

Wigner functions of s waves

J. P. Dahl,^{1,2} S. Varro,^{3,2} A. Wolf,² and W. P. Schleich²

¹*Chemical Physics, Department of Chemistry, Technical University of Denmark, DTU 207, DK-2800 Lyngby, Denmark*

²*Institut für Quantenphysik, Universität Ulm, D-89069 Ulm, Germany*

³*Research Institute for Solid State Physics and Optics, H-1525 Budapest, P.O. Box 49, Hungary*

(Received 15 January 2007; published 11 May 2007)

We derive explicit expressions for the Wigner function of wave functions in D dimensions which depend on the hyperradius—that is, of s waves. They are based either on the position or the momentum representation of the s wave. The corresponding Wigner function depends on three variables: the absolute value of the D -dimensional position and momentum vectors and the angle between them. We illustrate these expressions by calculating and discussing the Wigner functions of an elementary s wave and the energy eigenfunction of a free particle.

DOI: [10.1103/PhysRevA.75.052107](https://doi.org/10.1103/PhysRevA.75.052107)

PACS number(s): 03.65.Ca, 02.30.Uu, 42.50.–p

I. INTRODUCTION

Quantum phase-space distributions play an important role in various branches of physics ranging from nuclear and particle physics via quantum optics to quantum chaosology [1–4]. Due to its simplicity and neighborhood to classical phase-space distributions, the Wigner function [5] stands out most clearly from the wealth of such distributions and has attracted a lot of attention. However, most studies of the Wigner function have concentrated on one-dimensional quantum systems. In the present paper we analyze the Wigner function of quantum states in a D -dimensional configuration space where the wave function depends only on the hyperradius that is the square root of the sum of squares of the D Cartesian coordinates. By choice, the quantum states in question may refer to a single particle in D dimensions or to a system of 2 or more particles, each of which resides in $D/2$ or fewer dimensions [6].

A. Why s waves?

Wave functions which depend on the hyperradius only correspond to waves with angular momentum zero. They are commonly referred to as s waves. However, for a system of particles there also exist wave functions which correspond to a vanishing total angular momentum and do not have the property to depend exclusively on the hyperradius. In order to distinguish these s waves from the ones with hyperradius dependence we have introduced in Ref. [7] the name *hyper-radial s waves*. We emphasize that the present paper deals solely with hyper-radial s waves, but in order to keep the notation simple we use throughout the article the name s wave as a shorthand notation for a hyper-radial s wave.

The importance of our study derives from the close correspondence between entanglement and the negative parts of the Wigner function recently established [7] for s waves describing a system of particles. Indeed, due to the constraint on the dependence of the s wave on the coordinates, s waves also describe entangled quantum systems. In a recent paper [7] we have shown that the negative volume of the Wigner function is an excellent measure for the entanglement contained in such a state. Moreover, it does not suffer the restrictions [8] of the other measures on the number of particles.

Apart from this application of Wigner functions to the field of quantum information s waves are interesting in their own right. Indeed, the ground state of isotropic quantum systems such as a hydrogen atom [9] or a Bose-Einstein condensate is described by an s wave. Here the Wigner function provides a deeper insight into the physics of these states and, in particular, correlations between position \vec{r} and momentum \vec{p} spanning quantum phase space.

Since s waves correspond to angular momentum zero, a naive consideration would make one expect the position and momentum variables \vec{r} and \vec{p} to be either parallel or antiparallel—that is, the angle θ between \vec{r} and $\vec{p} \equiv \hbar \vec{k}$ to take on only the values $\theta=0$ or $\theta=\pi$. However, this classical picture of an s wave neglects interference. Indeed, the s wave is the result of the *interference* of a continuous superposition [10] of one-dimensional motions corresponding to \vec{r} and \vec{p} being parallel or antiparallel. Since the s wave depends on the hyperradius only and is independent of the directions, the individual elementary waves corresponding to the one-dimensional radial motion in all space directions contribute with equal weight. In this sense the s wave consists predominantly of interference.

As a consequence of this complex interference of elementary waves now motions which are not in the radial direction emerge. Since they originate from interference, their weight as described by the Wigner function may be negative. Thus, the Wigner function of an s wave may take on nonvanishing values for angles $\theta \neq 0$ or $\theta \neq \pi$ and, in particular, may display domains where W is negative.

At first sight this behavior is rather unusual. In a classical phase-space approximation of an eigenstate of angular momentum we expect the values of angular momentum to be constrained by a δ function to the eigenvalue. However, in the Wigner-function formalism we find that all values of angular momentum manifesting themselves in different angles can occur.

This behavior is reminiscent of the properties of the Wigner function of an energy eigenstate in one dimension. Here a classical picture would suggest a δ function along the classical phase-space trajectory determined by the energy eigenvalue. However, the quantum description by the Wigner function softens the δ function into an Airy function with

oscillations in the phase-space domain circumvented by the classical trajectory.

Phase space is usually described by position and momentum variables \vec{r} and \vec{p} , respectively. However, in order to simplify the formalism we express phase space in terms of position and wave vector $\vec{k} \equiv \vec{p}/\hbar$. In this way the resulting expressions for the Wigner function do not explicitly contain \hbar .

B. Outline and summary

Our paper is organized as follows. In Sec. II we start from the well-known definition of a Wigner function in D dimensions and demonstrate that the Wigner function W corresponding to an s wave can only depend on the absolute values r and k of the position and the wave vectors \vec{r} and \vec{k} and the angle θ between \vec{r} and \vec{k} . We then derive explicit expressions for W , either in terms of the position or the wave vector representation of the wave function. Both formulas clearly demonstrate that W depends on r , k , and θ only. Moreover, W results from a double integral over a position and an angle variable.

The next two sections illustrate these expressions for W for two specific quantum states. In Sec. III we calculate the Wigner functions of an elementary s wave followed in the Sec. IV by the example of an energy eigenstate of a free particle in D dimensions. Three arguments support this choice: (i) whereas the elementary s wave illustrates the expression for W in position space, the free particle follows from the complementary representation in wave vector space, (ii) the elementary s wave has already played a major role in a previous study [7] on the connection of entanglement and negative parts of the Wigner function, and (iii) the example of a free particle leads to differential equations in phase space [11] which can still be solved analytically in order to test the result obtained from the direct integration of the definition of the Wigner function. In this discussion of the two exemplary Wigner functions we focus on their dependence on all three space variables—that is, r , k , and θ . Moreover, we analyze in detail the influence of the number D of dimensions. We conclude in Sec. V by presenting a summary and an outlook.

In order to concentrate on the essential ideas while keeping the paper self-contained we have included all relevant calculations but have moved them to extended appendixes. Since our discussion of the Wigner function of an s wave relies heavily on the concept of hyperspherical coordinates, we dedicate Appendix A to a brief summary of this topic. Here we follow closely the treatment of Ref. [12]. In Appendix B we perform the integrations over $D-2$ angles in the original definition of the Wigner function, arriving at the two expressions for the Wigner distribution of an s wave central to the present paper. In order to lay the foundation for the discussion of the Wigner function of the free particle as well as to show that the wave-vector representation of an s wave only depends on k we recall in Appendix C the wave vector representation of an s wave. Appendix D is devoted to the evaluation of the two-dimensional integration in the definition of the Wigner function of the elementary s wave. In

Appendix E we finally turn to the example of a free particle in D dimensions. Here we first recall the wave function in position space and with the help of Appendix C we find that the corresponding wave-vector representation involves a δ function. We then perform the necessary integrations and obtain the Wigner function W_E of a free particle. As a test of this expression we verify that its marginal distribution leads to the probability density in position space. We conclude by deriving the differential equations in phase space determining W_E . One set of equations follows from application of the Weyl-Wigner transform to the energy eigenvalue equation. An additional set of equations emerges from the eigenvalue equation of the square of the angular momentum. Only these two sets together determine uniquely the Wigner function.

II. REPRESENTATION

In this section we show that the isotropy of a D -dimensional s wave with $2 \leq D$ permits us to express the corresponding Wigner function as a double integral. We derive two equivalent representations of the Wigner function: one is in terms of the position and the other in terms of the wave-vector wave function. For this purpose we first note that the corresponding Wigner function of an s state depends on three variables only.

A. Three variables only

The crucial ingredient of our proof is the fact that for any orthogonal matrix U with

$$UU^T = U^T U = \mathbb{1} \quad (1)$$

and

$$\det U = 1, \quad (2)$$

the value of the wave function

$$\psi(|U\vec{r}|) = \psi(|\vec{r}|) \quad (3)$$

of an s state does not change. When we now recall the definition

$$W(\vec{r}, \vec{k}) \equiv \frac{1}{(2\pi)^D} \int d^D \xi e^{-i\vec{k} \cdot \vec{\xi}} \psi^* \left(\vec{r} - \frac{1}{2} \vec{\xi} \right) \psi \left(\vec{r} + \frac{1}{2} \vec{\xi} \right) \quad (4)$$

of the Wigner function and include the identity matrix $U^T U = \mathbb{1}$ in the arguments of the Fourier term,

$$e^{-i\vec{k} \cdot \vec{\xi}} = e^{-i(U\vec{k})^T \cdot U\vec{\xi}}, \quad (5)$$

and in the wave functions,

$$\psi \left(\vec{r} \pm \frac{\vec{\xi}}{2} \right) = \psi \left(U^T \left[U\vec{r} \pm \frac{U\vec{\xi}}{2} \right] \right), \quad (6)$$

and establish the integration variable

$$\vec{\zeta} = U\vec{\xi}, \quad d^D \zeta = d^D \xi, \quad (7)$$

we realize that the value of the Wigner function

$$W(\vec{r}, \vec{k}) = W(U\vec{r}, U\vec{k}) \quad (8)$$

is independent of a simultaneous rotation of \vec{r} and \vec{k} . Due to this fact, the Wigner function can only depend on three vari-

ables: namely, the absolute value $r \equiv |\vec{r}|$ of the position vector, the absolute value $k \equiv |\vec{k}|$ of the wave vector, and the angle $\theta \equiv \arccos[(\vec{r} \cdot \vec{k})/rk]$ between them.

B. Explicit expressions

In the previous section we have shown that due to the isotropy of the s -wave function, the corresponding Wigner function W can depend on r , k , and the angle θ between \vec{r} and \vec{k} only. We now derive two completely equivalent expressions for W which bring out this fact explicitly. These two formulas are either in terms of the position or wave-vector representation of the s wave.

1. Position space

We start from the standard definition, Eq. (4), of the Wigner function in D dimensions and note that for an s wave the wave function $\psi = \psi(|\vec{r}|)$ depends on the hyperradius only. As a consequence Eq. (4) takes the form

$$W = \frac{1}{(2\pi)^D} \int d^D \xi e^{-i\vec{k} \cdot \vec{\xi}} \psi^*(r_-) \psi(r_+), \quad (9)$$

where we have introduced the abbreviation

$$r_{\pm} \equiv \left| \vec{r} \pm \frac{1}{2} \vec{\xi} \right| = \left[r^2 + \frac{1}{4} \xi^2 \pm \vec{r} \cdot \vec{\xi} \right]^{1/2}. \quad (10)$$

The special form of the integrand in Eq. (9) suggests that one perform the integration over the D components of $\vec{\xi}$ in hyperspherical coordinates summarized in Appendix A. This method involves as variables the hyperradius ξ , the angle ϑ , and the $D-2$ angles $\varphi_1, \dots, \varphi_{D-2}$. It is now possible to orient the coordinate system of $\vec{\xi}$ relative to \vec{r} and \vec{k} in such a way that it is possible to perform the integration over the $D-2$ angles $\varphi_1, \dots, \varphi_{D-2}$ explicitly. In Appendix B 1 we pursue this approach and derive the expression

$$\begin{aligned} W &= (2\pi)^{-(D+1)/2} (k \sin \theta)^{-(D-3)/2} \int_0^\infty d\xi \xi^{(D+1)/2} \\ &\times \int_0^\pi d\vartheta \sin^{(D-1)/2} \vartheta J_{(D-3)/2}(\xi k \sin \theta \sin \vartheta) \\ &\times e^{-i\vec{k} \cdot \vec{\xi}} \psi^*(r_-) \psi(r_+) \end{aligned} \quad (11)$$

for the Wigner function of an s wave. In hyperspherical coordinates the quantities r_{\pm} defined in Eq. (10) take the form

$$r_{\pm} = \left[r^2 + \frac{1}{4} \xi^2 \pm r \xi \cos \vartheta \right]^{1/2}. \quad (12)$$

Equations (11) and (12) bring out most clearly that W only depends on r , k , and the angle θ between \vec{r} and \vec{k} —that is, $W = W(r, k, \theta)$. Moreover, we emphasize that two integrations are necessary to obtain W . At first sight one might think that a single integration might suffice since the wave function of an s wave depends on a single variable. However, the above calculation demonstrates that this suspicion is wrong.

We conclude by noting that strictly speaking these expressions are not valid for $D=2$, since they emerge from the

expression, Eq. (11), which is only defined for $3 \leq D$. Nevertheless, we show in Appendix B that the expression, Eq. (11), holds also true for $D=2$.

2. Wave-vector space

We now turn to a representation of W in terms of the wave function $\tilde{\psi}$ in wave-vector space. For this purpose we start from the definition

$$W(r, k) = \frac{1}{(2\pi)^D} \int d^D q e^{i\vec{r} \cdot \vec{q}} \tilde{\psi}^* \left(\vec{k} - \frac{1}{2} \vec{q} \right) \tilde{\psi} \left(\vec{k} + \frac{1}{2} \vec{q} \right) \quad (13)$$

in terms of the wave function

$$\tilde{\psi}(\vec{k}) = \frac{1}{(2\pi)^{D/2}} \int d^D r \psi(\vec{r}) e^{-i\vec{k} \cdot \vec{r}} \quad (14)$$

in \vec{k} space.

When we compare the two definitions, Eqs. (4) and (13), of the Wigner function we recognize that the roles of \vec{r} and \vec{k} are interchanged. Therefore, we expect an expression for W in terms of $\tilde{\psi}$ similar to Eq. (11). In order to derive this formula we first recall in Appendix C that the wave-vector representation of an s wave depends only on the absolute value $k \equiv |\vec{k}|$ of the wave vector. As a consequence the Wigner function of an s wave takes the form

$$W(\vec{r}, \vec{k}) = \frac{1}{(2\pi)^D} \int d^D q e^{i\vec{r} \cdot \vec{q}} \tilde{\psi}^*(k_-) \tilde{\psi}(k_+), \quad (15)$$

with

$$k_{\pm} \equiv \left| \vec{k} \pm \frac{1}{2} \vec{q} \right| = \left[k^2 + \frac{1}{4} q^2 \pm \vec{k} \cdot \vec{q} \right]^{1/2}. \quad (16)$$

Again we can express the D integrations over the wave-vector components q_1, \dots, q_D in hyperspherical coordinates. However, in the choice of the orientation of the \vec{q} -coordinate system we have to take into account the fact that the roles of \vec{r} and \vec{k} are interchanged. In Appendix B 2 we perform the integration over the $D-2$ angles $\varphi_1, \dots, \varphi_{D-2}$ explicitly and derive the expression

$$\begin{aligned} W &= (2\pi)^{-(D+1)/2} (r \sin \theta)^{-(D-3)/2} \int_0^\infty dq q^{(D+1)/2} \\ &\times \int_0^\pi d\vartheta \sin^{(D-1)/2} \vartheta J_{(D-3)/2}(qr \sin \theta \sin \vartheta) \\ &\times e^{i\vec{r} \cdot \vec{q}} \tilde{\psi}^*(k_-) \tilde{\psi}(k_+) \end{aligned} \quad (17)$$

for the Wigner function of an s wave with

$$k_{\pm} = \left[k^2 + \frac{1}{4} q^2 \mp kq \cos \vartheta \right]^{1/2}. \quad (18)$$

Again the formula only depends on r , k , and θ .

