Equations for relativistic particles, binaries, and ternaries

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Relativistic differential equations for bound states of n Dirac particles ($n=1,2,3$) are put into forms that depend only on the square of the total center-of-mass system energy. Instead of coupling 4^n Dirac components, they decompose into two equivalent equations for $4^n/2$ components ψ and χ of total chiralities +1 and −1, respectively. Their time-dependent versions are of the Klein-Gordon type; they reproduce the relativistic kinematics for the emission of photons or pions. Although the equations are presently extended to mesons and baryons within perturbative QCD only, the necessity of free Klein-Gordon equations for closed systems implies E^2 spectra not only for mesons, but also for baryons. For binaries, there exists an intricate transformation which turns the equation for ψ into an effective one-body Dirac equation. The corresponding transformation for χ is derived here.

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I. INTRODUCTION

Some years ago, an eight-component equation has been derived for the bound states of two fermions, in the first place for fermions with small anomalous magnetic moments [1]. Similar equations have recently been found for n fermions, which have $4^n/2$ components instead of the 4^n components of Dirac-Breit and Bethe-Salpeter equations [2]. By eliminating half of the Dirac components, one finds that the equations actually have the square of the total center-of-mass system (c.m.s.) energy E as eigenvalues, not E itself. Finally, time-dependent forms have been found in which E^2 is replaced by $-\partial_t^2$ ($\partial_t=\partial/\partial_t$, units $\hbar=c=1$), precisely as in a Klein-Gordon equation [3]. It has also been shown that $-\partial_t^2$ is necessary for the relativistic recoil in radiative decays [4]. The energy of a photon emitted in the transition from an atomic state of c.m.s. energy E to the ground state of energy E_{gr} in its own c.m.s. is

$$\omega=(E^2-E_{gr}^2)/2E. \tag{1}$$

The deviations from the recoil-free $\omega=E-E_{gr}$ are so small in atoms that they can be treated as nonrelativistic corrections. Although this paper uses atomic notation, the E^2 spectrum must also apply to mesons and baryons. In Sec. IV, Eq. (1) will be generalized to pion emission by the baryon "resonances," which via unitarity and analyticity also affects the baryon mass splittings themselves, as a kind of "pionic Lamb shift."

The linear, 4^n-component equations $i\partial_t\psi=H\psi$ contradict relativity except for $n=1$, where the Dirac equation has an equivalent two-component Kramers version [5–7]. As the present formalism assumes the conservation of energy and momentum, the only relevant equation is that of a free electron, which is known to imply a free Klein-Gordon equation. However, a discrete electron spectrum does appear in Landau levels, where the energies of the emitted photons also follow Eq. (1) [3] (the relativistic Landau levels are equidistant in E^2).

The Kramers equation is recalled here because it displays an unusual property of all equations with only $4^n/2$ components: it is not Hermitian. For $n=1$, this follows simply from the elimination of components of the Dirac equation. Decomposition of the Dirac spinor ψ_D into components ψ_r and ψ_l of chirality +1 and −1 (eigenstates of the chirality matrix γ^5) produces two coupled equations. With the momentum operators $\pi^\mu=p^\mu+eA^\mu$, the Dirac equation assumes the form

$$(\pi^0-\boldsymbol{\sigma\pi})\psi_r=m\psi_l,\quad(\pi^0+\boldsymbol{\sigma\pi})\psi_l=m\psi_r, \tag{2}$$

where the $\boldsymbol{\sigma}$ are Pauli matrices. Using the first equation for the elimination of ψ_l, one obtains a second-order equation for ψ_r alone:

$$(\pi^0+\boldsymbol{\sigma\pi})(\pi^0-\boldsymbol{\sigma\pi})\psi_r=m^2\psi_r. \tag{3}$$

The product of operators in Eq. (3) is not Hermitian; the Hermitian conjugate operator appears in the equation for ψ_l. The expectation value of its anti-Hermitian component vanishes.

