Vacuum self-focusing of very intense laser beams

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We argue that long-range photon-photon attraction induced by the dipole interaction of two electron-positron loops can lead to "vacuum self-focusing" of very intense laser beams. The focusing angle θ_F is found to increase with the beam intensity I as $\theta_F \sim I^{4/3}$; for the laser beams available at present or in the near future, $\theta_F \simeq 10^{-10} - 10^{-7}$.

I. INTRODUCTION

The response of quantum vacuum to strong external fields is a problem of great fundamental interest. Recent technological developments made available very intense laser beams, and opened the possibility to study nonlinear QED interactions in experiment (for a recent review, see [1]). In particular, multiphoton effects have been already identified experimentally in light-by-light scattering [2]. Theoretical studies of light-by-light scattering so far have been limited to the one-fermion-loop effects—the scattering of photons induced by the coupling to one electron-positron pair. The studies of higher loop corrections so far have been limited to two-loop effects arising from an additional internal virtual photon interacting with the single fermion loop; the corresponding results are available in an integral form [3]; see [4] for a comprehensive review.[1]

The expansion in the number of fermion loops clearly converges very fast, since additional fermion loops bring into the effective low-energy interaction extra powers of $\alpha^2 F^2 / m^4 \ll 1$, where F is the field strength tensor, m is the electron mass, and $\alpha \simeq 1/137$ is the electromagnetic coupling. We thus come to the conclusion that one can safely approximate the photon-photon scattering amplitude by a one-fermion-loop result.

This conclusion however may be premature for the field of an intense laser. Indeed, let us compare the photon-photon scattering amplitudes at the one-fermion-loop level [Fig. 1(a)] and the two-fermion-loop level [Fig. 1(b)]. The amplitude of the box diagram of Fig. 1(a) in coordinate space falls off exponentially at large separations r as $\sim (\alpha^2/m^4 r^7) \exp(-2mr)$ since it involves the exchange of a massive fermion pair in the t channel. On the other hand, the amplitude of Fig. 1(b) behaves in coordinate space like $\sim \alpha^4/m^8 r^{11}$ (we will show this by an explicit calculation below). The comparison of the amplitudes of Figs. 1(a) and 1(b) thus leads us to the conclusion that the two-fermion-loop amplitude of Fig. 1(b) will dominate over the one-fermion-loop amplitude of Fig. 1(a) at distances larger than

$$r \sim -\ln \alpha/m. \qquad (1)$$

One may wonder whether multiphoton interactions can also be important. There are two types of such interactions: (i) when additional external beam photon lines are attached to a single fermion loop and (ii) when additional external beam photon lines are attached to different fermion loops connected by t-channel photon pair exchanges. The diagrams of type (i) for n external beam photon lines are suppressed by $(eF/m^2)^n$ and fall off exponentially as a function of interphoton separation; we thus expect that such interactions can be neglected when the field strength is smaller than the critical one, $F_c^2 \sim m^4/\alpha$. The diagrams of type (ii) are characterized by a power falloff and so are potentially relevant even at subcritical field strength; however, the interactions involving l beam photons are suppressed by $(\alpha^3/m^4 r^4)^l$. This parameter becomes of the order of unity at distances smaller than $r \sim \alpha^{3/4}/m$. In this paper we consider the dynamics of laser beams of smaller intensities, corresponding to larger interphoton distances (note that $-\ln \alpha \gg \alpha^{3/4}$ for $\alpha = 1/137$) [see Eq. (1)].

This situation is completely analogous to the interaction of atoms: at short distances, the interaction involving the exchange of constituent electrons dominates, but at the distances large compared to the size of the atoms, the interaction is dominated by the (attractive) two-photon exchange
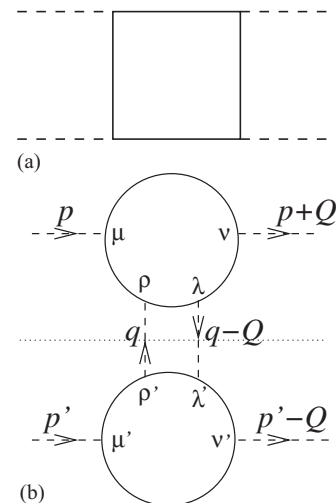

(a)

(b)

FIG. 1. The amplitudes of light-by-light scattering at the one-fermion loop (a) and two-fermion-loop (b) level.

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[1]Some nonperturbative effects such as those caused by the existence of bound electron-position states have also been considered [5].

despite the fact that it is of higher order in the coupling α. The key question now is whether the conditions for the dominance of two-fermion-loop diagrams of Fig. 1(b) can be realized in realistic laser beams.