We conclude by noting that again the expression, Eq. (17), contains the case $D=2$.

3. Kernel

We conclude by comparing the two expressions, Eqs. (11) and (17), for the Wigner function W of an s wave. First we note that the formal structure of both relations is identical. The difference occurs in the wave function ψ versus $\tilde{\psi}$ and the roles of r and k are interchanged.

Indeed, we can bring out the similarity even more clearly by introducing the kernel

$$\begin{aligned} \mathcal{K}(b, \theta | \zeta, \eta) &\equiv (2\pi)^{-(D+1)/2} (b \sin \theta)^{-(D-3)/2} \zeta^{(D+1)/2} \\ &\quad \times \sin^{(D-1)/2} \eta e^{-ib\zeta \cos \theta \cos \vartheta} \\ &\quad \times J_{(D-3)/2}(\zeta b \sin \theta \sin \eta), \end{aligned} \quad (19)$$

which casts Eqs. (11) and (17) into the rather symmetric form

$$W = \int_0^\infty d\zeta \int_0^\pi d\eta \mathcal{K}(k, \theta | \zeta, \eta) \psi^*(s_-(r)) \psi(s_+(r)) \quad (20)$$

and

$$W = \int_0^\infty d\zeta \int_0^\pi d\eta \mathcal{K}^*(r, \theta | \zeta, \eta) \tilde{\psi}^*(s_-(k)) \tilde{\psi}(s_+(k)), \quad (21)$$

where

$$s_\pm(s) \equiv \left[s^2 + \frac{1}{4} \zeta^2 \pm s \zeta \cos \eta \right]^{1/2}. \quad (22)$$

We note that the kernel \mathcal{K} depends on two pairs of variables. The first pair consists of the two integration variables ζ and η . Here η corresponds to the angle ϑ and ζ represents either the position variable ξ or the wave vector q . The second pair includes two out of the three phase-space variables. The angle θ is always present. However, the first slot of \mathcal{K} indicated by the variable b depends on the representation of the state. When we start from the wave function ψ in position space—that is, $\psi = \psi(r)$ —the first argument of \mathcal{K} is the variable k which is complementary to r . The r dependence of the Wigner function then results from the wave function ψ evaluated at the argument $s_\pm(r)$. When we start from the wave-vector representation with $\tilde{\psi} = \tilde{\psi}(k)$ the first argument of \mathcal{K} is r and the k dependence enters through the wave function $\tilde{\psi}$ evaluated at the argument $s_\pm(k)$.

Needless to say both formulas, Eqs. (20) and (21), are completely equivalent. However, one might be more convenient to perform the two integrations than the other. Indeed, in Sec. IV we discuss the Wigner function of a free particle in D dimensions. Whereas the calculation in position space is extremely difficult, the integrations in wave-vector space are rather elementary.

C. Marginal distributions

The total number of dimensions of the phase space is $2D$ consisting of the D coordinates of \vec{r} and D coordinates of \vec{k} . However, the Wigner function of an s wave depends on only three variables: namely, r , k , and θ .

From the original definitions, Eqs. (4) and (13), of the Wigner function we can easily deduce the marginal distributions

$$|\psi(\vec{r})|^2 = \int d^D k W(\vec{r}, \vec{k}) \equiv P(\vec{r}) \quad (23)$$

and

$$|\psi(\vec{k})|^2 = \int d^D r W(\vec{r}, \vec{k}) \equiv P(\vec{k}). \quad (24)$$

Obviously this property must also hold for the Wigner function of an s wave. However, it is not obvious from the expressions, Eqs. (20) and (21). We now verify the marginal distributions, Eqs. (23) and (24). For this purpose we substitute the Wigner distribution, Eq. (20), into the integral, Eq. (23), which yields

$$P(\vec{r}) = \int_0^\infty d\zeta \int_0^\pi d\eta \mathcal{M}(\zeta, \eta) \psi^*(s_-(r)) \psi(s_+(r)). \quad (25)$$

Here we have introduced the integrated kernel

$$\mathcal{M}(\zeta, \eta) \equiv \int d^D b \mathcal{K}(b, \theta | \zeta, \eta), \quad (26)$$

with the integration

$$\int d^D b = \int_0^\infty db b^{D-1} \int_0^\pi d\theta \sin^{D-2} \theta \int d\omega \quad (27)$$

extending over the D -dimensional space of b .

Likewise we find from Eq. (21)

$$P(\vec{k}) = \int_0^\infty d\zeta \int_0^\pi d\eta \mathcal{M}^*(\zeta, \eta) \tilde{\psi}^*(s_-(k)) \tilde{\psi}(s_+(k)).$$

In Appendix B 3 we derive the expression

$$\mathcal{M}(\zeta, \eta) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{D}{2}\right)}{\Gamma\left(\frac{D-1}{2}\right)} \sin^{D-2} \eta \delta(\zeta) \quad (28)$$

and the δ function in ζ allows us to perform the integration which yields

$$P(\vec{r}) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{D}{2}\right)}{\Gamma\left(\frac{D-1}{2}\right)} \int_0^\pi d\eta \sin^{D-2} \eta |\psi(r)|^2 \quad (29)$$

or

$$P(\vec{k}) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{D}{2}\right)}{\Gamma\left(\frac{D-1}{2}\right)} \int_0^\pi d\eta \sin^{D-2} \eta |\tilde{\psi}(k)|^2. \quad (30)$$

With the help of the integral relation [13]

$$\int_0^\pi d\varphi \sin^\nu \varphi = \sqrt{\pi} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu+2}{2}\right)}, \quad (31)$$

we arrive at the probability distributions, Eqs. (23) and (24), in position or wave-vector space.

III. ELEMENTARY s WAVE

Our first application of the expression, Eq. (11), for the Wigner function of an s wave results from the wave function

$$\Psi^{(D)}(r) = \mathcal{N} [1 + a(\kappa r)^2] \left(\frac{\kappa^2}{\pi}\right)^{D/4} e^{-(\kappa r)^2/2}, \quad (32)$$

with the normalization constant

$$\mathcal{N} \equiv \left[\left(1 + \frac{1}{2}aD\right)^2 + \frac{1}{2}a^2D \right]^{-1/2}. \quad (33)$$

Here κ is a real constant with a dimension of an inverse length and a is a dimensionless real parameter.

Despite its simplicity the elementary s wave $\Psi^{(D)}$ contains a wealth of physics. For $a=0$ we find the familiar Gaussian and in the limit of $a \rightarrow \pm\infty$ we arrive at the shell function which has attracted our attention in the context of the time evolution of s waves [14]. Most recently we have also concentrated on the state emerging for $a=-2/D$. In particular, we have shown [7] that this member of the class of elementary s waves corresponds to a maximally entangled state. Since the parameter a interpolates between these cases, we refer to it as the interpolation parameter.

When we substitute Eq. (32) into the expression, Eq. (11), for the Wigner function in terms of the position wave function and perform the integration over ξ and ϑ we arrive at the formula

$$W_a^{(D)} = W_0^{(D)}(r, k) (1 + a\mathcal{P}_a^{(D)}), \quad (34)$$

with the Gaussian

$$W_0^{(D)}(r, k) = \pi^{-D} e^{-(\kappa r)^2 - (k/\kappa)^2} \quad (35)$$

and the polynomial

$$\mathcal{P}_a^{(D)} \equiv \wp_- r^2 - \wp_+ k^2 + \wp(r^2 + k^2)^2 - 4\wp k^2 r^2 \sin^2 \theta \quad (36)$$

defined by the coefficients

$$\wp_\pm(a, D) \equiv 2 \frac{1 + aD/2 \pm a}{(1 + aD/2)^2 + a^2D/2} \quad (37)$$

and

$$\wp(a, D) \equiv \frac{a}{(1 + aD/2)^2 + a^2D/2}. \quad (38)$$

For the details of the calculation we refer to Appendix D.

A. Discussion

We now briefly discuss the Wigner function in its dependence on the phase-space variables \vec{r} and \vec{k} and the number D

of dimensions for special cases of a . First we note that indeed \vec{r} and \vec{k} enter $W_a^{(D)}$ only through r , k , and θ in complete agreement with the considerations of Sec. II. Next we recognize that $W_a^{(D)}$ is the product of two contributions: (i) the Gaussian $W_0^{(D)}$ in r and k and (ii) a polynomial $\mathcal{P}_a^{(D)}$ multiplied by the interpolation parameter a .

As a consequence the case $a=0$ corresponding to a Gaussian wave function yields the Gaussian Wigner function $W_0^{(D)}$. This distribution is always positive. However, when a is nonvanishing reflecting a non-Gaussian state the Wigner function of the elementary s wave can take on negative values. The origin of this behavior is the polynomial $\mathcal{P}_a^{(D)}$. It contains two terms which for $0 < a$ are always negative or at most zero: (i) the second term in Eq. (36) which is proportional to k^2 and (ii) the contribution containing the square of the sine function.

The latter not only involves even powers of r and k but also the angle θ between \vec{r} and \vec{k} . In this context it is worthwhile mentioning that this term is always positive or zero. Indeed, when \vec{r} and \vec{k} are parallel or antiparallel—that is, $\theta=0$ or $\theta=\pi$ —this contribution to $\mathcal{P}_a^{(D)}$ vanishes. For $0 < \theta < \pi$ it is nonvanishing and assumes its largest negative value for $\theta=\pi/2$ when \vec{r} and \vec{k} are orthogonal. Moreover, the combination $rk \sin \theta$ emerging in the polynomial can be interpreted as an angular momentum variable. This interpretation is rather remarkable since the elementary s wave is an eigenstate of the angular momentum operator corresponding to zero eigenvalue. Nevertheless, the Wigner function also involves nonvanishing values of the angular momentum in full accordance with the discussion in the Introduction.

B. Examples

In Figs. 1 and 2 we compare and contrast the Wigner functions $W_S \equiv W_\infty^{(D)}$ and $W_{\max} \equiv W_{-2/D}^{(D)}$ corresponding to the shell wave function and the maximally entangled state emerging from the elementary s wave Eq. (32) in the limit of $a \rightarrow \pm\infty$ and $a=-2/D$, respectively. Since the phase space of the Wigner function W of an s wave is three dimensional, we depict $W_a^{(D)}$ in its dependence on r and k in a single three-dimensional figure and then study the dependence of this figure on the angle θ . Moreover, we also discuss the influence of the number D of dimensions.

For this purpose each row of Figs. 1 and 2 shows the Wigner function $W_a^{(D)}(r, k, \theta)$ for three different angles θ but fixed number D of dimensions. Likewise for a fixed angle θ each column displays the dependence on D . In order to bring out the characteristic features we have chosen the angles $\theta_1 = \pi/6$, $\theta_2 = \pi/3$, and $\theta_3 = \pi/2$.

IV. FREE PARTICLE

We now turn to the second application of our expressions, Eqs. (11) and (17), for the Wigner function of an s wave. Here we derive and discuss the Wigner function W_E corresponding to the free nonrelativistic particle of mass M in an eigenstate of energy with $E \equiv (\hbar k_0)^2 / (2M)$ with vanishing angular momentum. We first recall the essential features of

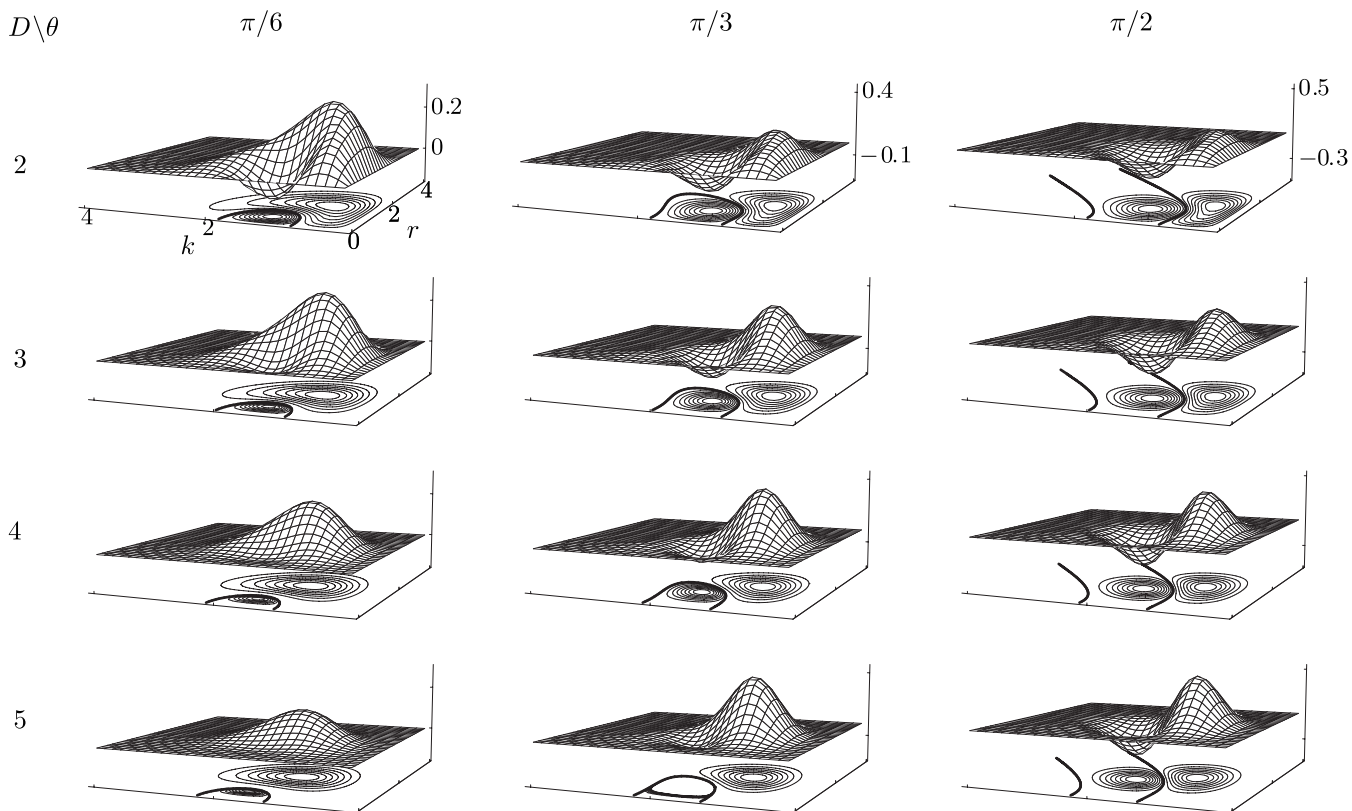


FIG. 1. Wigner function W_S of the shell wave function corresponding to the limit $a \rightarrow \pm\infty$ in Eq. (32) in its dependence on the angle θ between \vec{r} and \vec{k} (horizontal) and the number D of dimensions (vertical). Here we do not show the Wigner function W_S but $W_S(rk)^{D-1}(\sin^{D-2}\theta)S^{(D)}S^{(D-1)}$. In the bottom of each figure we display contour lines of W_S . The thick line marks the curve where the Wigner function vanishes, separating positive domains from negative domains. The horizontal axes r and k are identical in all figures. However, the vertical axis changes with increasing angles—that is, going from one row to the next—but is identical in each column.

the corresponding wave function ψ_E in position space. Then we start from the expression, Eq. (17), for the Wigner function in terms of the wave functions $\tilde{\psi}_E$ in wave-vector space and calculate W_E . This approach is most convenient since $\tilde{\psi}_E$ is given by a δ function which allows us to perform immediately the two integrations over q and ϑ defining W_E . For the relevant calculations we refer to Appendix E.