For $n>1$, one may similarly derive the second-order equations from the Dirac-Breit equations [8]. However, these derivations are only approximate. They fail for the inclusion of vacuum polatization and contain incorrect squares of Breit operators which are normally eliminated by "positive-energy projectors." The correct derivation of the interaction uses instead the Born series of quantum electrodynamics (QED). It produces a non-Hermitian operator M^2, which occurs in two equivalent equations,

$$\Box\psi=M^2\psi,\quad\Box\chi=(M^2)^\dagger\chi,\quad\Box=\nabla_R^2-\partial_{t,lab}^2. \tag{4}$$

ψ and χ have total chirality +1 and −1, but M^2 does not factorize as in Eq. (3).

The Kramers equation (3) is invariant under a somewhat complicated parity transformation, $\psi_r'=m^{-1}(\pi^{0\prime}+\boldsymbol{\sigma\pi}')\psi_r(-\mathbf{r})$, which is another way of exchanging $\psi_r(\mathbf{r})$ with $\psi_l(-\mathbf{r})$ [in Dirac notation, this is simply $\psi_D'=\beta\psi_D(-\mathbf{r})$, where $\beta=\gamma^0$ is the matrix which exchanges ψ_r with ψ_l]. As M^2 does not factorize for $n=3$, that equation is

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not parity invariant: ψ must be exchanged with χ. For even n, however, the total chirality operator $\gamma^5_{tot}=\gamma^5_1\ldots\gamma^5_n$ commutes with the parity operator $\beta=\beta_1\ldots\beta_n$, such that the separation into ψ and χ is parity invariant. The binary equation ($n=2$) is rederived in the next section. The derivation of the corresponding equation for χ is new to the best of my knowledge.

Difficulties with the 4^n-component formalism have promoted a very different approach for $n=2$, namely nonrelativistic expansions of the QED and QCD (quantum chromodynamics) field couplings ("NRQED," "NRQCD"), by which one can calculate the energy levels to a certain order in $\alpha=e^2$, using perturbation theory of the two-body Schrödinger equation, with the (nonrelativistic) reduced mass, $H=\mathbf{p}^2/2\mu_{nr}+V$, $V=-\alpha/r_{12}$ [9]. The method ends presently at second-order perturbation theory, which gives the energy levels to the order α^6. Encouraged by this success, an old claim [10] has been revived that relativistic equations for fixed numbers of particles cannot exist. It is true that the form (2) of the Dirac equation is not exact, but there exists a form with a more precise g factor, $g_e=2+\alpha/\pi+\ldots$. The corrections to $g_e=2$ are calculated by the relativistic Feynman rules with a precision that remains to be reached by NRQED. The leading α^8 term of the muonium ($e^-\mu^+$) hyperfine structure follows from first-order perturbation theory of the Dirac equation, and for uranium atoms, the equation can be solved numerically if necessary.

II. THE BINARY EQUATION

The equation is best constructed by first adding the Dirac Hamiltonians for two free fermions, then by reducing the result in the c.m.s. ($\mathbf{p}_1+\mathbf{p}_2=0$) to an eight-component equation, and finally by adding the interaction as the Fourier transform of the Born series of elastic c.m.s. scattering, after it has also been reduced to an 8×8 matrix [1]. However, this interaction has not yet been found from the scattering of three particles. We therefore use the less precise Dirac-Breit equation as a starting point [8]. It also contains the sum H_0 of the free Dirac Hamiltonians. With $\boldsymbol{\alpha}_i=\gamma^5_i\boldsymbol{\sigma}_i$,

$$H_0=m_+ +\Sigma_i\gamma^5_i\boldsymbol{\sigma}_i\mathbf{p}_i,\quad m_+=\Sigma_i m_i\beta_i. \tag{5}$$

The zero component of the total four-momentum operator enters as

$$\pi^0_{tot}=i\Sigma_i\partial t_i-\Sigma_{i<j}V_{ij},\quad V_{ij}=q_iq_j/r_{ij},\quad q_1=-e. \tag{6}$$

The only other interaction is provided by the Breit operator H_B, to be specified below,

$$(\pi^0_{tot}-\Sigma_i\gamma^5_i\boldsymbol{\sigma}_i\mathbf{p}_i-H_B)\psi_{DB}=m_+\psi_{DB}. \tag{7}$$

The equation is now split into two equations for the components $\psi^{(+)}$ and $\psi^{(-)}$ of ψ_{DB} which have total chirality +1 and −1, respectively,

$$(\pi^0_{tot}-\Sigma_i\gamma^5_i\boldsymbol{\sigma}_i\mathbf{p}_i-H_B)\psi^{(+)}=m_+\psi^{(-)}, \tag{8}$$

$$(\pi^0_{tot}-\Sigma_i\gamma^5_i\boldsymbol{\sigma}_i\mathbf{p}_i-H_B)\psi^{(-)}=m_+\psi^{(+)}. \tag{9}$$

In Eq. (8), $\Pi_i\gamma^5_i=1$ is understood, and in Eq. (9), $\Pi_i\gamma^5_i=-1$.

