A_t^y + e^{i\phi_t}A_t^{y*}.$$

Denote $\mathcal{C}_t=U_t\mathcal{Y}_tU_t^*$, i.e., $\mathcal{C}_t$ consists of the process $Z$ up to time $t$. It can easily be shown that the conditional expectations in the Heisenberg and Schrödinger pictures are related by $\mathbb{P}(j_t(X)|\mathcal{Y}_t)=U_t^*\mathbb{Q}^t(X|\mathcal{C}_t)U_t$.

To compute $\mathbb{Q}^t(X|\mathcal{C}_t)$ we would like to use the Bayes formula. Suppose $V_t$ satisfies the following QSDE ($V_0=I$):

$$dV_t = \left(e^{i\phi_t}\alpha\cos(\kappa F_z)dA_t^{x*} + \alpha\sin(\kappa F_z)dZ_t - \frac{\alpha^2}{2}dt\right)V_t. \tag{18}$$

Then, the coefficients of $dA_t^{x*}$, $dA_t^{y*}$, and $dt$ are the same as in Eq. (6). Therefore we have $\mathbb{Q}^t(S)=\mathbb{P}(U_t^*SU_t)=\mathbb{P}(V_t^*SV_t)$. The equation for $V_t$ is driven by $Z$ and $A^{x*}$, both commute with $\mathcal{C}_t$, i.e., $V_t$ commutes with $\mathcal{C}_t$. That means we can now apply Bayes formula with $V=V_t$ to obtain

$$\mathbb{P}(j_t(X)|\mathcal{Y}_t) = U_t^*\mathbb{Q}^t(X|\mathcal{C}_t)U_t = \frac{U_t^*\mathbb{P}(V_t^*XV_t|\mathcal{C}_t)U_t}{U_t^*\mathbb{P}(V_t^*V_t|\mathcal{C}_t)U_t}.$$

Using the quantum Itô rule we find

$$\begin{aligned} dV_t^*XV_t = {} & V_t^*\mathcal{L}(X)V_t dt + \alpha e^{-i\phi_t}V_t^*\cos(\kappa F_z)XV_t dA_t^x \\ & + \alpha e^{i\phi_t}V_t^*X\cos(\kappa F_z)V_t dA_t^{x*} + \alpha V_t^*[\sin(\kappa F_z)X \\ & + X\sin(\kappa F_z)]V_t dZ_t, \end{aligned}$$

where $\mathcal{L}$ is given by Eq. (16). Since $dA_t^x$ and $dA_t^{x*}$ are independent of $\mathcal{C}_t$ and since vacuum expectations of stochastic integrals with respect to $dA_t^x$ and $dA_t^{x*}$ are zero, we find in an analogous way as before

$$\begin{aligned} d\mathbb{P}(V_t^*XV_t|\mathcal{C}_t) = {} & \mathbb{P}(V_t^*\mathcal{L}(X)V_t|\mathcal{C}_t)dt + \alpha\mathbb{P}(V_t^*[\sin(\kappa F_z)X \\ & + X\sin(\kappa F_z)]V_t|\mathcal{C}_t)dZ_t. \end{aligned}$$

Now introduce $\sigma_t(X)=U_t^*\mathbb{P}(V_t^*XV_t|\mathcal{C}_t)U_t$ for all atomic gas operators $X$. Using the quantum Itô rule, we obtain the linear homodyne filtering equation

$$d\sigma_t(X) = \sigma_t(\mathcal{L}(X))dt + \alpha\sigma_t[\sin(\kappa F_z)X + X\sin(\kappa F_z)]dY_t. \tag{19}$$

Using $\pi_t(X)=\sigma_t(X)/\sigma_t(I)$ and the Itô rule, we find the following homodyne quantum filter:

$$\begin{aligned} d\pi_t(X) = {} & \pi_t(\mathcal{L}(X))dt + \alpha(\pi_t[\sin(\kappa F_z)X + X\sin(\kappa F_z)] \\ & - 2\pi_t[\sin(\kappa F_z)]\pi_t(X))\{dY_t - 2\alpha\pi_t[\sin(\kappa F_z)]dt\}. \end{aligned} \tag{20}$$

The process $dY_t-2\alpha\pi_t[\sin(\kappa F_z)]dt$ is again called the innovations or the innovating martingale. It can be shown [13,14] that the innovations are a continuous martingale with respect to the filtration $\mathcal{Y}_t$ and the measure induced by $\mathbb{P}$. It follows from Levy's theorem that the innovations for the homodyne detection setup form a Wiener process.