Consider a laser beam that is characterized by the wavelength λ, coherence length l_c, and the transverse diameter d. The photon density in such a beam is determined by the ratio of the field energy density to the energy of a single photon

$$\rho = \frac{E^2}{\hbar \omega}. \tag{2}$$

Let us approximate the beam by a cylinder of height l and cross sectional area S with the symmetry axis parallel to the beam direction. The beam energy contained in the cylinder is then $E^2 S l$. The beam intensity is the flux of energy through the unit surface, i.e.,

$$I = \frac{E^2 S l_c}{S(l_c/c)} = E^2 c, \tag{3}$$

hence

$$\rho = \frac{1}{\hbar c} \frac{I}{\omega}. \tag{4}$$

For example, for the Omega-EP laser at the Laboratory for Laser Energetics of the University of Rochester the parameters are [6], $I = 3 \times 10^{20}$ W/cm^2 at the wavelength $\lambda = 1054$ nm (corresponding to $\omega = 2\pi c/\lambda = 1.8 \times 10^{15}$ s^{-1}). This amounts to the mean photon density of $\rho = 5.3 \times 10^{28}$ cm^{-3}. Therefore, the typical interphoton distance in the beam is $R \sim \rho^{-1/3}$, which is $R = 2.6 \times 10^{-3}$ nm for the Omega-EP laser.

As long as the photons obey the linear Maxwell equations the laser beam is a coherent state. However, the coherent state can be destroyed by the mutual interaction of the photons. Such interactions appear as a result of quantum fluctuations of a photon into a virtual $e^+ e^-$ pair. These pairs can be considered as electric or magnetic dipoles of size $\lambdabar_e = \hbar/m_e c = 3.8 \times 10^{-4}$ nm, which can interact with virtual dipoles created by other photons in the beam. Thus, there are four dimensionful scales associated with the problem of photon interaction in the laser beam. These are

$$\lambdabar_e \ll R \ll \lambda \ll L, \tag{5}$$

where L is the macroscopic length of the beam (l_c or d). By comparing Eqs. (5) and (1) we conclude that (i) the condition of Eq. (1) for the dominance of two-fermion-loop interactions can be met in the available laser beams, and (ii) since the photon wavelength λ is the largest (microscopic) scale in the problem we can treat the two-photon interaction as a low-energy scattering problem.

II. PHOTON-PHOTON INTERACTION AT LARGE DISTANCES

Let us now turn to the calculation of the photon-photon potential density, which can be expressed through the scattering amplitude $\mathcal{M}(s, Q^2)$ as

$$V(\mathbf{r}, t) = -\int \frac{d^4 Q}{(2\pi)^4} e^{-iQx} \lim_{s \to 0} \mathcal{M}(s, Q^2), \tag{6}$$

where $x = (t, \mathbf{r})$ is the relative four-coordinate. This formula gives the potential density as a function of time. If we were considering the scattering of massive particles, we could have chosen their center-of-mass rest frame in which the potential would not depend on time t. This is not possible in the case of photon-photon interaction. Instead, we can obtain a meaningful quantity by averaging the potential (6) over the period of the electromagnetic wave oscillation $T = 2\pi/\omega$. We thus define the mean potential density as

$$\bar{V}(\mathbf{r}) = \frac{1}{2T} \int_{-T}^{T} V(\mathbf{r}, t) dt. \tag{7}$$

To calculate the two-photon interaction amplitude, Fig. 1(b), we use the t-channel dispersion relation

$$\mathcal{M}(s, Q^2) = \frac{1}{\pi} \int \frac{dM^2}{M^2 - Q^2 - i0} \, \mathrm{Im} \, \mathcal{M}(s, M^2), \tag{8}$$

where M is the invariant mass of two t-channel photons. The imaginary part of the amplitude can be calculated by making the cut shown by the dotted line in Fig. 1(b). This amounts to placing the cut photons on mass shell. According to the Mandelstam-Cutkosky rule the cut photon's propagator is replaced by

$$\frac{-ig_{\mu\mu'}}{q^2 + i0} \to 2\pi \delta(p^2) \sum_{\text{polariz.}} \epsilon_\mu^* \epsilon_{\mu'}. \tag{9}$$

Therefore, in the low-energy limit

$$\mathrm{Im} \, \mathcal{M}(0, Q^2) = \int \frac{d^4 q}{(2\pi)^2} \delta(q^2) \delta[(q - Q)^2] \sum_{\sigma\sigma'} |\mathcal{M}^{\mathrm{box}}_{++\sigma\sigma'}(M^2)|^2, \tag{10}$$

where we used the fact that all photons in the beam have the same polarization (assigned $+$ here).