Since we dedicate this section to a discussion of W_E , we also present a heuristic derivation starting from the differential equations in phase space determining W_E . This approach brings to light the origin of an intimate connection between the wave function in position space and the corresponding Wigner function. Indeed, we show that the Wigner function of a free particle in D dimensions is determined by the appropriately scaled energy wave function in $D-1$ dimensions. This hierarchy originates from a transversality condition of the Wigner function, which is very much analogous to the Coulomb gauge in electrodynamics. We conclude by analyzing the dependence of the Wigner function W_E on the number D of dimensions.

A. Position wave function

We start our discussion of the Wigner function of a free particle by first recalling the corresponding wave function. Here we do not solve the appropriate time-independent

Schrödinger equation in D dimensions but rather construct the wave function by interfering plane waves. In this way we lay the formulation for understanding the rather unusual hierarchical connection between the wave function in $D-1$ dimensions and the Wigner function in D dimensions.

The superposition

$$\psi_E(\vec{r}) \equiv N \int d^D q e^{-i\vec{q}\cdot\vec{r}} \delta(q - k_0) \quad (39)$$

of plane waves satisfies the time-independent Schrödinger equation

$$(\Delta^{(D)} + k_0^2)\psi_E(\vec{r}) = 0, \quad (40)$$

corresponding to the energy $E \equiv (\hbar k_0)^2 / (2M)$. Here $\Delta^{(D)}$ denotes the Laplacian in D -dimensional position space.

The normalization constant N of ψ_E follows from the orthonormality condition

$$\int d^D r \psi_E^*(\vec{r}) \psi_{E'}(\vec{r}) = \delta(E - E') \quad (41)$$

of energy eigenstates corresponding to a continuous spectrum.

In the definition, Eq. (39), of ψ_E we integrate the wave vector \vec{q} over all space directions with equal weight which according to Appendix E yields the expression

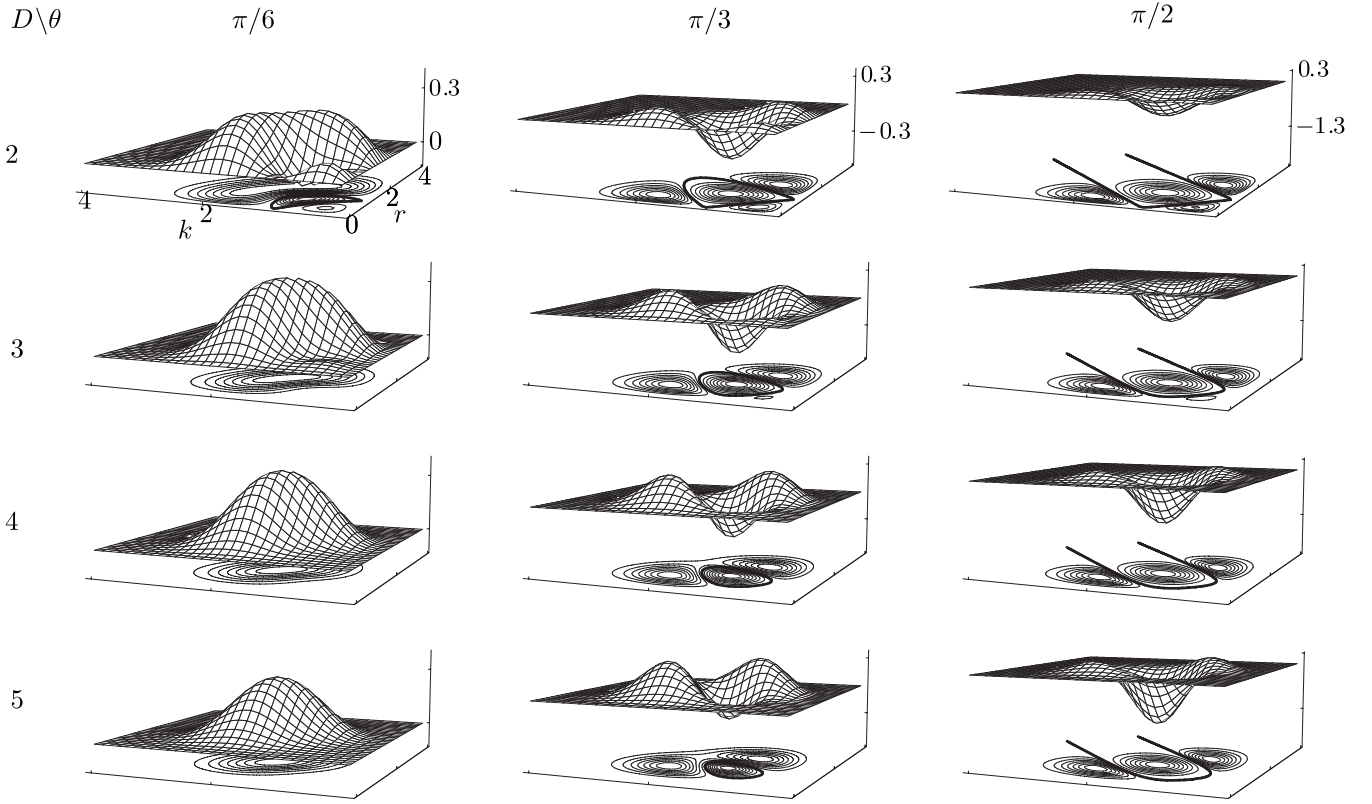


FIG. 2. Wigner function W_{\max} of the maximally entangled state corresponding to the choice $a = -2/D$ in Eq. (32). The arrangement of figures is as in Fig. 1.

$$\psi_E(\vec{r}) = \mathcal{N}^{(D)} f_{k_0}^{(D)}(r) \quad (42)$$

for the energy wave function in a D -dimensional position space with the wave

$$f_{k_0}^{(D)}(r) \equiv \frac{J_{(D-2)/2}(k_0 r)}{r^{(D-2)/2}} = \frac{\sqrt{r} J_{(D-2)/2}(k_0 r)}{r^{(D-1)/2}} \quad (43)$$

and the normalization constant

$$\mathcal{N}^{(D)} \equiv \frac{\sqrt{M}}{\hbar} \frac{1}{\sqrt{S^{(D)}}}, \quad (44)$$

where $S^{(D)}$, Eq. (A10), denotes the surface of a sphere in D dimensions. Since ψ_E only depends on the hyperradius, ψ_E corresponds to a state of angular momentum zero.

B. Wigner function

The form, Eq. (43), of the wave $f_{k_0}^{(D)}$ makes it difficult to perform the necessary integrations in the definition, Eq. (11), of the Wigner function W_E corresponding to ψ_E . However, the wave vector representation of $\tilde{\psi}_E$ discussed in Appendix E is rather elementary and contains a δ function in k . This feature allows us to obtain the Wigner function W_E in a straightforward way starting from Eq. (17). In Appendix E we perform the necessary integrations and arrive at

$$W_E(\vec{r}, \vec{k}) = (\mathcal{N}^{(D)})^2 g_{k_0}^{(D)}(k) f_{q_0}^{(D-1)}(r \sin \theta), \quad (45)$$

with

$$g_{k_0}^{(D)}(k) \equiv \frac{1}{S^{(D-1)}} \frac{2}{\pi \Gamma\left(\frac{D-1}{2}\right)} \frac{(k_0^2 - k^2)^{(D-3)/4}}{k_0^{D-2} k} \quad (46)$$

and

$$q_0 = 2(k_0^2 - k^2)^{1/2}. \quad (47)$$

This expression describes W_E for $k \leq k_0$, only. In the domain $k_0 < k$ the Wigner function W_E vanishes exactly, which is a rather unusual behavior.

The expression, Eq. (45), for W_E consists of three basic elements: (i) the square of the normalization constant $\mathcal{N}^{(D)}$ of the wave function ψ_E , (ii) the function $g_{k_0}^{(D)}$ which is independent of position and depends solely on the wave vector and the number D of dimensions, and (iii) the wave $f_{q_0}^{(D-1)}$ which also describes the position dependence of the energy wave function ψ_E .

We now discuss these constituents of W_E in more detail and start our analysis with the square of $\mathcal{N}^{(D)}$ and the function $g_{k_0}^{(D)}$. The familiar definition, Eq. (4), of the Wigner function brings out most clearly that the marginal distribution

$$P(\vec{r}) = \int d^D k W(\vec{r}, \vec{k}) = |\psi(\vec{r})|^2, \quad (48)$$

that is, the Wigner function integrated over wave-vector space, must yield the probability density $P(\vec{r}) = |\psi(\vec{r})|^2$ in position space. As a consequence the normalization constant

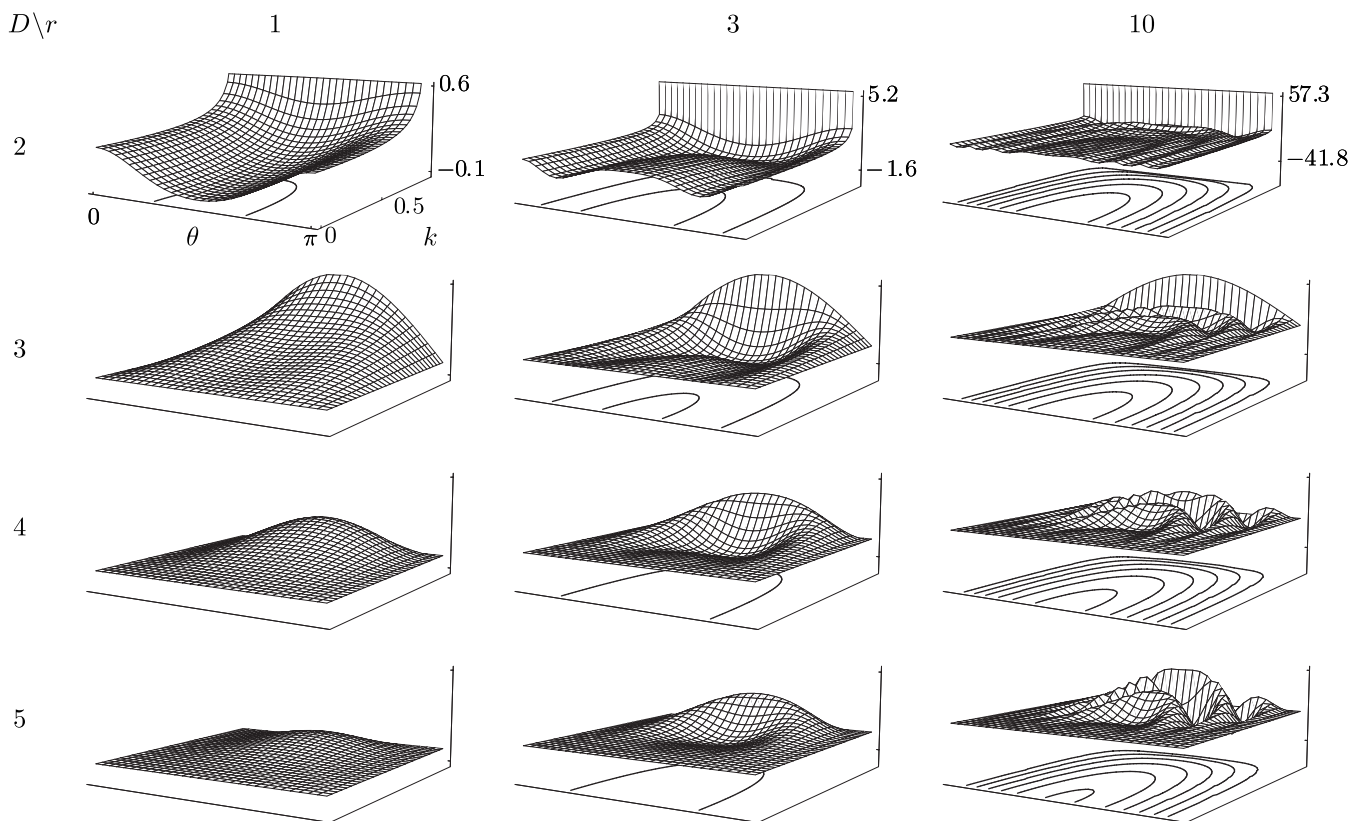


FIG. 3. Wigner function W_E corresponding to the free nonrelativistic particle in an eigenstate of energy $E \equiv (\hbar k_0)^2 / (2M)$ with $k_0 = 1$ in its dependence on the absolute value r of the position vector (horizontal) and the number D of dimensions (vertical). Here we do not show the Wigner function W_E but $V^{(D)}W_E = (rk)^{D-1}(\sin^{D-2} \theta)S^{(D)}S^{(D-1)}W_E$. Along the thick line the Wigner function vanishes, separating positive domains from negative domains. The horizontal axis θ and k are identical in all figures. However, the vertical axis changes with increasing angles—that is, going from one row to the next—but is identical in each column.

$\mathcal{N}^{(D)}$ has to appear quadratically in W_E . Moreover, the function $g_{k_0}^{(D)}$ is needed in order to obtain $P(\vec{r})$. In Appendix E we perform this integration over \vec{k} and verify that indeed we arrive at the position density.

However, the most important contribution to W_E originates from the wave $f_{q_0}^{(D-1)}$. Here it is remarkable that the position dependence of W_E enters through the same function which also determines the wave function ψ_E in position space. However, there are three subtleties: (i) Whereas the wave function ψ_E in D space dimensions follows from the wave $f_{q_0}^{(D)}$ in D dimensions, the corresponding Wigner function W_E follows from $f_{q_0}^{(D-1)}$ —that is, from the wave in $D-1$ dimensions—that is, a space whose dimensions are reduced by 1—(ii) the wave vector k_0 is replaced by q_0 defined by Eq. (47), and (iii) r is replaced by $r_2 \equiv r \sin \theta$.

C. Discussion

In order to compare and contrast the properties of a quantum free particle with its classical counterpart we choose the absolute value k of the wave vector and the angle θ between \vec{k} and \vec{r} as the variables of the system and analyze the behavior of W_E as a function of the number D of dimensions. The absolute value r of the position vector and the energy eigenvalue $E \equiv (\hbar k_0)^2 / (2M)$ are the characteristic parameters of

the system. We emphasize that this choice of variables is different from the one used in the discussion of the elementary s wave illustrated in Figs. 1 and 2.

