Twin paradox on the photon sphere

Marek A. Abramowicz,* Stanisław Bajtlik,† and Włodek Kluźniak‡
N. Copernicus Astronomical Center, Bartycka 18, PL-00-716 Warszawa, Poland
and Nordita, Copenhagen, Denmark
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We consider another version of the twin paradox. The twins move along the same circular free photon path around the Schwarzschild center. In this case, despite their different velocities, all twins have the same nonzero acceleration. On the circular photon path, the symmetry between the twins situations is broken not by acceleration (as it is in the case of the classic twin paradox), but by the existence of an absolute standard of rest (timelike Killing vector). The twin with the higher velocity with respect to the standard of rest is younger on reunion. This closely resembles the case of periodic motions in compact (nontrivial topology) three-dimensional (3D) space recently considered in the context of the twin paradox by Barrow and Levin [Phys. Rev. A **63**, 044104 (2001)], except that in that paper accelerations of all twins were equal to zero and that, in the case considered here, the 3D space has trivial topology.

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The resolution of the classical twin paradox is that the traveling twin must suffer an acceleration since he stops, turns around, and returns to his never accelerating twin brother. The physical situation of the twins is therefore not the same and, indeed, the one who accelerates is younger on reunion. Barrow and Levin [1], see also [2], discussed a clever version of the paradox in compact spaces, with the twins moving with zero acceleration on a closed path. Compactness of the space defines a global standard of rest, and the twin moving faster with respect to this standard is younger on reunion. In this Brief Report we consider still another example of the twin paradox. Twins move with constant velocity along a circular photon path in the Schwarzschild spacetime. In this case, as shown by Abramowicz and Lasota [3], see also [4], they all have the same nonzero acceleration. We will show that the one who moves faster with respect to the standard of rest (given by the Killing symmetry of the Schwarzschild spacetime) is younger on reunion.

In Schwarzschild coordinates, the Schwarzschild metric has the form

$$ds^2 = \left(1 - \frac{r_G}{r}\right)dt^2 - \left(1 - \frac{r_G}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{1}$$

The metric depends neither on the time coordinate t, nor on the azimuthal coordinate φ. It is convenient to invariantly express these time and azimuthal symmetries in terms of two Killing vector fields,

$$\eta^i = \delta^i_t, \quad \xi^i = \delta^i_\varphi, \quad \eta^i \xi_i = 0. \tag{2}$$

Note that $\eta^i \eta_i = g_{tt} = 1 - r_G/r$, and $\xi^i \xi_i = g_{\varphi\varphi} = -\sin^2\theta r^2$. The unit timelike vector,

$$N^i = (\eta^k \eta_k)^{-1/2} \eta^i, \tag{3}$$

defines the static observers outside the event horizon $r = r_G$. They provide the absolute standard of rest in Schwarzschild spacetime.

Circular orbits r=const on equatorial plane $\theta = \pi/2$ have the only nonzero components $U^t = dt/ds \equiv A$ and $U^\varphi = d\varphi/ds \equiv A\Omega$. They are invariantly defined by

$$U^i = A(\eta^i + \Omega \xi^i). \tag{4}$$

Since $U^i U_i = 1$, the normalization is $1/A^2 = \eta^i \eta_i + \Omega^2 \xi^i \xi_i$.

The velocity with respect to the absolute standard of rest (static observers) is defined by

$$V^i = (\delta^i_k - N^i N_k) U^k \tag{5}$$

and the corresponding speed V (or $\tilde{V}$) is defined by

$$\tilde{V}^2 = \frac{V^2}{1 - V^2} = -V^i V_i. \tag{6}$$

Note that $V = c\beta$ and $\tilde{V} = c\beta\gamma$ in terms of the Lorentz β and $\gamma \equiv (1 - \beta^2)^{-1/2}$. From the above definitions one calculates

$$V^2 = -\Omega^2 \frac{\xi^i \xi_i}{\eta^k \eta_k} \equiv \Omega^2 \mathcal{R}^2 = \Omega^2 \frac{r^3}{r - r_G}. \tag{7}$$

The quantity $\mathcal{R}$ is the "distance from the rotation axis."