The low-energy on-shell amplitudes corresponding to the box diagram Fig. 1(a) can be found elsewhere (see, e.g., in [7]).

$$\sum_{\sigma\sigma'} |\mathcal{M}^{\mathrm{box}}_{+\sigma\sigma'}(M^2)|^2 = \left(\frac{11\alpha^2 M^4}{45m^4}\right)^2 + \left(-\frac{2\alpha^2 M^4}{15m^4}\right)^2$$

$$= 157 \left(\frac{\alpha^2 M^4}{45m^4}\right)^2. \tag{11}$$

Taking the phase space integral in Eq. (10) and substituting it together with (11) into (8) yields the amplitude in the t channel. However, we are interested in the s-channel amplitude. It can be obtained by analytic continuation to the region of spacelike momentum Q. Therefore, the s-channel amplitude is

$$\mathcal{M}(0, Q^2) = \frac{157}{8\pi^2} \left(\frac{\alpha^2}{45m^4}\right)^2 \int \frac{dM^2}{M^2 - Q^2} M^8, \tag{12}$$

where now $Q^2 < 0$. The potential density can be calculated by substituting Eq. (12) into Eq. (6). We have

$$V(x) = -\frac{1}{(2\pi)^4}\frac{157}{8\pi^2}\left(\frac{\alpha^2}{45m^4}\right)^2 \int_0^\infty dM^2 M^8$$

$$\times \int d^4 Q e^{-iQx} \frac{1}{M^2 - Q^2 - i0}. \tag{13}$$

Integration over Q yields

$$\int d^4 Q e^{iQx} \frac{1}{M^2 - Q^2 - i0} = 4\pi^2 i \frac{M^2}{\sqrt{(-x^2 + i0)M^2}}$$

$$\times K_1[\sqrt{(-x^2 + i0)M^2}], \tag{14}$$

where K_1 is the modified Bessel function. Now we have to average the potential density (13) over the oscillation period according to Eq. (7). Since the field oscillation period $T = \lambda$ is much larger than the interphoton distance R (the typical exchanged mass is $M \sim R^{-1}$), we can set the integration limits in Eq. (7) to infinity. The integral over t can be done by rotating the integration contour by π in the plane of complex t. The result is

$$\bar{V}(\mathbf{r}) = -\frac{1}{(2\pi)^4}\frac{157}{8\pi^2}\left(\frac{\alpha^2}{45m^4}\right)^2 \int_0^\infty dM^2 M^8 \frac{1}{2T}\frac{4\pi^2}{r}e^{-Mr}. \tag{15}$$

After integration over M we finally derive

$$\bar{V}(\mathbf{r}) = -\frac{\omega}{(2\pi)^4}\frac{157}{16\pi^2}\left(\frac{\alpha^2}{45m^4}\right)^2 \frac{2\pi^2}{r^{11}} 2 \times 9! \tag{16}$$

This is the mean potential density between two photons at large distances.

Let us compare this to the corresponding result for the interaction between atoms, where the potential falls off as $\sim r^{-6}$ at distances larger than the size of the atom a but smaller than a/α ("Van der Waals interaction") and as $\sim r^{-7}$ at distances larger than a/α due to relativistic retardation effects ("Casimir-Polder interaction").[2] Our result falls off even faster, as r^{-11}, which can be traced back through the calculation to the gauge invariance of QED and the fact that the dipoles in our case are virtual, unlike in the interaction of atoms.

III. DISPERSION RELATION OF THE BEAM AND SELF-FOCUSING

In this section, we would like to calculate the refraction index n of the intense laser beam taking into account the photon-photon attraction as described in the previous section. Assuming the dominance of two-particle interactions, the Hamiltonian describing the dynamics of interacting particles can be represented in the secondary quantization approach in the following general form (see, e.g., [10]):

$$\hat{H} = \sum \omega_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{1}{2}\sum \langle \mathbf{p}, \mathbf{p}'|\hat{U}|\mathbf{p} + \mathbf{Q}, \mathbf{p}' - \mathbf{Q}\rangle$$

$$\times \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}'}^\dagger \hat{a}_{\mathbf{p}+\mathbf{Q}} \hat{a}_{\mathbf{p}'-\mathbf{Q}}$$

$$\approx \sum \omega_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{1}{2}U\sum_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} \hat{a}_{\mathbf{p}}. \tag{17}$$