In Fig. 3 we show the Wigner function W_E of a free particle for a fixed energy E determined by $k_0 = 1$ for three values of r and D . Moreover, the Wigner function is multiplied by the factor $V^{(D)} \equiv (rk)^{D-1}(\sin^{D-2} \theta)S^{(D)}S^{(D-1)}$ due to the D -dimensional volume element.

According to Eqs. (45)–(47) the Wigner function in two dimensions, shown in first row in Fig. 3, reads

$$V^{(2)}W_E = \frac{2r \cos(2\sqrt{k_0^2 - k^2}r \sin \theta)}{\sqrt{k_0^2 - k^2}} \quad (49)$$

and becomes singular for $k \rightarrow k_0$. This feature is reminiscent of the classical phase-space representation of a free particle with zero angular momentum. Here we find a singularity at the classical energy—that is, $k = k_0$. Furthermore, W_E assumes a maximum for $\theta = 0$ and $\theta = \pi$ which corresponds to the fact that the wave vector \vec{k} of the particle is parallel or antiparallel to the position vector \vec{r} .

In contrast the Wigner function W_E changes dramatically in higher dimensions. Already for $D = 3$ the Wigner function

$$V^{(3)}W_E = \frac{2}{k_0\pi}kr^2(\sin\theta)J_0(2\sqrt{k_0^2-k^2}r\sin\theta) \quad (50)$$

vanishes for $\theta=0$ and $\theta=\pi$. Furthermore, there is no cusp at $k=k_0$ anymore, but a maximum.

For even larger dimension—that is $D \geq 4$ —the deviation between the classical phase-space representation and the Wigner function of a free particle is more pronounced. In $D=5$ dimensions the Wigner function

$$V^{(5)}W_E = \frac{2}{k_0^3\pi}k^3r^3(\sin^2\theta)\sqrt{k_0^2-k^2}J_1(2\sqrt{k_0^2-k^2}r\sin\theta) \quad (51)$$

vanishes for $k=k_0$ and for $\theta=0$ or $\theta=\pi$. It reaches its maximal value for $\theta=\pi/2$, which corresponds to the classically forbidden movement along a circle.

In summary the borderline $k=k_0$ of phase space is rather special and crucially depends on the number of dimensions. In $D=2$ it is a cusp with an inverse square-root dependence. For $D=3$ the singularity is softened but still a maximum. For $D \leq 4$ there is an exact zero. Moreover, the values of the Wigner function at $\theta=0$, $\theta=\pi/2$, and $\theta=\pi$ show a strong dependence on the number of dimensions.

D. Phase-space equations

All features—that is, the hierarchy of wave functions and Wigner functions with respect to dimensions and the special form in which position and wave vector enter into the expression of W_E through q_0 and $r\sin\theta$ —follow from the differential equations

$$[4(k_0^2-k^2) + \Delta^{(D)}]W_E(\vec{r}, \vec{k}) = 0 \quad (52)$$

and

$$\vec{k} \cdot \frac{\partial}{\partial \vec{r}} W_E(\vec{r}, \vec{k}) = 0 \quad (53)$$

in phase space derived in Appendix E. The first equation corresponds to the Schrödinger equation whereas the second one is the remnant of the Liouville equation for a stationary state. We now rederive the expression, Eq. (45), for the Wigner function W_E by solving these two equations using a Fourier transform and by making use of the symmetry relation of the Fourier transform of the Wigner function of an s wave discussed in Appendix E 5.

1. Solution by Fourier transform

In order to solve the set of equations, Eqs. (52) and (53), we make a Fourier ansatz

$$W_E(\vec{r}, \vec{k}) = \int d^Dq e^{-i\vec{q}\cdot\vec{r}} \tilde{W}_E(\vec{q}, \vec{k}), \quad (54)$$

which when substituted into Eq. (52) immediately yields

$$[q_0^2 - q^2] \tilde{W}_E(\vec{q}, \vec{k}) = 0, \quad (55)$$

that is,

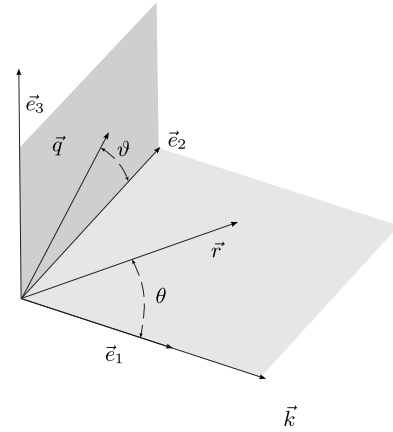


FIG. 4. Integration over the wave vector \vec{q} in the Fourier ansatz, Eq. (54), for the Wigner function W_E of an energy eigenstate. Since the Schrödinger equation (52) for W_E determines the length q_0 of \vec{q} , we are left with an integration over a sphere in D dimensions. Due to the transversality condition, Eq. (57), this integration takes place in the space orthogonal to \vec{k} , leading to a reduction of the number of dimensions. For the case of $D=3$ depicted here we find a circle.

$$\tilde{W}_E(\vec{q}, \vec{k}) = \delta(q_0^2 - q^2) \tilde{w}_E(\vec{q}, \vec{k}). \quad (56)$$

Consequently it is the Schrödinger-type equation Eq. (52) for the Wigner function W_E which determines the length of the wave vector \vec{q} to be q_0 as defined by Eq. (47).

The function $\tilde{W}_E(\vec{q}, \vec{k})$ still depends on the direction of \vec{q} and \vec{k} . However, Eq. (53) puts a constraint on the direction of \vec{q} with respect to \vec{k} which specifies W_E and explains the above-mentioned hierarchy. Indeed, when we substitute the ansatz, Eq. (54), into Eq. (53) we find the constraint

$$(\vec{k} \cdot \vec{q}) \tilde{W}_E(\vec{q}, \vec{k}) = 0, \quad (57)$$

which translates into the condition that the wave vector \vec{q} has to be always orthogonal on the wave vector \vec{k} . This requirement is reminiscent of the transversality condition enforced on the vector potential \vec{A} of electrodynamics by the Coulomb gauge

$$\vec{\nabla} \cdot \vec{A} = 0. \quad (58)$$

We are now in the position to perform the integration over \vec{q} in D dimensions. We first note that the Schrödinger-type equation (52) for W_E determines the length of \vec{q} to be q_0 , leaving us with $D-1$ integrations over angles. However, due to the transversality condition, Eq. (57), these angle integrations are restricted to the $D-1$ dimensions orthogonal to the wave vector \vec{k} .

In order to bring out this geometry most clearly we summarize the situation in Fig. 4 for the case of $D=3$. We align the coordinate system of \vec{q} integration such that the \vec{e}_1 axis is along the wave vector. The \vec{e}_2 axis is orthogonal to \vec{k} and the plane defined by \vec{r} and \vec{k} . The other $D-2$ dimensions are orthogonal to these axes. With this choice of the coordinate system the vectors \vec{q} and \vec{r} take the form

$$\vec{q} = (0, q_2, q_3, \dots, q_D) = (0, q_0 \cos \vartheta, q_3, \dots, q_D) \quad (59)$$

and

$$\vec{r} = (r_1, r_2, 0, \dots, 0) = r(\cos \theta, \sin \theta, 0, \dots, 0) \quad (60)$$

and the phase in the Fourier representation, Eq. (54), reads

$$\vec{q} \cdot \vec{r} = q_2 r_2 = q_0 \cos \vartheta r \sin \theta. \quad (61)$$

As a result we find the representation

$$W_E = \int d^{D-1} q e^{-i\vec{q} \cdot \vec{r}_2} \delta(q^2 - q_0^2) \tilde{w}_E(\vec{q}, \vec{k}) \quad (62)$$

of the Wigner function where $\vec{r}_2 \equiv r_2 \vec{e}_2$ and the function \tilde{w}_E depends only on the direction of \vec{q} and the wave vector \vec{k} .

In contrast to the original Fourier ansatz, Eq. (54), the integration over the directions of \vec{q} is now restricted to the unit sphere in $D-1$ dimensions. When we recall the well-known identity

$$\delta(x^2 - y^2) = \frac{1}{2|x|} [\delta(x - y) + \delta(x + y)] \quad (63)$$

and note that $0 \leq q$ and $0 \leq q_0$ we arrive at

$$W_E = \frac{1}{2q_0} \int d^{D-1} q e^{-i\vec{q} \cdot \vec{r}_2} \delta(q - q_0) \tilde{w}_E(\vec{q}, \vec{k}). \quad (64)$$

So far this calculation is valid for an energy eigenstate of a free particle of arbitrary angular momentum. We now specify this expression for an s wave.

2. Symmetry property of s wave

In Appendix E 4 we show that the Fourier transform \tilde{W} of the Wigner function W of any s wave depends only on three variables: namely, on the $q \equiv |\vec{q}|$, $k \equiv |\vec{k}|$ and on the angle $\chi \equiv \arccos[(\vec{q} \cdot \vec{k})/rq]$ between them—that is,

$$\tilde{W}(\vec{q}, \vec{k}) = \tilde{W}(q, k, \chi). \quad (65)$$

When we apply this property to the Fourier transform \tilde{W}_E of the energy eigenstate defined in Eq. (56) we find

$$\tilde{W}_E(\vec{q}, \vec{k}) = \delta(q_0^2 - q^2) \tilde{w}_E(q, k, \chi).$$

Due to the transversality condition, Eq. (57), the scalar product between \vec{q} and \vec{k} vanishes—that is, $\chi=0$. As a consequence, the expression, Eq. (64), of the Wigner function W_E reduces to

$$W_E = \frac{1}{2q_0} \tilde{w}_E(q_0, k, 0) \int d^{D-1} q e^{-i\vec{q} \cdot \vec{r}_2} \delta(q - q_0). \quad (66)$$

When we recall from Appendix E 1 the relation

$$f_{k_0}^{(D)}(r) = (2\pi k_0)^{-D/2} \int d^D q e^{-i\vec{q} \cdot \vec{r}} \delta(q - k_0), \quad (67)$$

we arrive at

$$W_E = \frac{1}{2q_0} \tilde{w}_E(q_0, k, 0) (2\pi q_0)^{(D-1)/2} f_{q_0}^{(D-1)}(r_2). \quad (68)$$

This expression is identical to the one obtained by direct integration, Eq. (45), when we identify

$$(\mathcal{N}^{(D)})^2 g_{k_0}^{(D)}(k) \equiv \frac{1}{2q_0} \tilde{w}_E(q_0, k, 0) (2\pi q_0)^{(D-1)/2}. \quad (69)$$

The right-hand side still depends on k through q_0 and \tilde{w}_E . The corresponding functional dependence can only be determined from arguments which go beyond the scope of the present paper.

V. SUMMARY

The Wigner function of an s wave in D dimensions depends only on three variables: the modulus of the position vector, the modulus of the wave vector, and the angle between them. We have derived integral representations of the s -wave Wigner function in terms of the wave functions in position representation or wave-vector representation. In each case we have expressed the Wigner function as a double integral with respect to a radial variable and a polar angle in the D -dimensional hyperspace. We have illustrated our formalism using the two examples of the elementary s wave and the energy eigenstate of a free particle with vanishing angular momentum. Here we have discussed the dependence of the results on the number of dimensions.

ACKNOWLEDGMENTS

We thank I. Białyński-Birula and D. Kobe for many fruitful discussions. One of us (S.V.) is grateful to the DAAD Grant Nos. A/01/19250 and A/04/21056 and to the Hungarian National Science Foundation (OTKA) Project No. T048324. Moreover, we appreciate the support of the Ministry of Science, Research and the Arts of Baden-Württemberg and the Landesstiftung Baden-Württemberg in the framework of the Quantum Information Highway A8 and the Center of Quantum Engineering.

APPENDIX A: HYPERSPHERICAL COORDINATES

In this appendix we briefly summarize the essential ingredients of the concept of hyperspherical coordinates. Here we follow very closely the treatment of Ref. [12].

We consider a vector \vec{b} with D components (b_1, b_2, \dots, b_D) in a Cartesian coordinate system defined by mutually orthogonal unit vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_D$. In hyperspherical coordinates $b \equiv |\vec{b}|$, ϑ , and $\varphi_1, \dots, \varphi_{D-2}$ the Cartesian components b_1, \dots, b_D take the form

$$b_1 = b \cos \vartheta,$$

$$b_2 = b \sin \vartheta \cos \varphi_1,$$

$$b_3 = b \sin \vartheta \sin \varphi_1 \cos \varphi_2,$$

⋮

$$b_{D-1} = b \sin \vartheta \sin \varphi_1 \sin \varphi_2 \cdots \cos \varphi_{D-2},$$

$$b_D = b \sin \vartheta \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{D-2}. \quad (\text{A1})$$

Here the range of b is $0 \leq b \leq \infty$. The $D-1$ angles $\vartheta, \varphi_1, \dots$, and φ_{D-2} cover different domains. Indeed, we have $0 \leq \varphi_i \leq \pi$ for $i=1, 2, 3, \dots, D-3$ but $-\pi \leq \varphi_{D-2} \leq \pi$.

In these coordinates the D -dimensional volume element $d^D b$ reads

$$d^D b = (db b^{D-1})(d\vartheta \sin^{D-2} \vartheta) \times (d\varphi_1 \sin^{D-3} \varphi_1) \cdots (d\varphi_{D-3} \sin \varphi_{D-3}) d\varphi_{D-2}. \quad (\text{A2})$$

The area

$$S^{(D)} \equiv \int_0^\pi d\vartheta \sin^{D-2} \vartheta \int d\omega \equiv \Omega_\vartheta \Omega_\varphi \quad (\text{A3})$$

of the unit sphere in D dimensions consists of the integral

$$\Omega_\vartheta \equiv \int_0^\pi d\vartheta \sin^{D-2} \vartheta \quad (\text{A4})$$

over the angle ϑ and the integral

$$\Omega_\varphi \equiv \int d\omega = \int_0^\pi d\varphi_1 \sin^{D-3} \varphi_1 \int_0^\pi d\varphi_2 \sin^{D-4} \varphi_2 \cdots \int_0^\pi d\varphi_{D-3} \sin \varphi_{D-3} \int_{-\pi}^\pi \varphi_{D-2} \quad (\text{A5})$$

over $D-2$ angles $\varphi_1, \varphi_2, \dots, \varphi_{D-2}$.