For $n=2$, one may define $\gamma^5_1=\gamma^5$, which implies $\gamma^5_2=+\gamma^5$ and $-\gamma^5$ in Eqs. (8) and (9), respectively. The Breit operator may be written as

$$H_B=\gamma^5_1\gamma^5_2 b,\quad b=-\frac{1}{2}V_{12}(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2+\sigma_{1r}\sigma_{2r}), \tag{10}$$

with $\sigma_{ir}=\mathbf{r}\boldsymbol{\sigma}_i/r$ and $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$. This leads to the following forms in Eqs. (8) and (9):

$$[\pi^0_{tot}-\gamma^5(\boldsymbol{\sigma}_1\mathbf{p}_1+\boldsymbol{\sigma}_2\mathbf{p}_2)-b]\psi^{(+)}=m_+\psi^{(-)}, \tag{11}$$

$$[\pi^0_{tot}-\gamma^5(\boldsymbol{\sigma}_1\mathbf{p}_1-\boldsymbol{\sigma}_2\mathbf{p}_2)+b]\psi^{(-)}=m_+\psi^{(+)}. \tag{12}$$

Elimination of $\psi^{(-)}$ by means of the first equation gives an equation for $\psi^{(+)}$ alone. With the abbreviations

$$\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2=\boldsymbol{\sigma},\quad \boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2=\Delta\boldsymbol{\sigma},\quad \mathbf{p}=\tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2),$$

$$\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2, \tag{13}$$

the equation becomes

$$\left[\pi^0_{tot}-\gamma^5\left(\boldsymbol{\sigma}\mathbf{p}+\tfrac{1}{2}\Delta\boldsymbol{\sigma}\mathbf{P}\right)+b\right]m_+^{-1}\times\left[\pi^0_{tot}-\gamma^5\left(\Delta\boldsymbol{\sigma}\mathbf{p}+\tfrac{1}{2}\boldsymbol{\sigma}\mathbf{P}\right)-b\right]\psi^{(+)}=m_+\psi^{(+)}. \tag{14}$$

For $\mathbf{p}_1^2=\mathbf{p}_2^2$, the product of the two inner brackets follows from the original expressions as

$$(\boldsymbol{\sigma}_1\mathbf{p}_1)^2-(\boldsymbol{\sigma}_2\mathbf{p}_2)^2=\mathbf{p}_1^2-\mathbf{p}_2^2=0. \tag{15}$$

This cancellation reduces the second-order differential operator to a first-order one, which has no longer the product form. The stationary solutions have $\pi^0_{tot}=E-V$, $V=V_{12}$. The V^2 of $(E-V)^2$ is eliminated by a shift of the distance, $r_{12}=r+\alpha/2E$ [11]. This entails a change in the angular part of b, which is canceled by the "retardation" part of the Breit operator. There remains the unretarded Gaunt interaction,

$$b_G=-V\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2. \tag{16}$$

The b^2 does not cancel, but it must be omitted anyway. One reason is that b is expected to be valid as a first-order perturbation only. The decisive fact, however, is that b^2 is as large as V^2 and leads to wrong results [12]. In the standard Dirac-Breit equation, b^2 is eliminated by "positive-energy projectors." An alternative approach, avoiding these problems with the Gaunt term was done by Van Alstine and Crater [15]. They constructed a form of the two-body Dirac equation which has an interaction structure quite similar to that of the Gaunt term but differing at higher orders so that the same spectral results as the Breit interaction would be produced. In the QED derivation, the interaction of Eq. (14) arises from the first Born term of elastic scattering,

$$T^{(1)}_{if}=-4\pi q_1q_2 j^\mu_1 g_{\mu\nu} j^\nu_2/(-\mathbf{q}^2),\quad j^\mu_i=\bar{\psi}'_i\gamma^\mu_i\psi_i, \tag{17}$$