## VI. STRONG DRIVING, WEAK COUPLING

Define the *measurement strength* as the product $M=\alpha^2\kappa^2$. In a typical experimental setting $\alpha$ will be very large (strong driving) and $\kappa$ will be very small (weak coupling). The idea in this section will be to exaggerate this by taking the limit $\alpha\to\infty$ while keeping the product $M=\alpha^2\kappa^2$ constant.

Let us introduce the following scaled sum and difference processes:

$$Y_t^+ = \frac{Y_t^\xi + Y_t^\eta}{\alpha^2}, \quad Y_t^- = \frac{Y_t^\xi - Y_t^\eta}{\alpha}. \tag{21}$$

Note that we scaled the sum by $\alpha^2$ and the difference by $\alpha$. We will see that with these scalings we get finite output processes in the limit. In practice the scalings are determined by the experiment, i.e., they are chosen in such a way that the photocurrents nicely fill the scales on the read out devices. We are interested in the statistics of the processes $Y_t^+$ and $Y_t^-$. Therefore, following [24], we introduce their characteristic functionals

$$\begin{aligned} \Phi^+(k,t) &= \mathbb{P}\left(\exp\left(-i\int_0^t k(s)dY_s^+\right)\right) \\ &= \mathbb{P}\left(U_t^*\exp\left(-i\int_0^t\frac{k(s)}{\alpha^2}(d\Lambda_s^{xx}+d\Lambda_s^{yy})\right)U_t\right), \end{aligned}$$

$$\begin{aligned} \Phi^-(k,t) &= \mathbb{P}\left(\exp\left(-i\int_0^t k(s)dY_s^-\right)\right) \\ &= \mathbb{P}\left(U_t^*\exp\left(-i\int_0^t\frac{k(s)}{\alpha}(d\Lambda_s^{xy}+d\Lambda_s^{yx})\right)U_t\right), \end{aligned}$$

where $k$ is an arbitrary function in $L^2(\mathbb{R}^+)$. The characteristic functionals $\Phi^+$ and $\Phi^-$ faithfully encode the complete statistics of the processes $Y_t^+$ and $Y_t^-$.

Using the quantum Itô rule and the fact that vacuum expectations of stochastic integrals are zero, we find the following differential equation for $\Phi^+(k,t)$:

$$\frac{d\Phi^+}{dt}(k,t) = \alpha^2\left[\exp\left(-i\frac{k(t)}{\alpha^2}\right)-1\right]\Phi^+(k,t),$$

In the limit $\alpha$ to infinity (while keeping $M=\alpha^2\kappa^2$ constant), we therefore obtain

$$\Phi^+(k,t) = \exp\left(-i\int_0^t k(s)ds\right).$$

This is the characteristic functional of the deterministic time process $t$. In short, as $\alpha$ tends to infinity, $dY_t^+$ tends to $dt$.

To calculate $\Phi^-(k,t)$, define for all atomic gas operators $X$,

$$\Phi^-(X,k,t) = \mathbb{P}\left( U_t^* X \exp\left( -i \int_0^t \frac{k(s)}{\alpha} (d\Lambda_s^{xy} + d\Lambda_s^{yx}) \right) U_t \right).$$

Note that $\Phi^-(k,s)=\Phi^-(I,k,s)$. Using the quantum Itô rule and the fact that vacuum expectations of stochastic integrals are zero, we find the following system of differential equations $(n \geqslant 0)$:

$$\frac{d\Phi^-}{dt}(\alpha^n \sin^n(2\kappa F_z),k,t) = \alpha^2 \left[ \cos\left(\frac{k(t)}{\alpha}\right) - 1 \right] \times \Phi^-(\alpha^n \sin^n(2\kappa F_z),k,t) - i\alpha \sin\left(\frac{k(t)}{\alpha}\right) \times \Phi^-(\alpha^{n+1} \sin^{n+1}(2\kappa F_z),k,t).$$