With the help of the integral formula [13]

$$\int_0^\pi d\varphi \sin^\nu \varphi = \sqrt{\pi} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu+2}{2}\right)}, \quad (\text{A6})$$

we find

$$\Omega_\vartheta = \sqrt{\pi} \frac{\Gamma\left(\frac{D-1}{2}\right)}{\Gamma\left(\frac{D}{2}\right)} \quad (\text{A7})$$

and

$$\Omega_\varphi = \sqrt{\pi} \frac{\Gamma\left(\frac{D-2}{2}\right)}{\Gamma\left(\frac{D-1}{2}\right)} \sqrt{\pi} \frac{\Gamma\left(\frac{D-3}{2}\right)}{\Gamma\left(\frac{D-2}{2}\right)} \cdots \sqrt{\pi} \frac{\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)} 2\pi, \quad (\text{A8})$$

that is,

$$\Omega_\varphi = 2 \frac{\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)}. \quad (\text{A9})$$

As a consequence the area $S^{(D)}$ of the unit sphere reads

$$S^{(D)} = \Omega_\vartheta \Omega_\varphi = 2 \frac{\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)}. \quad (\text{A10})$$

In Appendix B we perform the integral

$$\bar{\Omega}_\varphi \equiv \int_0^\pi d\varphi_2 \sin^{D-4} \varphi_2 \cdots \int_0^\pi d\varphi_{D-3} \sin \varphi_{D-3} \int_{-\pi}^\pi d\varphi_{D-2} \quad (\text{A11})$$

over the $D-3$ angles $\varphi_2, \dots, \varphi_{D-2}$ which with the help of the integral relation, Eq. (A6), takes the form

$$\bar{\Omega}_\varphi = \sqrt{\pi} \frac{\Gamma\left(\frac{D-3}{2}\right)}{\Gamma\left(\frac{D-2}{2}\right)} \cdots \sqrt{\pi} \frac{\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)} 2\pi,$$

that is,

$$\bar{\Omega}_\varphi = 2 \frac{\pi^{(D-2)/2}}{\Gamma\left(\frac{D-2}{2}\right)}. \quad (\text{A12})$$

APPENDIX B: INTEGRATION OVER ANGLES

In this appendix we perform the integration over $D-2$ angles in the definition of the Wigner function corresponding to an s wave. Here we pursue two different approaches: Whereas the first treatment starts from the definition of the Wigner function in terms of the position wave function, the second technique relies on the Wigner function in wave-vector representation. We demonstrate that the treatment presented in this appendix is also valid for $D=2$.

1. Position space

We start our derivation of the Wigner function of an s wave $\psi = \psi(r)$ from the expression

$$W(\vec{r}, \vec{k}) = (2\pi)^{-D} \int d^D \xi e^{-i\vec{k} \cdot \vec{\xi}} \psi^*(r_-) \psi(r_+), \quad (\text{B1})$$

with

$$r_\pm \equiv \left[\vec{r}^2 + \frac{1}{4} \vec{\xi}^2 \pm \vec{r} \cdot \vec{\xi} \right]^{1/2}. \quad (\text{B2})$$

a. Case of $3 \leq D$

The key element in performing the integration over $\vec{\xi}$ in Eq. (B1) is the appropriate alignment of the ξ -coordinate system with respect to the vectors \vec{r} and \vec{k} . Indeed, it is convenient to choose the \vec{e}_1 axis to be parallel to the position vector \vec{r} and the $\vec{e}_1 - \vec{e}_2$ plane to be defined by the position vector \vec{r} and the wave vector \vec{k} as shown in Fig. 5. With this choice we have

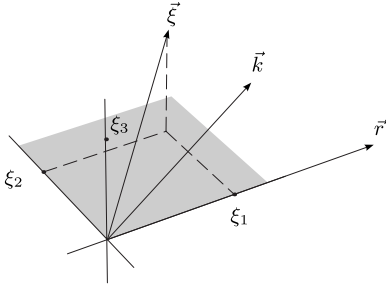


FIG. 5. Orientation of the $\vec{\xi}$ -coordinate system employed to perform the integration over the D Cartesian variables ξ_1, \dots, ξ_D relative to the given vectors \vec{r} and \vec{k} . The \vec{e}_1 axis defining the component ξ_1 of $\vec{\xi}$ is along \vec{r} whereas the \vec{e}_2 axis defining the component ξ_2 of $\vec{\xi}$ is orthogonal to \vec{r} and in the plane defined by \vec{r} and \vec{k} . Here we have only depicted the case $D=3$.

$$\vec{r} = r(1, 0, 0, \dots, 0) \quad (\text{B3})$$

and

$$\vec{k} = (k_1, k_2, 0, \dots, 0) = k(\cos \theta, \sin \theta, 0, \dots, 0). \quad (\text{B4})$$

In the so-selected coordinate system the scalar product $\vec{k} \cdot \vec{\xi}$ of the Fourier term $e^{-i\vec{k} \cdot \vec{\xi}}$ appears to be

$$\vec{k} \cdot \vec{\xi} = k_1 \xi_1 + k_2 \xi_2. \quad (\text{B5})$$

When we now use the hyperspherical coordinates discussed in Appendix A and express $\xi_1 = \xi \cos \vartheta$ and $\xi_2 = \xi \sin \vartheta \cos \varphi_1$ we find

$$\vec{k} \cdot \vec{\xi} = \xi k_1 \cos \vartheta + \xi k_2 \sin \vartheta \cos \varphi_1 \quad (\text{B6})$$

and with the choice Eq. (B3) for \vec{r} leading to $\vec{r} \cdot \vec{\xi} = r \xi_1 = r \xi \cos \vartheta$ the arguments r_{\pm} of the wave function ψ read

$$r_{\pm} = \left[r^2 + \frac{\xi^2}{4} \pm r \xi (\cos \vartheta) \right]^{1/2}. \quad (\text{B7})$$

Therefore, the integrand $e^{-i\vec{k} \cdot \vec{\xi}} \psi^*(r_-) \psi(r_+)$ of the D -dimensional integral defining the Wigner function, Eq. (B1), of an s wave depends only on the three integration variables ξ , ϑ , and φ_1 . Consequently we can cast the integral for W in the form

$$W = \int_0^{\infty} d\xi \int_0^{\pi} d\vartheta \mathcal{K}(k_1, k_2 | \xi, \vartheta) \psi^*(r_-) \psi(r_+), \quad (\text{B8})$$

where we have introduced the kernel

$$\mathcal{K} \equiv (2\pi)^{-D} \xi^{D-1} (\sin^{D-2} \vartheta) e^{-i\xi k_1 \cos \vartheta} \mathcal{C}_{(D-3)/2}(\xi k_2 \sin \vartheta) \bar{\Omega}_{\varphi} \quad (\text{B9})$$

together with the integral

$$\mathcal{C}_{\nu}(z) \equiv \int_0^{\pi} d\varphi_1 (\sin^{2\nu} \varphi_1) e^{-iz \cos \varphi_1} \quad (\text{B10})$$

and

$$\bar{\Omega}_{\varphi} \equiv \int_0^{\pi} d\varphi_2 (\sin^{D-4} \varphi_2) \cdots \int_0^{\pi} d\varphi_{D-3} (\sin \varphi_{D-3}) \int_{-\pi}^{\pi} d\varphi_{D-2}. \quad (\text{B11})$$

When we recall the identity

$$\int_0^{\pi} d\varphi (\sin^{2\nu} \varphi) e^{\pm iz \cos \varphi} = \sqrt{\pi} \left(\frac{2}{z} \right)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right) J_{\nu}(z), \quad (\text{B12})$$

where $J_{\nu}(z)$ denotes the Bessel function of order ν together with Eq. (A12) we find for the product

$$\mathcal{C}_{(D-3)/2} \bar{\Omega}_{\varphi} = (2\pi)^{(D-1)/2} (\xi k_2 \sin \vartheta)^{-(D-3)/2} J_{(D-3)/2}(\xi k_2 \sin \vartheta) \quad (\text{B13})$$

and the kernel Eq. (B9) reduces to

$$\mathcal{K} = \frac{\xi^{(D+1)/2} \sin^{(D-1)/2} \vartheta}{(2\pi)^{(D+1)/2} k_2^{(D-3)/2}} e^{-i\xi k_1 \cos \vartheta} J_{(D-3)/2}(\xi k_2 \sin \vartheta), \quad (\text{B14})$$

which upon substitution into Eq. (B8) yields the expression

$$W = (2\pi)^{-(D+1)/2} k_2^{-(D-3)/2} \int_0^{\infty} d\xi \xi^{(D+1)/2} \int_0^{\pi} d\vartheta (\sin^{(D-1)/2} \vartheta) \times J_{(D-3)/2}(\xi k_2 \sin \vartheta) e^{-i\xi k_1 \cos \vartheta} \psi^*(r_-) \psi(r_+) \quad (\text{B15})$$

for the Wigner function of an s wave. When we recall the definitions, Eq. (B4), of k_1 and k_2 we arrive at the result Eq. (11)

b. Case of $D=2$

At first sight the expression, Eq. (B15), is only valid for $3 \leq D$. We now show that it also contains the case $D=2$. For this purpose we start from the original definition, Eq. (B1), of the Wigner function

$$W = \frac{1}{(2\pi)^2} \int_0^{\infty} d\xi \xi \int_0^{2\pi} d\vartheta e^{-i\xi k \cos(\vartheta-\theta)} \psi^*(r_-) \psi(r_+) \quad (\text{B16})$$

in two dimensions and polar coordinates and show that the expression

$$W = (2\pi)^{-(D+1)/2} k_2^{-(D-3)/2} \int_0^{\infty} d\xi \xi^{(D+1)/2} \int_0^{\pi} d\vartheta (\sin^{(D-1)/2} \vartheta) \times J_{(D-3)/2}(\xi k_2 \sin \vartheta) e^{-i\xi k_1 \cos \vartheta} \psi^*(r_-) \psi(r_+) \quad (\text{B17})$$

following from Eq. (B15) for $D=2$ reduces to Eq. (B16). When we use the relation [13]

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi}} \frac{\cos x}{\sqrt{x}}, \quad (\text{B18})$$

we find

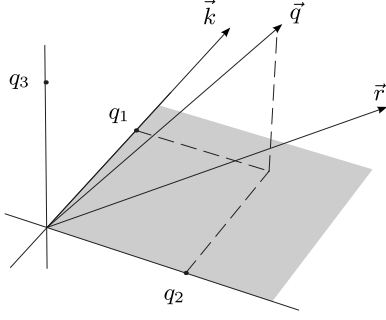


FIG. 6. Orientation of \vec{q} -coordinate system employed to perform the integration over the D Cartesian variables q_1, \dots, q_D relative to the given vectors \vec{r} and \vec{k} . In contrast to Fig. 5 the \vec{e}_1 axis defining the component q_1 of \vec{q} is along the wave vector \vec{k} and the \vec{e}_2 axis is orthogonal to \vec{k} and in the plane defined by \vec{r} and \vec{k} . Here we have only depicted the case $D=3$.

$$W = \frac{2}{(2\pi)^2} \int_0^\infty d\xi \xi \int_0^\pi d\vartheta \cos(\xi k \sin \theta \sin \vartheta) \times e^{-ik\xi \cos \theta \cos \vartheta} \psi^*(r_-) \psi(r_+) \quad (\text{B19})$$

or

$$W = \frac{1}{(2\pi)^2} \int_0^\infty d\xi \xi \int_0^\pi d\vartheta \psi^*(r_-) \psi(r_+) \times (e^{-ik\xi \cos(\theta-\vartheta)} + e^{-ik\xi \cos(\theta+\vartheta)}). \quad (\text{B20})$$

When we now use the substitution $\vartheta \rightarrow 2\pi - \vartheta$ for the second exponential and the fact that by definition, Eq. (B7), $r_\pm(\vartheta) = r_\pm(2\pi - \vartheta)$ we obtain the final result, Eq. (B16).

2. Wave-vector space

In the previous section we have derived the Wigner function of an s wave in terms of the position wave function. We now find a completely equivalent formulation in terms of the wave-vector representation.

Our starting point is the expression

$$W(\vec{r}, \vec{k}) = \frac{1}{(2\pi)^D} \int d^D q e^{i\vec{r} \cdot \vec{q}} \tilde{\psi}^*(k_-) \tilde{\psi}(k_+) \quad (\text{B21})$$

for the Wigner function based on the wave-vector representation $\tilde{\psi}$ with

$$k_\pm \equiv \left[\vec{k}^2 + \frac{1}{4} \vec{q}^2 \pm \vec{k} \cdot \vec{q} \right]^{1/2}. \quad (\text{B22})$$

As in the previous section we perform the integrations in hyperspherical coordinates. However, we now choose the coordinate system such that

$$\vec{r} = (r_1, r_2, 0, \dots, 0) \equiv r(\cos \theta, \sin \theta, 0, \dots, 0) \quad (\text{B23})$$

and

$$\vec{k} = k(1, 0, 0, 0, \dots, 0) \quad (\text{B24})$$

as shown in Fig. 6. We emphasize that compared to the in-

tegration in the position representation now \vec{r} and \vec{k} have changed their role in the coordinate system. Consequently \vec{k} points in the \vec{e}_1 direction and \vec{r} lies in the $\vec{e}_1 - \vec{e}_2$ plane.

As a consequence we find

$$\vec{r} \cdot \vec{q} = r_1 q_1 + r_2 q_2 = r_1 q \cos \vartheta + r_2 q \sin \vartheta \cos \varphi_1 \quad (\text{B25})$$

and

$$k_\pm = \left[k^2 + \frac{1}{4} q^2 \pm k q \right]^{1/2} = \left[k^2 + \frac{1}{4} q^2 \pm k q \cos \vartheta \right]^{1/2}. \quad (\text{B26})$$

Here we have expressed the components $q_1 \equiv q \cos \vartheta$ and $q_2 \equiv q \sin \vartheta \cos \varphi_1$ by the hyperspherical coordinates q , ϑ , and φ_1 .

Thus the integral, Eq. (B21), for W takes the form

$$W = \int_0^\infty dq \int_0^\pi d\vartheta \tilde{\mathcal{K}}(r_1, r_2 | q, \vartheta) \tilde{\psi}^*(k_-) \tilde{\psi}(k_+), \quad (\text{B27})$$

with the kernel

$$\tilde{\mathcal{K}} \equiv (2\pi)^{-D} q^{D-1} (\sin^{D-2} \vartheta) e^{iqr_1 \cos \vartheta} C_{(D-3)/2}^*(qr_2 \sin \vartheta) \bar{\Omega}_\varphi. \quad (\text{B28})$$

When we compare $\tilde{\mathcal{K}}$ to \mathcal{K} we find the elementary connection

$$\tilde{\mathcal{K}}(r_1, r_2 | q, \vartheta) = \mathcal{K}^*(r_1, r_2 | q, \vartheta). \quad (\text{B29})$$

As a consequence we find

$$W = (2\pi)^{-(D+1)/2} r_2^{-(D-3)/2} \int_0^\infty dq q^{(D+1)/2} \int_0^\pi d\vartheta (\sin^{(D-1)/2} \vartheta) \times J_{(D-3)/2}(qr_2 \sin \vartheta) e^{iqr_1 \cos \vartheta} \tilde{\psi}^*(k_-) \tilde{\psi}(k_+). \quad (\text{B30})$$

When we recall the definitions, Eq. (B23), of r_1 and r_2 we arrive at the result, Eq. (17).