after its reduction to an 8×8 matrix. The second Born term $T^{(2)}_{if}$ does not contribute to V^2, i.e., all V^2 terms including b^2 are absent. The consistent equation derived from Eq. (14) is thus

$$\left[E^2-2EV-\left(\boldsymbol{\sigma}\mathbf{p}+\tfrac{1}{2}\Delta\boldsymbol{\sigma}\mathbf{P}\right)(E-V_-)\tilde{\gamma}^5-(E-V_+)\gamma^5\left(\Delta\boldsymbol{\sigma}\mathbf{p}+\tfrac{1}{2}\boldsymbol{\sigma}\mathbf{P}\right)\right]\psi^{(+)}=m_+^2\psi^{(+)}, \tag{18}$$

$$V_\pm = V(1 \pm \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2), \quad \tilde{\gamma}^5 = m_+\gamma^5/m_+. \tag{19}$$

The algebra of Pauli matrices gives

$$(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2 + i\boldsymbol{\sigma}^\times, \quad (\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_1 - i\boldsymbol{\sigma}^\times,$$

$$\boldsymbol{\sigma}^\times = \boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2. \tag{20}$$

Next, we use the c.m. system, $\mathbf{P}\psi=0$. The remaining combinations of Pauli matrices are

$$\boldsymbol{\sigma}(1 - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2) = 0, \quad (1 + \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\Delta\boldsymbol{\sigma} = 2i\boldsymbol{\sigma}^\times. \tag{21}$$

As a result, Eq. (18) is simplified as follows:

$$[E^2 - 2EV - E\mathbf{p}(\tilde{\gamma}^5\boldsymbol{\sigma} + \gamma^5\Delta\boldsymbol{\sigma}) + 2i\gamma^5 V\mathbf{p}\boldsymbol{\sigma}^\times]\psi^{(+)} = m_+^2\psi^{(+)}. \tag{22}$$

The $\tilde{\gamma}^5$ of Eq. (18) satisfies the same Dirac algebra as γ^5, but it is not Hermitian,

$$(\tilde{\gamma}^5)^2 = 1, \quad \{\tilde{\gamma}^5, \beta\} = 0, \quad \beta \equiv \beta_1\beta_2. \tag{23}$$

Notice that β_1 and β_2 occur in m_+^2 in the combination

$$m_+^2 = m_1^2 + m_2^2 + 2m_1m_2\beta. \tag{24}$$

There exists a very special transformation which makes $\tilde{\gamma}^5$ Hermitian,

$$\psi^{(+)} = C_1\psi, \quad C_1 = (m_+m_-)^{-1/2}(m_2\beta_2 + m_1\beta_1P_{spin}), \tag{25}$$

$$m_\pm = m_1\beta_1 \pm m_2\beta_2, \quad P_{spin} = \frac{1}{2}(1 + \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2). \tag{26}$$

The spin exchange operator P_{spin} is diagonal in the triplet-singlet spin basis, with eigenvalues +1 and −1, respectively. As $\boldsymbol{\sigma}$ operates only between triplet states, one may here set $P_{spin}=1$, $C_1=(m_+/m_-)^{1/2}$. Both $\Delta\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^\times$ are antidiagonal in this basis, which implies opposite signs of P_{spin} to their right and left, $C_1\Delta\boldsymbol{\sigma}=\Delta\boldsymbol{\sigma}C_1^{-1}$. Multiplication by C_1^{-1} from the left and using $C_1^{-1}\gamma^5=\gamma^5C_1$ thus removes the tilde from $\tilde{\gamma}^5\boldsymbol{\sigma}$, without affecting the other operators. The result is a one-body Dirac equation with somewhat different units,

$$[E^2 - m_1^2 - m_2^2 - 2EV - 2E\gamma^5\mathbf{p}\boldsymbol{\sigma}_1 + 2i\gamma^5V\mathbf{p}\boldsymbol{\sigma}^\times]\psi = 2m_1m_2\beta\psi. \tag{27}$$