The acceleration on an orbit described by Eq. (4) is

$$a_i \equiv U^k \nabla_k U_i = -\frac{1}{2} \frac{\nabla_i(\eta^k \eta_k) + \Omega^2 \nabla_i(\xi^k \xi_k)}{(\eta^k \eta_k) + \Omega^2(\xi^k \xi_k)}. \tag{8}$$

One may define the "gravitational potential" Φ by the invariant expression

*Also at Physics Department, Göteborg University, SE-412-96 Göteborg, Sweden. Electronic address: marek@fy.chalmers.se
†Electronic address: bajtlik@camk.edu.pl
‡Also at Zielona Góra University, Wieża Braniborska, Lubuska 2, PL-65-265 Zielona Góra, Poland. Electronic address: wlodek@camk.edu.pl

$$\Phi = -\frac{1}{2}\ln(\eta^k \eta_k), \tag{9}$$

and then put the acceleration formula (8) into a form identical with its Newtonian version,

$$a_i = \nabla_i \Phi + \frac{\tilde{V}^2}{\mathcal{R}} \nabla_i \mathcal{R}. \tag{10}$$

From this equation one calculates that for the free (geodesic) motion, the orbital velocity V^2 equals, in the Schwarzschild spacetime,

$$V^2 = \frac{1}{2}\frac{r_G}{r - r_G}. \tag{11}$$

In particular, for $r=(3/2)r_G$ it is $V^2=1$, which therefore corresponds to a free, circular motion of photons. The $r=(3/2)r_G$ circles are called the photon circles, and one also speaks about the photon sphere.

Abramowicz and Lasota [3] noticed that $\nabla_i \mathcal{R}=0$ on the photon circle, and they deduced from Eq. (10) that on the photon circle acceleration does not depend on the orbital velocity: For all steady, nongeodesic, circular motions along $r=(3/2)r_G$ the acceleration takes the universal value (we give its r component only, as all other components are zero),

$$(a_r)^* = -\frac{1}{2}\left[\frac{d\ln(1 - r_G/r)}{dr}\right]_{(3/2)r_G} = -\frac{2}{3r_G}. \tag{12}$$

This suggests another version of the twin paradox. At the time $t=t_0$ two identical twins meet. Because they move with two different, constant, orbital velocities V_1 and V_2 along the same photon circle, they will certainly meet again. Who will be younger on reunion at the time $t=t_0+\Delta t$? Unlike in the classical version of the twin paradox, but similarly as in the case of compact space considered by Barrow and Levin [1], the acceleration a cannot be the answer here, as the two twins have *the same acceleration*. In the compact space they all have $a=0$, and on the photon circle they have $a=a^*$ given by Eq. (12). Indeed, like in the compact space, the answer is the velocity, the *absolute* velocity with respect to the *absolute* standard of rest: As we shall see, on reunion the twin who moves faster with respect to the standard of rest is younger. In the compact space, as explained by Barrow and Levin, there is no local standard of rest, but there is a global one. In the case of photon circle, the standard of rest is local, and given by the Killing symmetry.

The proper time elapsed from separation to reunion is

$$\tau = \int_{t_0}^{t_0+\Delta t} \frac{ds}{dt} dt = \int_{t_0}^{t_0+\Delta t} \frac{dt}{U^t} = \frac{\Delta t}{g_{tt}^{1/2}\gamma}. \tag{13}$$

From this, one finally deduces

$$\tau_1 = \frac{\gamma_2}{\gamma_1}\tau_2, \tag{14}$$

which proves the point: The twin who moves faster is younger on reunion.

The acceleration-free version of the twin paradox has a long history. We are aware of several papers which discussed the paradox in a cylindrical Minkowski spacetime [5–7], and of several papers which discussed the paradox for geodesic observers orbiting the Sun [8] and the Kerr black hole [9,10].

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