3. Marginals

In Sec. II we show that the Wigner function of an s wave satisfies the marginal property. For this purpose we need to evaluate the integral

$$\mathcal{M} \equiv \int_0^\infty db b^{D-1} \int_0^\pi d\theta (\sin^{D-2} \theta) \int d\omega \mathcal{K}(b, \theta | \zeta, \eta), \quad (\text{B31})$$

that is, the kernel

$$\mathcal{K}(b, \theta | \zeta, \eta) \equiv (2\pi)^{-(D+1)/2} (b \sin \theta)^{-(D-3)/2} \zeta^{(D+1)/2} \sin^{(D-1)/2} \times \eta e^{-ib\zeta \cos \theta \cos \vartheta} J_{(D-3)/2}(\zeta b \sin \theta \sin \eta) \quad (\text{B32})$$

integrated over the first pair (b, θ) of variables together with the integral Ω_φ over the angles $\varphi_1, \dots, \varphi_{D-2}$ which yields

$$\mathcal{M} = (2\pi)^{-(D+1)/2} \Omega_\varphi \zeta^{(D+1)/2} (\sin^{(D-1)/2} \eta) \int_0^\infty db b^{(D+1)/2} \times I^{(D-3)/2}(\zeta b \sin \eta, \zeta \cos \eta). \quad (\text{B33})$$

Here we have recalled the definition of Ω_φ , Eq. (A9), and introduced the integral

$$I^{(\nu)}(\alpha, \beta) \equiv \int_0^\pi d\theta (\sin^{\nu+1} \theta) J_\nu(\alpha \sin \theta) e^{-i\beta \cos \theta}. \quad (\text{B34})$$

Since $\sin \theta$ is symmetric with respect to $\pi/2$, whereas $\cos \theta$ is antisymmetric, $I^{(\nu)}$ takes the form

$$I^{(\nu)}(\alpha, \beta) = \int_0^\pi d\theta (\sin^{\nu+1} \theta) J_\nu(\alpha \sin \theta) \cos(\beta \cos \theta), \quad (\text{B35})$$

which with the help of the identity [13]

$$I^{(\nu)}(\alpha, \beta) = \sqrt{2\pi} \alpha^\nu \frac{J_{\nu+1/2}(\sqrt{\alpha^2 + \beta^2})}{(\sqrt{\alpha^2 + \beta^2})^{\nu+1/2}} \quad (\text{B36})$$

leads to

$$\mathcal{M} = \frac{\pi^{-1/2}}{\Gamma\left(\frac{D-1}{2}\right)} \sin^{D-2} \eta \mathcal{B}(\zeta). \quad (\text{B37})$$

Here we have recalled the expression, Eq. (A9), for Ω_φ and have introduced the abbreviation

$$\mathcal{B}(\zeta) \equiv 2^{-(D-2)/2} \zeta^{D/2} \int_0^\infty db b^{(D-2)/2+1} J_{(D-2)/2}(b\zeta). \quad (\text{B38})$$

We now show that \mathcal{B} is a δ function. For this purpose we recall [13] the asymptotic behavior

$$2^\nu \Gamma(\nu + 1) \lim_{z \rightarrow 0} \frac{J_\nu(z)}{z^\nu} = 1 \quad (\text{B39})$$

of the Bessel function at the origin and insert this relation into the integral over b , which yields

$$\mathcal{B} = \Gamma\left(\frac{D}{2}\right) \lim_{\epsilon \rightarrow 0} \left(\frac{\zeta}{\epsilon}\right)^{(D-2)/2} \zeta \int_0^\infty db b J_{(D-2)/2}(b\zeta) J_{(D-2)/2}(\epsilon b). \quad (\text{B40})$$

In Appendix E 1 we rederive the orthogonality relation

$$\int_0^\infty dr r J_{(D-2)/2}(k_0 r) J_{(D-2)/2}(k'_0 r) = \frac{\delta(k_0 - k'_0)}{k'_0} \quad (\text{B41})$$

of the Bessel functions which leads to

$$\mathcal{B} = \Gamma\left(\frac{D}{2}\right) \lim_{\epsilon \rightarrow 0} \left(\frac{\zeta}{\epsilon}\right)^{(D-2)/2} \delta(\zeta - \epsilon) \quad (\text{B42})$$

or

$$\mathcal{B} = \Gamma\left(\frac{D}{2}\right) \delta(\zeta). \quad (\text{B43})$$

When we substitute this result into Eq. (B37) we obtain the final formula

$$\mathcal{M} = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{D}{2}\right)}{\Gamma\left(\frac{D-1}{2}\right)} (\sin^{D-2} \eta) \delta(\zeta) \quad (\text{B44})$$

for the integrated kernel.

APPENDIX C: WAVE-VECTOR REPRESENTATION

In this appendix we derive the wave function $\tilde{\psi}$ in wave number corresponding to the s wave

$$\Psi(\vec{r}) \equiv \frac{1}{\sqrt{S^{(D)}}} \frac{u(r)}{r^{(D-1)/2}} \quad (\text{C1})$$

in the D -dimensional hyperspace.

We emphasize that our derivation does not depend on the definition of a radial momentum operator but solely relies on quantum mechanics in Cartesian coordinates. In particular, we establish the relation

$$\tilde{\psi}(\vec{k}) = \frac{1}{\sqrt{S^{(D)}}} \frac{\tilde{u}(k)}{k^{(D-1)/2}}, \quad (\text{C2})$$

where the wave functions $u(r)$ and $\tilde{u}(k)$ are connected via the relation

$$\tilde{u}(k) = \int_0^\infty dr u(r) \sqrt{kr} J_{(D-2)/2}(kr). \quad (\text{C3})$$

Our derivation starts from the standard definition

$$\tilde{\psi}(\vec{k}) = (2\pi)^{-D/2} \int d^D r \psi(\vec{r}) e^{-i\vec{k}\cdot\vec{r}} \quad (\text{C4})$$

of the wave function $\tilde{\psi}(\vec{k})$ as the D -dimensional Fourier transform of the position wave function $\psi(\vec{r})$. When we substitute the special form, Eq. (C1), of the s wave together with the D -dimensional volume element [12]

$$\int d^D r = \int_0^\infty dr r^{D-1} \int_0^\pi d\vartheta (\sin^{D-2} \vartheta) \int d\omega \quad (\text{C5})$$

into the Fourier transform Eq. (C4), we arrive at

$$\tilde{\psi}(\vec{k}) = \frac{1}{\sqrt{S^{(D)}}} (2\pi)^{-D/2} \int_0^\infty dr r^{(D-1)/2} u(r) C_{(D-2)/2}(kr) \Omega_\varphi, \quad (\text{C6})$$

which with the help of the integral relations, Eqs. (A9) and (B12), for Ω_φ and C_ν reduces to

$$\tilde{\psi}(\vec{k}) = \frac{1}{\sqrt{S^{(D)}}} \frac{1}{k^{(D-1)/2}} \int_0^\infty dr u(r) \sqrt{kr} J_{(D-2)/2}(kr). \quad (\text{C7})$$

Here we note that due to the s wave nature of the wave function, only the absolute value k of the wave vector enters.

In the complete analogy to the radial wave function $u(r)$ determined by Eq. (C1) we can define the corresponding wave function $\tilde{u}(k)$ in wave-vector space via Eq. (C2) which yields

$$\tilde{u}(k) = \int_0^\infty dr u(r) \sqrt{kr} J_{(D-2)/2}(kr). \quad (\text{C8})$$

Hence, the two wave functions \tilde{u} and u are not connected via a simple Fourier transform, but rather a Hankel transform, which depends on the dimension of the hyperspace. This feature is a consequence of fact that the variable r can only assume positive values. For large distances r the influence of the origin of the r axis can be neglected. Indeed, when we use the asymptotic expression

$$J_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right), \quad (\text{C9})$$

all factors combine to the familiar factor $(2\pi)^{-1/2}$ and a plane wave.

APPENDIX D: ELEMENTARY s WAVE

In this appendix we perform in detail the calculations leading to the Wigner function $W_a^{(D)}$ of the elementary s wave:

$$\Psi^{(D)}(r) = \mathcal{N}[1 + a(\kappa r)^2] \left(\frac{\kappa^2}{\pi}\right)^{D/4} e^{-(\kappa r)^2/2}, \quad (\text{D1})$$

where the coefficients a and κ are real and with

$$\mathcal{N} \equiv \left[\left(1 + \frac{1}{2}aD\right)^2 + \frac{1}{2}a^2D \right]^{-1/2}. \quad (\text{D2})$$

Here we use the expression, Eq. (B15), for the Wigner function as a double integral over position space. In the course of this calculation we encounter an integral which we calculate in the second part of this appendix.

1. Formulation of problem

From the definition, Eq. (D1), of $\Psi^{(D)}$ together with the relation

$$r_\pm^2 = r^2 + \left(\frac{\xi}{2}\right)^2 \pm 2r\left(\frac{\xi}{2}\right) \cos \vartheta, \quad (\text{D3})$$

we find with the dimensionless hyperradius $\bar{r} \equiv \kappa r$ the expression

$$\begin{aligned} (\Psi^{(D)}(r_-))^* \Psi^{(D)}(r_+) &= \pi^{-D/2} e^{-\bar{r}^2} \mathcal{N}^2 \left\{ 1 + 2a\bar{r}^2 + a^2\bar{r}^4 \right. \\ &\quad + 2a(1 + a\bar{r}^2) \left(\frac{\kappa\xi}{2}\right)^2 + a^2 \left(\frac{\kappa\xi}{2}\right)^4 \\ &\quad \left. - 4a^2\bar{r}^2 \left(\frac{\kappa\xi}{2}\right)^2 \cos^2 \vartheta \right\} \kappa^D e^{-(\kappa\xi/2)^2}, \end{aligned} \quad (\text{D4})$$

which upon substitution into the double integral

$$\begin{aligned} W_a^{(D)} &= (2\pi)^{-(D+1)/2} k_2^{-(D-3)/2} \int_0^\infty d\xi \xi^{(D+1)/2} \int_0^\pi d\vartheta (\sin^{(D-1)/2} \vartheta) \\ &\quad \times J_{(D-3)/2}(\xi k_2 \sin \vartheta) e^{-i\xi k_1 \cos \vartheta} (\Psi^{(D)}(r_-))^* \Psi^{(D)}(r_+) \end{aligned} \quad (\text{D5})$$

for the Wigner function yields

$$\begin{aligned} W_a^{(D)} &= \pi^{-D} e^{-\bar{r}^2} \mathcal{N}^2 \{ (1 + 2a\bar{r}^2 + a^2\bar{r}^4) \mathcal{F}_{00} + 2a(1 + a\bar{r}^2) \mathcal{F}_{20} \\ &\quad + a^2 \mathcal{F}_{40} - 4a^2\bar{r}^2 \mathcal{F}_{22} \}. \end{aligned} \quad (\text{D6})$$

Here we have introduced the integral

$$\begin{aligned} \mathcal{F}_{2m2n} &\equiv \frac{2}{\sqrt{\pi}} \bar{k}_2^{-(D-3)/2} \int_0^\infty dy y^{(D+1)/2+2m} \int_0^\pi d\vartheta (\sin^{(D-1)/2} \vartheta) \\ &\quad \times e^{-y^2} \cos^{2n} \vartheta J_{(D-3)/2}(2y\bar{k}_2 \sin \vartheta) e^{-i2y\bar{k}_1 \cos \vartheta}, \end{aligned} \quad (\text{D7})$$

with the integration variable $y \equiv \kappa\xi/2$ and the dimensionless wave vector $\bar{k}_1 \equiv k_1/\kappa$ and $\bar{k}_2 \equiv k_2/\kappa$.

When we decompose the Fourier term

$$e^{-i\beta \cos \vartheta} = \cos(\beta \cos \vartheta) - i \sin(\beta \cos \vartheta), \quad (\text{D8})$$

the imaginary part does not contribute to the integral since $\sin \vartheta$ is symmetric with respect to $\pi/2$ whereas $\cos \vartheta$ is asymmetric. As a consequence, we arrive at the integral

$$\begin{aligned} \mathcal{F}_{2m2n} &\equiv 2\bar{k}_2^{-(D-3)/2} \pi^{-1/2} \int_0^\infty dy y^{(D+1)/2+2m} \\ &\quad \times e^{-y^2} I_{2n}^{(D-3)/2}(2y\bar{k}_2, 2y\bar{k}_1), \end{aligned} \quad (\text{D9})$$

where

$$I_{2n}^{(\nu)}(\alpha, \beta) \equiv \int_0^\pi d\vartheta (\sin^{\nu+1} \vartheta) \cos^{2n} \vartheta J_\nu(\alpha \sin \vartheta) \cos(\beta \cos \vartheta). \quad (\text{D10})$$

2. Calculation of \mathcal{F}_{2m0}

With the integral relation [13]

$$I_0^{(\nu)}(\alpha, \beta) = \sqrt{2\pi} \alpha^\nu \frac{J_{\nu+1/2}(\sqrt{\alpha^2 + \beta^2})}{(\alpha^2 + \beta^2)^{\nu+1/2}}, \quad (\text{D11})$$

we find for $\nu \equiv (D-3)/2$, $\alpha \equiv 2y\bar{k}_2$, and $\beta \equiv 2y\bar{k}_1$ together with $\bar{k}_1^2 + \bar{k}_2^2 \equiv \bar{k}^2 \equiv (k/\kappa)^2$ following from the definition Eq. (B4) of \bar{k} the result

$$I_0^{(D-3)/2}(2y\bar{k}_2, 2y\bar{k}_1) = \sqrt{\pi} \bar{k}_2^{-(D-3)/2} \bar{k}^{-(D-2)/2} y^{-1/2} J_{(D-2)/2}(2y\bar{k}). \quad (\text{D12})$$

When we substitute this expression into the definition, Eq. (D9), for \mathcal{F}_{2m2n} we arrive at the integral

$$\mathcal{F}_{2m0} = 2\bar{k}^{-(D-2)/2} \int_0^\infty dy y^{(D-2)/2+1+2m} e^{-y^2} J_{(D-2)/2}(2y\bar{k}), \quad (\text{D13})$$

which with the relation

$$\mathcal{L}_{2m}^{(\nu)} \equiv \int_0^\infty dy y^{\nu+1+2m} e^{-y^2} J_\nu(2y\gamma) = \frac{m!}{2} e^{-\gamma^2} \gamma^\nu L_m^{(\nu)}(\gamma^2) \quad (\text{D14})$$

reduces to

$$\mathcal{F}_{2m0} = m! e^{-\bar{k}^2} L_m^{(D-2)/2}(\bar{k}^2). \quad (\text{D15})$$

Here $L_m^{(\nu)}$ denotes the n th associated Laguerre polynomial.

When we recall the definitions [13]

$$L_0^{(\nu)}(x) = 1 \quad (\text{D16})$$

and

$$L_1^{(\nu)}(x) = 1 + \nu - x, \quad (\text{D17})$$

together with

$$L_2^{(\nu)}(x) = \frac{1}{2} [2 + 3\nu + \nu^2 - (4 + 2\nu)x + x^2], \quad (\text{D18})$$

we arrive at the explicit expressions

$$\mathcal{F}_{00} = e^{-\bar{k}^2} \quad (\text{D19})$$

and

$$\mathcal{F}_{20} = \left(\frac{D}{2} - \bar{k}^2 \right) e^{-\bar{k}^2}, \quad (\text{D20})$$

with

$$\mathcal{F}_{40} = \left[\frac{1}{4} D(D+2) - (D+2)\bar{k}^2 + \bar{k}^4 \right] e^{-\bar{k}^2}, \quad (\text{D21})$$

for the integrals \mathcal{F}_{2m0} .