It may be compared with the nonrelativistic one-body reduction, which uses $\mathbf{p}_1^2/2m_1+\mathbf{p}_2^2/2m_2=\mathbf{p}^2/2\mu_{nr}$. In both cases, the reduction breaks down in the presence of a magnetic field, but not for an electric dipole field. After division by $2E$, Eq. (27) takes a familiar form,

$$(\epsilon - V - \gamma^5\mathbf{p}\boldsymbol{\sigma}_1 - V_{hf})\psi = \mu\beta\psi, \quad \mu = m_1m_2/E, \tag{28}$$

where $V_{hf}=-i\gamma^5V\mathbf{p}\boldsymbol{\sigma}^\times/E$ is a hyperfine operator. The solutions of Eq. (27) have been discussed extensively [8]. They are symmetric under the exchange $m_1\leftrightarrow m_2$ and contain the signs of m_1 and m_2 only in the combination m_1m_2. The sum $m_{12}=m_1+m_2$ which occurs in the "external field approximation" [13] becomes replaced by E, as in the reduced mass μ of Eq. (28). The only exception is another factor (m_1-m_2), in which case even powers of $m_1^2-m_2^2$ are allowed, as in the "Salpeter shift." The first-order perturbative energy shift $\delta E^2=2E\delta E$ contains only even powers of E.

An extra factor P_{12} in front of C_1 in Eq. (25) exchanges $\boldsymbol{\sigma}_1$ with $\boldsymbol{\sigma}_2$. The E^2 dependence appears after a rescaling of $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$:

$$\mathbf{r}_E = \mathbf{r}/E, \quad \mathbf{p}_E = E\mathbf{p} = -i\boldsymbol{\nabla}_{\mathbf{r}E}, \quad V_E = -\alpha/r_E. \tag{29}$$

The time-independent version of M^2 in Eq. (4) is thus

$$M_0^2 = m_+^2 + 2V_E + 2\gamma^5\mathbf{p}_E\boldsymbol{\sigma}_1 + 2i\gamma^5V_E\mathbf{p}_E\boldsymbol{\sigma}^\times/E^2. \tag{30}$$

The spinor notation has been modified in this paper; the $\psi^{(+)}$ and $\psi^{(-)}$ of Eqs. (8) and (9) were originally called ψ and χ; the ψ of Eq. (27) was called ψ_1. This entailed an error in the definition of the χ entering Eq. (4) [3], which will now be corrected.

As in the Kramers equation for ψ_l, the equation for $\psi^{(-)}$ follows from Eq. (14) by exchange of the two brackets. The resulting changes in Eq. (4) are the replacement of $V\mathbf{p}$ by $\mathbf{p}V$ and the exchange of γ^5 with $\tilde{\gamma}^5$. As explained above, the tilde can only be removed from the combination $\tilde{\gamma}^5\boldsymbol{\sigma}$. This requires an extra transformation,

$$\psi^{(-)} = m_+\tilde{\psi}^{(-)}. \tag{31}$$

Multiplication by m_+^{-1} from the left now replaces $\gamma^5\boldsymbol{\sigma}$ by $m_+^{-1}\gamma^5m_+\boldsymbol{\sigma}=(\tilde{\gamma}^5)^\dagger\boldsymbol{\sigma}$. The $(\tilde{\gamma}^5)^\dagger$ is tranformed to γ^5 by the inverse of Eq. (25),

$$\tilde{\psi}^{(-)} = (m_+m_-)^{-1/2}(m_2\beta_2 - m_1\beta_1P_{spin})\chi = C_1^{-1}\chi. \tag{32}$$

As a result, χ now obeys the Hermitian conjugate of Eq. (27),

$$[E^2 - m_1^2 - m_2^2 - 2EV - 2E\gamma^5\mathbf{p}\boldsymbol{\sigma}_1 - 2i\gamma^5\mathbf{p}\boldsymbol{\sigma}^\times V]\chi = 2m_1m_2\beta\chi. \tag{33}$$

This guarantees identical eigenvalues E^2 of Eqs. (27) and (33).

The QED Born series is now brought into the form of an 8×8 matrix,

$$T_{if} = \tilde{\psi}_f^{(-)\dagger}T_8\psi_i^{(+)} = \psi_f^{(-)\dagger}m_+^{-1}T_8\psi_i^{(+)}, \tag{34}$$

where the indices i and f refer to in- and outgoing free two-body states in the c.m.s. The m_+^{-1} arises from Eq. (31); it was previously justified only by the successful construction of the interaction from T_8 [1].