The mean effective potential U for scattering of two photons corresponding to the density (16) is given by

$$U = \int d^3 r \bar{V}(r)$$

$$= \frac{1}{2T}\int_{-\infty}^\infty dt \int d^3 r \int \frac{d^4 Q}{(2\pi)^4} e^{-iQx} \mathcal{M}(0, Q^2)$$

$$= -\frac{\omega}{4\pi}\frac{157}{8\pi^2}\left(\frac{\alpha^2}{45m^4}\right)^2 \int_0^{1/R^2} dM^2 M^8$$

$$= -\frac{\omega}{4\pi}\frac{157}{8\pi^2}\left(\frac{\alpha^2}{45m^4}\right)^2 \frac{1}{4R^8}. \tag{18}$$

The same result can be obtained directly by taking the four-dimensional integral of the right-hand side of Eq. (14). Denoting by $4\pi A$ the forward elastic scattering amplitude we obtain from Eq. (18) $U = -A\omega$.

By analogy with the treatment of superfluidity we consider the quasiparticle fluctuations $\hat{a}_{\mathbf{p}} = \hat{c}_{\mathbf{p}} + \hat{a}_{\mathbf{p}}'$ and $\hat{a}_{\mathbf{p}}^\dagger = \hat{c}_{\mathbf{p}}^\dagger + \hat{a}_{\mathbf{p}}'^\dagger$ where the background classical laser field is described by the commuting c-number operators $\hat{c}_{\mathbf{p}}$ and $\hat{c}_{\mathbf{p}}^\dagger$. Expanding the Hamiltonian up to quadratic terms in $\hat{a}_{\mathbf{p}}'$'s and diagonalizing the Hamiltonian yields

$$\hat{H} = \omega N - \frac{1}{2}N_0 N \omega A + \sum_p \varepsilon(\omega)\hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}}, \tag{19}$$

where $\hat{b}_{\mathbf{p}}$'s are the quasiparticle operators and N_0 is a mean number of particles in a sphere of radius R: $N_0 \simeq (NR^3)/V \simeq \rho R^3 \simeq 1$. The dispersion law is given by

$$\varepsilon(\omega) = \omega\sqrt{1 - AN_0/2} \simeq \omega\sqrt{1 - A/2}. \tag{20}$$

The redshift of the laser beam occurs since a fraction of the beam's kinetic energy density $E^2 = \rho\omega$ [see Eq. (2)] gets transformed into the interaction energy density $(NN_0/2)/U/V = U\rho/2$, where N is the total number of photons in the beam. Clearly, this fraction is $|U|/2\omega = A/2$.

The phase velocity of the laser beam reads

$$u = \frac{\varepsilon(\omega)}{p} \approx 1 - \frac{A}{4}. \tag{21}$$

Equation (18) implies that $A \sim R^{-8} \sim \rho^{8/3}$. The density of the beam ρ decreases toward the beam edges. Therefore, the beam phase velocity u increases towards the beam periphery. This implies that the photon beam will self-focus: the refraction index $n = u^{-1}$ is largest in the center of the beam, which has the "focusing lens" effect. The focusing angle can be determined from the following simple observation. Focusing means that the photons in the beam center having phase velocity $1 - A/4$ will arrive to the focusing point simulta-

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[2]The dispersive method that we use is analogous to the one used previously to descirbe the interaction of atoms and dipoles in Refs. [8,9].

neously with the photons at the beam periphery having unit phase velocity. This can only happen if the relative ("focusing") angle between these bunches of photons satisfies $\cos \theta_F = 1 - A/4$. Since the focusing angle is small we have

$$\theta_F = \sqrt{A/2} = \left(\frac{157}{16\pi^3}\right)^{1/2}\left(\frac{\alpha^2}{180 m^4 R^4}\right). \qquad (22)$$

For the Omega-EP laser beam we have the following estimate: $\theta_F \simeq 1 \times 10^{-10}$. We expect an increase in the focusing angle with the increase of the beam intensity according to the law $\theta_F \propto R^{-4} \propto \rho^{4/3} \propto I^{4/3}$ [see Eqs. (18) and (4)]. The maximal possible effect can be achieved at $R \sim \lambda_e$ in which case $\theta_F \simeq 1 \times 10^{-7}$. This value should be considered as an upper bound on the self-focusing angle due to the mechanism considered in this paper since our approach breaks down for $R \lesssim \lambda_e$. We have to emphasize that these estimates are admittedly qualitative, since they are derived in the mean field approximation in which $R \sim \rho^{-1/3}$.

Our calculation shows that the focusing effect is present even for the plane wave, for which the leading order contribution from the box diagram of Fig. 1(a) vanishes. The magnitude of the expected effect, however, makes its experimental observation challenging. In particular, in a realistic experimental setup it would have to be distinguished from the nonlinear effects caused by the presence of the air.

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