3. Calculation of \mathcal{F}_{22}

We now turn to the higher-order integral

$$\mathcal{F}_{22} = 2k_2^{-(D-3)/2} \frac{1}{\sqrt{\pi}} \int_0^\infty dy y^{(D+1)/2+2} e^{-y^2} I_2^{(D-3)/2}(2y\bar{k}_2, 2y\bar{k}_1). \quad (\text{D22})$$

With the help of the relation

$$I_2^{(\nu)}(\alpha, \beta) = \sqrt{2\pi} \alpha^\nu \left[\frac{J_{\nu+3/2}(\sqrt{\alpha^2 + \beta^2})}{(\sqrt{\alpha^2 + \beta^2})^{\nu+3/2}} - \beta^2 \frac{J_{\nu+5/2}(\sqrt{\alpha^2 + \beta^2})}{(\sqrt{\alpha^2 + \beta^2})^{\nu+5/2}} \right] \quad (\text{D23})$$

derived in Sec. D5 we find

$$I_2^{(D-3)/2}(2y\bar{k}_2, 2y\bar{k}_1) = \frac{1}{2} \sqrt{\pi} \bar{k}_2^{-(D-3)/2} [\bar{k}^{-D/2} y^{-3/2} J_{D/2}(2y\bar{k}) - 2\bar{k}^{-(D+2)/2} \bar{k}_1^2 y^{-1/2} J_{(D+2)/2}(2y\bar{k})] \quad (\text{D24})$$

and Eq. (D22) reduces to

$$\mathcal{F}_{22} = \bar{k}^{-D/2} \mathcal{L}_0^{D/2} - 2\bar{k}^{-(D+2)/2} \bar{k}_1^2 \mathcal{L}_0^{(D+2)/2}. \quad (\text{D25})$$

Together with the integral relation Eq. (D14), that is,

$$\mathcal{L}_0^{D/2} = \frac{1}{2} e^{-\bar{k}^2} \bar{k}^{D/2} \quad (\text{D26})$$

and

$$\mathcal{L}_0^{(D+2)/2} = \frac{1}{2} e^{-\bar{k}^2} \bar{k}^{(D+2)/2}, \quad (\text{D27})$$

we arrive at

$$\mathcal{F}_{22} = e^{-\bar{k}^2} \left[\frac{1}{2} - \bar{k}^2 \cos^2 \theta \right], \quad (\text{D28})$$

where we have recalled the definition $k_1 = k \cos \theta$.

4. Summary

We are now in the position to substitute the explicit formulas Eqs. (D19)–(D21) and (D28) for \mathcal{F}_{00} , \mathcal{F}_{20} , \mathcal{F}_{40} , and \mathcal{F}_{22} , respectively, into the expression, Eq. (D6), for the Wigner function $W_a^{(D)}$. After minor algebra we arrive at

$$W_a^{(D)} = \pi^{-D} e^{-\bar{r}^2 - \bar{k}^2} \mathcal{N}^2 \left\{ \left(1 + \frac{aD}{2} \right)^2 + \frac{a^2 D}{2} + 2a \left[\left(1 + \frac{aD}{2} - a \right) \bar{r}^2 - \left(1 + \frac{aD}{2} + a \right) \bar{k}^2 \right] + a^2 (\bar{r}^4 + \bar{k}^4 - 2\bar{r}^2 \bar{k}^2 + 4\bar{r}^2 \bar{k}^2 \cos^2 \theta) \right\} \quad (\text{D29})$$

or

$$W_a^{(D)} = \pi^{-D} e^{-\bar{r}^2 - \bar{k}^2} (1 + a\mathcal{P}_a^{(D)}), \quad (\text{D30})$$

where we have recalled the definition, Eq. (D2), of the normalization constant \mathcal{N} and introduced the polynomial

$$\mathcal{P}_a^{(D)} \equiv \wp_- r^2 - \wp_+ k^2 + \wp(r^2 + k^2)^2 - 4\wp k^2 r^2 \sin^2 \theta, \quad (\text{D31})$$

with the coefficients

$$\wp_\pm(a, D) \equiv 2 \frac{1 + aD/2 \pm a}{(1 + aD/2)^2 + a^2 D/2} \quad (\text{D32})$$

and

$$\wp(a, D) \equiv \frac{a}{(1 + aD/2)^2 + a^2 D/2}. \quad (\text{D33})$$

5. Integral relation

We conclude by evaluating the integral

$$I_{2n}^{(\nu)}(\alpha, \beta) \equiv \int_0^\pi d\vartheta (\sin^{\nu+1} \vartheta) \cos^{2n} \vartheta J_\nu(\alpha \sin \vartheta) \cos(\beta \cos \vartheta), \quad (\text{D34})$$

which arises in Appendix D 1 in the calculation of the Wigner function corresponding to the elementary s wave using the integral relation [13]

$$I_0^{(\nu)}(\alpha, \beta) = \sqrt{2\pi} \alpha^\nu \frac{J_{\nu+1/2}(\sqrt{\alpha^2 + \beta^2})}{(\sqrt{\alpha^2 + \beta^2})^{\nu+1/2}} \quad (\text{D35})$$

together with the Feynman trick of “integration by differentiation.” Indeed, we find immediately the connection

$$I_{2n}^{(\nu)}(\alpha, \beta) = (-1)^n \frac{\partial^{2n}}{\partial \beta^{2n}} I_0^{(\nu)}(\alpha, \beta), \quad (\text{D36})$$

which with the explicit expression, Eq. (D35), for $I_0^{(\nu)}$ yields

$$I_{2n}^{(\nu)}(\alpha, \beta) = (-1)^n \sqrt{2\pi} \alpha^\nu \frac{\partial^{2n}}{\partial \beta^{2n}} \frac{J_{\nu+1/2}(\sqrt{\alpha^2 + \beta^2})}{(\sqrt{\alpha^2 + \beta^2})^{\nu+1/2}}. \quad (\text{D37})$$

In order to perform the differentiation in a convenient way we first recall the chain rule

$$\frac{\partial}{\partial \beta} = \frac{\partial z}{\partial \beta} \frac{\partial}{\partial z} = \beta \frac{\partial}{z \partial z} \quad (\text{D38})$$

for

$$z \equiv \sqrt{\alpha^2 + \beta^2}, \quad (\text{D39})$$

which provides us with the second derivative

$$\frac{\partial^2}{\partial \beta^2} = \frac{\partial}{\partial \beta} \left(\beta \frac{\partial}{z \partial z} \right) = \frac{\partial}{z \partial z} + \beta \frac{\partial}{\partial \beta} \left(\frac{\partial}{z \partial z} \right), \quad (\text{D40})$$

with respect to β —that is,

$$\frac{\partial^2}{\partial \beta^2} = \frac{\partial}{z \partial z} + \beta^2 \left(\frac{\partial}{z \partial z} \right)^2. \quad (\text{D41})$$

We then obtain from the recurrence relation [13]

$$\frac{\partial}{z \partial z} \left(\frac{J_\nu(z)}{z^\nu} \right) = - \frac{J_{\nu+1}(z)}{z^{\nu+1}} \quad (\text{D42})$$

of Bessel functions the expression

$$\frac{\partial^2}{\partial \beta^2} \left(\frac{J_{\nu+1/2}(z)}{z^{\nu+1/2}} \right) = - \frac{J_{\nu+3/2}(z)}{z^{\nu+3/2}} + \beta^2 \frac{J_{\nu+5/2}(z)}{z^{\nu+5/2}}, \quad (\text{D43})$$

which upon substitution into Eq. (D37) yields the result

$$I_{2n}^{(\nu)}(\alpha, \beta) = \sqrt{2\pi} \alpha^\nu \left[\frac{J_{\nu+3/2}(\sqrt{\alpha^2 + \beta^2})}{(\sqrt{\alpha^2 + \beta^2})^{\nu+3/2}} - \beta^2 \frac{J_{\nu+5/2}(\sqrt{\alpha^2 + \beta^2})}{(\sqrt{\alpha^2 + \beta^2})^{\nu+5/2}} \right], \quad (\text{D44})$$

for the case $n=1$.

APPENDIX E: FREE PARTICLE

In this appendix we first summarize the essential properties of an energy eigenstate of a free particle in D dimen-

sions. Here we derive the eigenfunctions ψ_E and $\tilde{\psi}_E$ in position and wave-vector space, respectively. We then use the latter to evaluate the integral, Eq. (B30), defining the Wigner function W_E of an energy eigenstate of a free particle. In order to test the result we verify the marginal property of the Wigner function. Since the energy wave function is normalized to a δ function, we only integrate over wave-vector space and obtain the square of the position wave function. We conclude by deriving equations in phase space determining the Wigner function W_E .

1. Wave-function essentials

We start by obtaining the normalization constant N of the energy wave function

$$\psi_E(\vec{r}) \equiv N \int d^D q e^{-i\vec{q}\cdot\vec{r}} \delta(q - k_0), \quad (\text{E1})$$

with $q \equiv |\vec{q}|$ from the orthonormality relation

$$\int d^D r \psi_E^*(\vec{r}) \psi_{E'}(\vec{r}) = \delta(E - E') = \frac{M}{\hbar^2} \frac{\delta(k_0 - k'_0)}{k_0}. \quad (\text{E2})$$

For this purpose we substitute the wave function ψ_E^* and $\psi_{E'}$ given by Eq. (E1) into the relation, Eq. (E2), and interchange the order of integrations, resulting in a δ function which allows us to cast the orthonormality condition into the form

$$(2\pi)^D |N|^2 \int d^D q \delta(q - k_0) \delta(q - k'_0) = \frac{M}{\hbar^2} \frac{\delta(k_0 - k'_0)}{k_0}. \quad (\text{E3})$$

Then we decompose the integration using hyperspherical coordinates discussed in Appendix A:

$$\int d^D q = \int_0^\infty dq q^{D-1} \int d\Omega = S^{(D)} \int_0^\infty dq q^{D-1}, \quad (\text{E4})$$

which yields

$$(2\pi)^D |N|^2 k_0^{D-1} S^{(D)} \delta(k_0 - k'_0) = \frac{M}{\hbar^2} \frac{\delta(k_0 - k'_0)}{k_0}, \quad (\text{E5})$$

that is,

$$N = \frac{\sqrt{M}}{\hbar} \frac{1}{\sqrt{S^{(D)}}} (2\pi k_0)^{-D/2} \equiv \mathcal{N}^{(D)} (2\pi k_0)^{-D/2}. \quad (\text{E6})$$

Next we perform the D integrations in the wave function ψ_E using hyperspherical coordinates, resulting in

$$\psi_E(\vec{r}) = N \int d\omega \int_0^\infty dq q^{D-1} \int_0^\pi d\vartheta (\sin^{D-2} \vartheta) \times e^{-iqr \cos \vartheta} \delta(q - k_0), \quad (\text{E7})$$

or

$$\psi_E(\vec{r}) = N k_0^{D-1} \Omega_\varphi \int_0^\pi d\vartheta (\sin^{D-2} \vartheta) e^{-ik_0 r \cos \vartheta}, \quad (\text{E8})$$

where according to Appendix A

$$\Omega_\varphi \equiv \int d\omega = 2 \frac{\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)} \quad (\text{E9})$$

is the integral over the $D-2$ angles $\varphi_1, \dots, \varphi_{D-2}$.

With the help of the integration relation, Eq. (B12), we can perform the integration over the angle ϑ and arrive at

$$\psi_E(\vec{r}) = N k_0^{D-1} \Omega_\varphi \sqrt{\pi} \frac{\Gamma\left(\frac{D-1}{2}\right)}{\left(\frac{k_0 r}{2}\right)^{(D-2)/2}} J_{(D-2)/2}(k_0 r) \quad (\text{E10})$$

or

$$\psi_E(\vec{r}) = \frac{\sqrt{M}}{\hbar} \frac{1}{\sqrt{S^{(D)}}} \frac{J_{(D-2)/2}(k_0 r)}{r^{(D-2)/2}} \equiv \mathcal{N}^{(D)} f_{k_0}^{(D)}(r). \quad (\text{E11})$$

In the last step we have made use of the expressions, Eqs. (E6) and (E9), for N and Ω_φ , respectively.

When we now compare the representation, Eq. (E1), of the energy wave function ψ_E to the explicit form, Eq. (E11), and recall the normalization constant N defined in Eq. (E6) we find

$$f_{k_0}^{(D)}(r) = (2\pi k_0)^{-D/2} \int d^D q e^{-i\vec{q}\cdot\vec{r}} \delta(q - k_0). \quad (\text{E12})$$

We now substitute the result, Eq. (E11), for ψ_E back into the orthonormality condition, Eq. (E2), to obtain the integral relation

$$\frac{M}{\hbar^2} \int_0^\infty dr r J_{(D-2)/2}(k_0 r) J_{(D-2)/2}(k'_0 r) = \frac{M}{\hbar^2} \frac{\delta(k_0 - k'_0)}{k_0}, \quad (\text{E13})$$

that is,

$$\int_0^\infty dr r J_{(D-2)/2}(k_0 r) J_{(D-2)/2}(k'_0 r) = \frac{\delta(k_0 - k'_0)}{k_0}, \quad (\text{E14})$$

which is important for the derivation of the wave-vector representation $\tilde{\psi}_E(k)$.

For this purpose we first identify, comparing Eqs. (C1) and (E11), the radial wave function

$$u_E(r) = \frac{\sqrt{M}}{\hbar} \sqrt{r} J_{(D-2)/2}(k_0 r) \quad (\text{E15})$$

of the energy eigenstate which when substituted into Eq. (C7) yields

$$\tilde{\psi}_E(k) = \frac{\sqrt{M}}{\hbar} \frac{1}{\sqrt{S^{(D)}}} k^{-(D-2)/2} \int_0^\infty dr r J_{(D-2)/2}(k_0 r) J_{(D-2)/2}(kr). \quad (\text{E16})$$

Hence, the integral relation, Eq. (E14), yields the expression

$$\tilde{\psi}_E(k) = \frac{\sqrt{M}}{\hbar} \frac{1}{\sqrt{S^{(D)}}} \frac{\delta(k - k_0)}{k_0^{D/2}} \quad (\text{E17})$$

for the wave vector representation $\tilde{\psi}_E$ of the energy eigenstate.