Turning now to the quarkonium model of heavy mesons, it seems clear that the calculation of energy levels from perturbative QCD should also profit from the binary formulation. A major improvement is the strict absence of special retardation operators. In principle, the equation could be solved numerically when the coupling becomes too strong, as in the case of uranium atoms. However, QCD calculations on a space-time lattice [14] show essential deviations from this picture. For the light mesons, they indicate an E^2 spectrum which is linear in m_i in the limit $m_i\to 0$, $E^2=B_0(m_1+m_2)$. As the invariance of Eq. (33) under the sign change of any of the m_i follows directly from the QCD Lagrangian, the terms $B_{0i}m_i$ require that B_{0i} automatically changes sign with m_i. Simply adding such terms in Eq. (27) is forbidden. A valid extension of the quarkonium model to the light mesons seems not possible.

On the other hand, one may obtain terms like $B_{0i}m_i$ by using different right- and left-handed masses in the free single-particle Dirac equations, $m_i^2=m_{ir}m_{il}$. Taking, for example, $m_{1r}=m_{2r}=B_0\gg m_{il}$, one obtains $m_+^2=B_0[m_{1l}+m_{2l}+2(m_{1l}m_{2l})^{1/2}\beta]$. This gives the desired result only for $\langle\beta\rangle=0$. If on the other hand one adds the Dirac operators of particles 1 and 2 without using the explicitly parity-invariant forms [3,8], m_+^2 becomes

$$m_+^2=B_0(m_{1l}+m_{2l})(1+2\beta). \tag{35}$$

A more explicit connection of this trick with broken chiral symmetry is still missing, however.

III. TERNARY EQUATIONS

The derivation of equations for $n=3$ uses again the Dirac-Breit equation (7), for which there is presently no alternative. The H_B of Eq. (10) is replaced by three Breit operators,

$$H_{Bi}=\gamma_i^5\gamma_j^5 b_{ij},\quad b_{ij}=-\frac{1}{2}V_{ij}(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j+\sigma_{ir}\sigma_{jr}), \tag{36}$$

In Eqs. (8) and (9), one may then replace $\gamma_i^5\gamma_j^5$ by γ_k^5 and $-\gamma_k^5$, respectively, with $k\neq i\neq j$. The product (14) is replaced by

$$[\pi_{tot}^0-\Sigma_i\tilde{\gamma}_i^5\boldsymbol{\sigma}_i\mathbf{p}_i+\Sigma_{i<j}\tilde{\gamma}_k^5 b_{ij}][\pi_{tot}^0-\Sigma_i\gamma_i^5\boldsymbol{\sigma}_i\mathbf{p}_i-\Sigma_{i<j}\gamma_k^5 b_{ij}]\psi^{(+)}=m_+\psi^{(+)}, \tag{37}$$

$$\tilde{\gamma}_i^5=m_+\gamma_i^5/m_+=\gamma_i^5(1-2m_i\beta_i/m_+). \tag{38}$$

A useful combination is

$$\tilde{\gamma}_i^5+\gamma_i^5=2\gamma_i^5 m_{jk+}/m_+,\quad m_{jk+}=m_j\beta_j+m_k\beta_k. \tag{39}$$

Several simplifications arise from $[\mathbf{p}_k,V_{ij}]=0$, for example,

$$[E-V_{12}+\tilde{\gamma}_3^5 b_{12}]\gamma_3^5\boldsymbol{\sigma}_3\mathbf{p}_3+\tilde{\gamma}_3^5\boldsymbol{\sigma}_3\mathbf{p}_3[E-V_{12}-\gamma_3^5 b_{12}]=2(E-V_{12})\gamma_3^5\boldsymbol{\sigma}_3\mathbf{p}_3 m_{12+}/m_+. \tag{40}$$

The generalization of Eq. (30) to $\mathbf{r}_{ij}$ brings also Eq. (37) into a form with only even powers of E.

The number of remaining products is still large and becomes familiar only in one of the static limits, for example, for $m_3\gg m_1+m_2$. This limit has

$$\tilde{\gamma}_1^5=\gamma_1^5,\quad \tilde{\gamma}_2^5=\gamma_2^5,\quad \tilde{\gamma}_3^5=-\gamma_3^5. \tag{41}$$

The operator (40) vanishes in this limit, and the Dirac-Breit operator for heliumlike systems appears as expected.