Note that although the atomic gas system might be very high dimensional, the dimension is finite. That means that the above system of differential equation is closed and consists only of a finite number of equations. In the limit $\alpha$ to infinity (while keeping $M=\alpha^2\kappa^2$ constant), we obtain the following finite system of coupled differential equations $(n \geqslant 0)$:

$$\frac{d\Phi^-}{dt}((\sqrt{M}F_z)^n,k,t) = -\frac{k(t)^2}{2}\Phi^-((\sqrt{M}F_z)^n,k,t) - 2ik(t)\Phi^-((\sqrt{M}F_z)^{n+1},k,t). \quad (22)$$

In principle we could now try to solve this system of equations. However, instead of finding an explicit solution, let us compare this with the statistics of the homodyne observations $Y_t$ defined in Eq. (17). In analogy to the discussion above, we define for all atomic gas operators $X$,

$$\Phi(X,k,t) = \mathbb{P}\left( U_t^* X \exp\left( -i \int_0^t k(s) d(e^{-i\phi_t} A_s^y + e^{i\phi_t} A_s^{y*}) \right) U_t \right).$$

Using the quantum Itô rule and the fact that vacuum expectations of stochastic integrals are zero, we find the following system of differential equations $(n \geqslant 0)$:

$$\frac{d\Phi}{dt}(\alpha^n \sin^n(\kappa F_z),k,t) = -\frac{k(t)^2}{2}\Phi(\alpha^n \sin^n(\kappa F_z),k,t) - 2ik(t)\Phi(\alpha^{n+1} \sin^{n+1}(\kappa F_z),k,t).$$

Taking the limit $\alpha \to \infty$ (while $M=\alpha^2\kappa^2$ is held constant) then again leads to the system of differential equations (22). Therefore we conclude that in the limit the processes $Y_t^-$ and $Y_t$ have exactly the same statistics. This means that from the point of view of statistical inference of the atomic gas system from the observations, the balanced polarimetry experiment and the $y$-channel homodyne detection experiment are equivalent.

Rearranging terms, we can write the linear quantum filtering equation (15) as

$$d\sigma_t(X) = \alpha^2(\sigma_t[\sin(\kappa F_z)X \sin(\kappa F_z)] + \sigma_t[\cos(\kappa F_z)X \cos(\kappa F_z)] - \sigma_t(X))dY_t^+ + \alpha(\sigma_t[\cos(\kappa F_z)X \sin(\kappa F_z)] + \sigma_t[\cos(\kappa F_z)X \sin(\kappa F_z)])dY_t^-.$$

Writing $\bar{Y}_t$ for the limit process of $Y_t^-$, and taking the limit of the above equation, we obtain the following linear quantum filtering equation:

$$d\sigma_t(X) = \sigma_t(\bar{\mathcal{L}}(X))dt + \sqrt{M}\sigma_t(F_z X + X F_z)d\bar{Y}_t, \quad (23)$$

where

$$\bar{\mathcal{L}}(X) = M\left[ F_z X F_z - \tfrac{1}{2}(F_z^2 X + X F_z^2) \right].$$

Moreover, we obtain the following normalized quantum filter:

$$d\pi_t(X) = \pi_t(\bar{\mathcal{L}}(X))dt + \sqrt{M}[\pi_t(F_z X + X F_z) - 2\pi_t(F_z)\pi_t(X)] \times [d\bar{Y}_t - 2\sqrt{M}\pi_t(F_z)dt]. \quad (24)$$

Since $d\bar{Y}_t - 2\sqrt{M}\pi_t(F_z)dt$ is a continuous martingale [13,14], it follows from Levy's theorem that it is a Wiener process. That is, we find that $d\bar{Y}_t = dW_t + 2\sqrt{M}\pi_t(F_z)dt$, with $W_t$ a Wiener process.

Furthermore, note that if we start from Eq. (19), taking $\alpha$ to infinity while $M=\alpha^2\kappa^2$ is held constant, then we also obtain the linear filter Eq. (23). Likewise, the homodyne filter Eq. (20) converges to the filter in Eq. (24) when $\alpha$ is taken to infinity while $M=\alpha^2\kappa^2$ is held constant.