2. Wigner function

We are now in the position to calculate the Wigner function using the wave-vector representation, Eq. (B30). For this purpose we first evaluate the product

$$\tilde{\psi}_E^*(k_-) \tilde{\psi}_E(k_+) = (\mathcal{N}^{(D)})^2 \frac{1}{k_0^D} \delta(k_- - k_0) \delta(k_+ - k_0), \quad (\text{E18})$$

with

$$\mathcal{N}^{(D)} \equiv \frac{\sqrt{M}}{\hbar} \frac{1}{\sqrt{S^{(D)}}}, \quad (\text{E19})$$

and use the δ functions for

$$k_\pm(q, \vartheta) \equiv \left[k^2 + \frac{1}{4} q^2 \pm kq \cos \vartheta \right]^{1/2} \quad (\text{E20})$$

to perform the integrations over q and ϑ with the help of the relation

$$\begin{aligned} & \int dq \int d\vartheta g(q, \vartheta) \delta(k_-(q, \vartheta) - k_0) \delta(k_+(q, \vartheta) - k_0) \\ &= \int dq \int d\vartheta \mathcal{D}^{-1}(q, \vartheta) g(q, \vartheta) \delta(q - q_0) \delta(\vartheta - \vartheta_0) \\ &= \mathcal{D}^{-1}(q_0, \vartheta_0) g(q_0, \vartheta_0), \end{aligned} \quad (\text{E21})$$

where

$$\mathcal{D} \equiv \left| \det \begin{pmatrix} \frac{\partial k_+}{\partial q} & \frac{\partial k_+}{\partial \vartheta} \\ \frac{\partial k_-}{\partial q} & \frac{\partial k_-}{\partial \vartheta} \end{pmatrix} \right| = \frac{\partial k_+}{\partial q} \frac{\partial k_-}{\partial \vartheta} - \frac{\partial k_-}{\partial q} \frac{\partial k_+}{\partial \vartheta} \quad (\text{E22})$$

denotes the Jacobian determinant.

As a consequence the Wigner function, Eq. (30), reduces to

$$\begin{aligned} W_E &= (\mathcal{N}^{(D)})^2 \frac{1}{k_0^D} (2\pi)^{-(D+1)/2} \mathcal{D}^{-1}(q_0, \vartheta_0) q_0^{(D+1)/2} \sin^{(D-1)/2} \\ &\times \vartheta_0 \frac{J_{(D-3)/2}(2q_0 r_2)}{r_2^{(D-3)/2}} e^{iq_0 r_1 \cos \vartheta_0}. \end{aligned} \quad (\text{E23})$$

Here we obtain the quantities q_0 and ϑ_0 from Eq. (E20), that is, from

$$k^2 + \frac{1}{4} q_0^2 + kq_0 \cos \vartheta_0 = k_0^2 \quad (\text{E24})$$

and

$$k^2 + \frac{1}{4} q_0^2 - kq_0 \cos \vartheta_0 = k_0^2, \quad (\text{E25})$$

which yields

$$\vartheta_0 = \frac{\pi}{2} \quad (\text{E26})$$

and

$$q_0^2 = 4(k_0^2 - k^2). \quad (\text{E27})$$

The last equation is of particular importance. Since q is a real integration variable, q_0 has to be real as well, which implies $0 \leq q_0^2$. This condition translates into the condition $k \leq k_0$ on the variable k . On the other hand, when $k_0 < k$ we cannot satisfy Eq. (E27) for a real-valued q_0 . As a consequence the product $\delta(k_+ - k_0)\delta(k_- - k_0)$ of δ functions enforces that the Wigner function of an energy eigenstate vanishes for $k_0 < k$.

For this reason we now concentrate on the domain $k \leq k_0$ and evaluate

$$\left. \frac{\partial k_{\pm}}{\partial q} \right|_{\substack{q=q_0 \\ \vartheta=\vartheta_0}} = \frac{q_0/2 \pm k \cos \vartheta_0}{2k_{\pm}(q_0, \vartheta_0)} = \frac{(k_0^2 - k^2)^{1/2}}{2k_0} \quad (\text{E28})$$

and

$$\left. \frac{\partial k_{\pm}}{\partial \vartheta} \right|_{\substack{q=q_0 \\ \vartheta=\vartheta_0}} = \mp \frac{kq_0 \sin \vartheta}{2k_{\pm}(q_0, \vartheta_0)} = \mp \frac{k}{k_0} (k_0^2 - k^2)^{1/2}, \quad (\text{E29})$$

which yields

$$\mathcal{D}(q_0, \vartheta_0) = \frac{k}{k_0^2} (k_0^2 - k^2) \quad (\text{E30})$$

and Eq. (E23) simplifies to

$$W = (\mathcal{N}^{(D)})^2 \frac{1}{S^{(D-1)}} \times \frac{2}{\pi \Gamma\left(\frac{D-1}{2}\right)} \frac{(k_0^2 - k^2)^{(D-3)/4} J_{(D-3)/2}(2\sqrt{k_0^2 - k^2} r_2)}{k_0^{D-2} k r_2^{(D-3)/2}}, \quad (\text{E31})$$

where we have made use of the definition, Eq. (A10), of $S^{(D-1)}$.

3. Marginal

We now verify that the so-calculated Wigner function satisfies the marginal property

$$P(\vec{r}) \equiv |\psi_E(\vec{r})|^2 = \int d^D k W_E(\vec{r}, \vec{k}); \quad (\text{E32})$$

that is, we obtain the position probability density when we integrate W_E over \vec{k} in D -dimensional space. For this purpose we decompose the integration over \vec{k} into one over k and θ together with the unit sphere in $(D-1)$ dimensions. Since W_E only depends on k and θ , this integration immediately yields $S^{(D-1)}$ and thus

$$P = S^{(D-1)} \int_0^{k_0} dk k^{D-1} \int_0^{\pi} d\theta (\sin^{D-2} \theta) W_E(r, k, \theta). \quad (\text{E33})$$

When we substitute the explicit expression, Eq. (E31), for W_E we find

$$P = (\mathcal{N}^{(D)})^2 \frac{1}{r^{(D-3)/2}} \frac{2}{\pi \Gamma\left(\frac{D-1}{2}\right)} \int_0^{k_0} dk \int_0^{\pi} d\theta \sin^{(D-1)/2} \times \theta \left(\frac{k}{k_0}\right)^{D-2} (k_0^2 - k^2)^{(D-3)/2} J_{(D-3)/2}(2(k_0^2 - k^2)^{1/2} r \sin \theta). \quad (\text{E34})$$

With the help of the integral relation

$$\int_0^{\pi} d\theta (\sin^{\nu+1} \theta) J_{\nu}(\alpha \sin \theta) = \left(\frac{2\pi}{\alpha}\right)^{1/2} J_{\nu+1/2}(\alpha), \quad (\text{E35})$$

following from Eq. (D11) for $\beta=0$, we arrive at

$$P = (\mathcal{N}^{(D)})^2 \frac{1}{r^{(D-2)/2}} \frac{4}{\sqrt{\pi} \Gamma\left(\frac{D-1}{2}\right)} \int_0^{k_0} dk \left(\frac{k}{k_0}\right)^{D-2} \times (k_0^2 - k^2)^{(D-4)/4} J_{(D-2)/2}(2(k_0^2 - k^2)^{1/2} r). \quad (\text{E36})$$

In order to perform the remaining integration we introduce the transformation $k \equiv k_0 \cos \vartheta$ which yields $dk = -k_0 \sin \vartheta d\vartheta$ and thus

$$P = \frac{M}{\hbar^2} \frac{1}{S^{(D)}} \frac{1}{r^{(D-2)/2}} \frac{k_0^{(D-2)/2}}{\sqrt{\pi} \Gamma\left(\frac{D-1}{2}\right)} \int_0^{\pi/2} d\vartheta \cos^{D-2} \times \vartheta (\sin^{(D-2)/2} \vartheta) J_{(D-2)/2}(2k_0 r \sin \vartheta).$$

With the help of the integral relation

$$\int_0^{\pi/2} d\vartheta \cos^{2\nu} \vartheta \sin^{\nu} \vartheta J_{\nu}(z \sin \vartheta) = \frac{\sqrt{\pi}}{2} \left(\frac{2}{z}\right)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right) J_{\nu}^2\left(\frac{z}{2}\right), \quad (\text{E37})$$

we find for $\nu=(D-2)/2$ and $z=2k_0 r$ the final result

$$P = \frac{M}{\hbar^2} \frac{1}{S^{(D)}} \frac{J_{(D-2)/2}^2(k_0 r)}{r^{D-2}} = |\psi_E(\vec{r})|^2, \quad (\text{E38})$$

which is indeed the desired probability density.

4. Phase-space equations for W_E

We now derive differential equations in phase space determining the Wigner function W_E of an energy eigenstate $|E\rangle$ defined by

$$\hat{H}|E\rangle\langle E| = E|E\rangle\langle E|, \quad (\text{E39})$$

where

$$\hat{H} \equiv \frac{\hbar^2}{2M} \hat{\vec{k}}^2 \quad (\text{E40})$$

denotes the Hamiltonian of the free particle with the energy eigenvalue $E \equiv (\hbar k_0)^2 / (2M)$.

When we take the Weyl-Wigner transform of both sides of Eq. (E39) in the form

$$\hat{\vec{k}}^2 |E\rangle\langle E| = k_0^2 |E\rangle\langle E|, \quad (\text{E41})$$

we find with the help of the formula [15]

$$(\hat{A} \cdot \hat{B})(\vec{r}, \vec{k}) = A \left(\vec{r} - \frac{1}{2i} \frac{\partial}{\partial \vec{k}}, \vec{k} + \frac{1}{2i} \frac{\partial}{\partial \vec{r}} \right) B(\vec{r}, \vec{k}) \quad (\text{E42})$$

for two operators \hat{A} and \hat{B} with Weyl-Wigner representations A and B the relation

$$\left(\vec{k} + \frac{1}{2i} \frac{\partial}{\partial \vec{r}} \right)^2 W_E(\vec{r}, \vec{k}) = k_0^2 W_E(\vec{r}, \vec{k}). \quad (\text{E43})$$

Here we have made use of the fact that the Weyl-Wigner transform of the operator $\hat{\vec{k}}^2$ is \vec{k}^2 .

We take real and imaginary parts of Eq. (E43) and arrive at

$$\left[4(k_0^2 - k^2) - \frac{\partial}{\partial \vec{r}} \frac{\partial}{\partial \vec{r}} \right] W_E(\vec{r}, \vec{k}) = 0 \quad (\text{E44})$$

and

$$\vec{k} \cdot \frac{\partial}{\partial \vec{r}} W_E(\vec{r}, \vec{k}) = 0. \quad (\text{E45})$$

In Sec. IV D we solve these equations by a Fourier ansatz.

5. Symmetry of the Fourier transform

It is interesting to note that Eqs. (E44) and (E45) involve the position but not the wave-vector variable in the differentiations. Indeed, \vec{k} enters as a parameter only. This feature suggests that there must be more conditions determining the Wigner function of an energy eigenstate in higher dimensions. In a future publication we will derive these equations

from the Weyl-Wigner transform of the angular momentum operator. However, for the purpose of obtaining the Wigner function of the energy eigenstate of a free particle with vanishing angular momentum a more elementary approach is sufficient.

From Sec. II A we recall that the Wigner function of an s wave is invariant under a simultaneous rotation of \vec{r} and \vec{k} —that is,

$$W(\vec{r}, \vec{k}) = W(U\vec{r}, U\vec{k}), \quad (\text{E46})$$

where U denotes an arbitrary normal matrix.

We now show that this condition implies that the Fourier transform

$$\tilde{W}(\vec{q}, \vec{k}) \equiv \int d^D r W(\vec{r}, \vec{k}) e^{i\vec{q} \cdot \vec{r}} \quad (\text{E47})$$

of the corresponding Wigner function is also invariant under a simultaneous rotation of \vec{q} and \vec{k} . Indeed, when we substitute the isotropy condition, Eq. (E46), of the Wigner function into the definition, Eq. (E47), of the Fourier transform \tilde{W} we arrive at

$$\tilde{W}(\vec{q}, \vec{k}) = \int d^D r W(U\vec{r}, U\vec{k}) e^{i\vec{q} \cdot \vec{r}}, \quad (\text{E48})$$

which upon the substitution

$$\vec{\eta} \equiv U\vec{r}, \quad d^D \eta = d^D r, \quad (\text{E49})$$

yields

$$\tilde{W}(\vec{q}, \vec{k}) = \int d^D \eta W(\vec{\eta}, U\vec{k}) e^{i(U\vec{q})^T \cdot \vec{\eta}} = \tilde{W}(U\vec{q}, U\vec{k}). \quad (\text{E50})$$

Hence, the Fourier transform of the Wigner function of an s wave is invariant under a simultaneous rotation of \vec{q} and \vec{k} . In complete analogy to the Wigner function the Fourier transform

$$\tilde{W}(\vec{q}, \vec{k}) = \tilde{W}(|\vec{q}|, |\vec{k}|, \chi) \quad (\text{E51})$$

depends on the three variables $|\vec{q}|$, $|\vec{k}|$, and the angle χ between \vec{q} and \vec{k} only.

-
- [1] M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, *Phys. Rep.* **106**, 121 (1984).
 [2] W. P. Schleich, *Quantum Optics in Phase Space* (Wiley-VHC, Weinheim, 2001).
 [3] C. Zachos, D. Fairlie, and T. Curtright, *Quantum Mechanics in Phase Space: An Overview with Selected Papers* (World Scientific, Singapore, 2005).
 [4] M. V. Berry, *Philos. Trans. R. Soc. London, Ser. A* **287**, 237 (1977).
 [5] E. Wigner, *Phys. Rev.* **40**, 749 (1932).
 [6] Of course we assume that the number of particles is a factor of D .

- [7] J. P. Dahl, H. Mack, A. Wolf, and W. P. Schleich, *Phys. Rev. A* **74**, 042323 (2006).
 [8] See, for example, S. Stenholm and K.-A. Suominen, *Quantum Approach to Informatics* (Wiley, Hoboken, NJ, 2005).
 [9] For the evaluation of the Wigner function of the ground state of the hydrogen atom see, for example, J. P. Dahl and M. Springborg, *Mol. Phys.* **47**, 1001 (1982); M. Springborg and J. P. Dahl, *Phys. Rev. A* **36**, 1050 (1987); J. P. Dahl and M. Springborg, *ibid.* **59**, 4099 (1999); L. Praxmeyer and K. Wódkiewicz, *ibid.* **67**, 054502 (2003); L. Praxmeyer, J. Mostowski, and K. Wódkiewicz, *J. Phys. A* **39**, 14143 (2006).
 [10] J. P. Dahl, D. M. Greenberger, M. J. W. Hall, G. Süßmann, A.

- Wolf, and W. P. Schleich, *Laser Phys.* **15**, 18 (2005).
- [11] D. Fairlie, *Proc. Cambridge Philos. Soc.* **60**, 581 (1964); W. Kundt, *Z. Naturforsch. A* **22**, 1333 (1967); J. P. Dahl, in *Energy Storage and Redistribution in Molecules*, edited by J. Hinze (Plenum, New York, 1983), p. 557; L. Wang and R. O'Connell, *Found. Phys.* **18**, 1023 (1988).
- [12] A. Sommerfeld, *Partielle Differentialgleichungen der Physik*, 7th ed. (Verlag Harry Deutsch, Thun, Switzerland, 1992).
- [13] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed. (Academic, San Diego, 1980).
- [14] I. Białynicki-Birula, M. A. Cirone, J. P. Dahl, M. V. Fedorov, and W. P. Schleich, *Phys. Rev. Lett.* **89**, 060404 (2002).
- [15] S. de Groot and L. Suttorp, *Foundations of Electrodynamics* (North-Holland, Amsterdam, 1972).