IV. THE ENERGIES OF EMITTED PIONS AND PHOTONS

The energy of a particle emitted from a closed system of total c.m.s. energy E follows from the conservation of momentum and energy alone, without any knowledge of the interaction. The particle will be called "pion," with $E_\pi^2-\mathbf{P}_\pi^2=m_\pi^2$. Photons are included by setting $m_\pi=0$.

The final state of the emitting system will be called "ground state," with $E_{gr,lab}^2-\mathbf{P}_{gr}^2=E_{gr}^2$. The decay leaves the ground state with a recoiling momentum $\mathbf{P}_{gr}=-\mathbf{P}_\pi$, such that $E_{gr,lab}$ exceeds E_{gr} which refers to the new c.m.s. With $\mathbf{P}_{gr}^2=\mathbf{P}_\pi^2$, one finds $E_{gr,lab}^2-E_{gr}^2=E_\pi^2-m_\pi^2$, or $(E_{gr,lab}-E_\pi)(E_{gr,lab}+E_\pi)=E_{gr}^2-m_\pi^2$. Energy conservation requires $E_{gr,lab}+E_\pi=E$, which then gives $E_{gr,lab}$ and E_π separately,

$$E_{gr,lab}=(E^2+E_{gr}^2-m_\pi^2)/2E,\quad E_\pi=(E^2-E_{gr}^2+m_\pi^2)/2E. \tag{42}$$

The second equation reduces to Eq. (1) for $m_\pi=0$.

It will now be shown that this consequence of momentum and energy conservation follows from Eq. (4) using first-order time-dependent Dirac perturbation theory. $\Box$ is Lorentz transformed to the c.m.s. where it reduces to $-\partial_t^2$ ($t=$ c.m.s. time). Separation of a factor e^{-iEt} from ψ and χ gives

$$-\partial_t^2 e^{-iEt}=e^{-iEt}(E^2+2iE\partial_t-\partial_t^2). \tag{43}$$

The second-order derivative is neglected, and after a time scaling,

$$t=E\tau,\quad E\partial_t=\partial_\tau, \tag{44}$$

Dirac's perturbation theory for the τ dependence can be adopted. The result is $2EE_\pi=E^2-E_{gr}^2$, which gives the factor $(2E)^{-1}$ in Eq. (1). As both equations (42) contain $E_{gr}^2-m_\pi^2$, the time-independent equation for the final state must be generalized to

$$M_0^2\psi_{gr}=(E_{gr}^2-m_\pi^2)\psi_{gr}. \tag{45}$$

Reabsorption of the pion or photon is treated accordingly. The residual E^2 dependence of M_0^2 is not converted to ∂_t^2. In the hyperfine operator of Eq. (30) and in the running coupling constant $\alpha_s(E^2)$ of QCD, one may take $E^2=E_{gr}^2$.

V. SUMMARY

The relativistic two-body equation can be reduced to a one-body equation, under the same conditions as the nonrelativistic one. In first-order perturbation theory, the anti-Hermitian part of its differential operator must be included only for degenerate states. The free eight-component spinors $\psi_i^{(\pm)}$ of the Dirac equation are both needed for the construction of the interaction; the general relation between between $\psi^{(-)}$ and the χ of Eq. (4) is derived in Eqs. (31) and (32) as $\psi^{(-)}=m_+C_1^{-1}\chi$. The stationary ternary equation has also E^2 as eigenvalues. This implies that symmetry relations within baryon multiplets contain the squares of masses, just as in meson mulitplets. The equation remains complicated, but becomes manageable in the static limit of heliumlike states. The extension to states containing light quarks is questionable but perhaps not impossible. However, the E^2 result follows already from Lorentz invariance and should apply to all states.

For binaries, the absence of energy transfer between the external fermion legs in the c.m.s. eliminates the extra retardation operators of Breit [14]. Corresponding simplifications are expected for the ternary equation, but they remain to be derived. Finally, first-order perturbation theory of the time-dependent equations (4) gives simple expressions for the energies of emitted particles which cannot be derived from first-order time-dependent equations of the type $i\partial_t\psi = H\psi$.

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