## VII. DECOUPLING THE $x$ CHANNEL

Let us give a brief formal discussion to show what happens in the strong driving, weak coupling limit. As $\alpha$ increases and $\kappa=\sqrt{M}/\alpha$ decreases, the relative effect of the atoms on the $x$-polarized channel also decreases. Therefore, we can reasonably expect that the $x$ channel remains in a coherent state. Instead of working with respect to the state $\mathbb{P}=\rho\otimes\phi$ we will now work with respect to the state

$$\mathbb{Q} = \rho \otimes \langle \psi^x(f), \cdot\, \psi^x(f) \rangle,$$

with $f(t)=\alpha e^{i\phi_t}$. Note that this means that the $y$ channel is still in the vacuum state. Working with respect to the coherent state on the $x$ channel means that the time evolution is given by Eq. (4). Formally we can write for $\beta,\ \gamma \in \{x,y\}$,

$$d\Lambda_t^{\beta\gamma} = a_t^{\beta*} a_t^{\gamma} dt, \quad dA_t^{\beta} = a_t^{\beta} dt, \quad dA_t^{\beta*} = a_t^{\beta*} dt,$$

and since for large $\alpha$ and small $\kappa$ the $x$ channel is approximately in the coherent state $\psi(f)$, we can replace $a_t^x$ by $\alpha e^{i\phi_t}$ and $a_t^{x*}$ by $\alpha e^{-i\phi_t}$. This means that we obtain for large $\alpha$ and small $\kappa$,

$$dU_t^0 = \{[\cos(\kappa F_z) - 1](\alpha^2 dt + d\Lambda_t^{yy}) - \sin(\kappa F_z)\alpha(e^{-i\phi_t} dA_t^y - e^{i\phi_t} dA_t^{y*})\} U_t^0.$$

Now, if we replace $\kappa$ by $\sqrt{M}/\alpha$ and take the limit $\alpha$ to infinity, then the time evolution satisfies the following QSDE:

$$d\bar{U}_t = \left( \sqrt{M} F_z (e^{-i\phi_t} dA_t^y - e^{i\phi_t} dA_t^{y*}) - \frac{M}{2} F_z^2 dt \right) \bar{U}_t.$$

That is, the $x$ channel has been decoupled from the interaction, see also [9,25].

In a similar way we easily see that in the strong driving, weak coupling limit we have $dY_t^+ = dt$ and for $\bar{Y}_t$, the limit of $Y_t^-$, we obtain

$$\bar{Y}_t = \frac{\bar{U}_t^*(\Lambda_t^{xy} + \Lambda_t^{yx})\bar{U}_t}{\alpha} = \frac{\bar{U}_t^*(\alpha e^{-i\phi_t}A_t^y + \alpha e^{i\phi_t}A_t^{y*})\bar{U}_t}{\alpha} = \bar{U}_t^*(e^{-i\phi_t}A_t^y + e^{i\phi_t}A_t^{y*})\bar{U}_t. \tag{25}$$

This shows once more the equivalence of the balanced polarimetry experiment and the $y$-channel homodyne detection experiment. It is easy to see that the characteristic functional of the process $\bar{Y}$ satisfies the set of coupled differential equations of Eq. (22). Moreover, after decoupling the $x$ channel the system is given by $j_t(X) = \bar{U}_t^* X \bar{U}_t$ and the observations by Eq. (25). Following essentially the same steps as in Sec. V, we again obtain Eqs. (23) and (24) as the linear and normalized filters, respectively.

## VIII. DISCUSSION

We have provided a quantum stochastic model Eq. (6) to describe recent polarimetry experiments in which polarized laser light interacts with an atomic gas via the Faraday interaction. In our description the gauge process plays a prominent role. It represents the scattering between different channels in the field and it provides us with counting processes that can be observed. As in [21], our quantum stochastic model presents an application to quantum optics of the gauge terms in a QSDE.

Once we set the model, we derived quantum filtering equations for balanced polarimetry and homodyne detection experiments, studied the statistics of output processes, and obtained filters in the strong driving, weak coupling limit. Our results in the limit confirm the ad hoc filter for the balanced polarimetry experiment that has already been in use in the literature [4]. Moreover, we showed that from the point of view of statistical inference the balanced polarimetry experiment and the homodyne detection experiment are equivalent.

Using formal arguments we have seen that in the strong driving, weak coupling limit the $x$ channel decouples from the description. Rigorous results on this decoupling are still to be obtained.

Having an underlying model from which rigorous derivations can depart, is likely to be advantageous in future further investigations. In particular, combining the model presented in this paper with the results in [26] could prove useful for investigating the situation where the laser beam passes through the gas multiple times [10,27].

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