Shear viscosity and damping for a Fermi gas in the unitarity limit

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(Received 19 December 2006; revised manuscript received 6 March 2007; published 17 April 2007)

The shear viscosity of a two-component Fermi gas in the normal phase is calculated as a function of temperature in the unitarity limit, taking into account strong-coupling effects that give rise to a pseudogap in the spectral density for single-particle excitations. The results indicate that recent measurements of the damping of collective modes in trapped atomic clouds can be understood in terms of hydrodynamics, with a decay rate given by the viscosity integrated over an effective volume of the cloud.

DOI: 10.1103/PhysRevA.75.043612 PACS number(s): 03.75.Ss, 05.30.Fk, 51.20.+d

I. INTRODUCTION

Strongly interacting Fermi systems play a central role in physics over a vast range of energies, from cold atoms over condensed matter systems to quark-gluon plasmas. For atomic gases, the regime of strong interaction is reached by the use of Feshbach resonances at which the scattering length diverges. This so-called unitarity limit has been studied experimentally through the expansion of a two-component Fermi gas [1] and by measuring its collective modes [2,3]. These experiments indicate that under certain conditions the dynamic properties of atomic gases in the unitarity limit are well described by hydrodynamics, both in the superfluid and in the normal phase. Related results have been reported for the strongly interacting quark-gluon plasma produced in heavy-ion collisions at RHIC [4,5].

In this paper we shall carry out a quantitative analysis of the hydrodynamic damping for the normal phase in the unitarity limit and compare our results to the measured rate of decay of the collective modes [2,3]. In the hydrodynamic limit, the rate of decay is related to the shear viscosity integrated over the volume of the trapped atomic cloud. As we shall see, under the given experimental conditions, hydrodynamics applies only in a limited temperature region above the superfluid transition temperature T_c. Furthermore, hydrodynamics necessarily fails in the outer parts of the atomic cloud, where the density is low and the mean free path therefore long. Since for a classical gas the viscosity is independent of density, one must introduce an explicit cutoff in the spatial integration of the viscosity, as shown in Ref. [6].

In the unitarity limit, we can use a dimensional argument [7] to write η as

$$\eta = n\hbar\alpha(T/T_F). \tag{1}$$

Here α is a dimensionless quantity which can only depend on temperature through T/T_F. The Fermi temperature is $T_F = k_F^2/2m$, where k_F is the magnitude of the Fermi momentum given by $k_F=(3\pi^2 n)^{2/3}$ with n being the density of the gas [with the exception of Eq. (1), we use units such that $\hbar = k_B = 1$]. Our aim is to obtain an approximate expression for the universal function $\alpha(T/T_F)$ which will allow us to compare theory with experiment.

The present work is a continuation of two previous papers [8,9], in the following referred to as papers I and II, respectively, on the damping of collective modes in Fermi gases. Before we describe the results of our present calculation we shall therefore summarize the approach taken in papers I and II, and indicate their limits of applicability.

In paper I we employed a Boltzmann equation for the fermion distribution function $f(\mathbf{r},\mathbf{p},t)$, taking into account the dependence of the scattering cross section on the energy in the relative motion of two particles, as well as the effect of the mean field in the streaming terms of the Boltzmann equation. The collective mode frequencies were calculated by taking moments of the Boltzmann equation, which introduced the spatially averaged viscous relaxation rate as the effective collision rate entering the imaginary part of the (complex) mode frequencies. At low temperatures, well below the Fermi temperature, the validity of this approach is restricted to the limit of weak coupling $k_F|a| \ll 1$, where a is the scattering length and k_F is the magnitude of the Fermi wave vector. In this limit the method is accurate within a few percent, since the viscous relaxation rate used in paper I is obtained from a trial function which yields viscosities that differ by only a few percent from those obtained from exact solutions to the Boltzmann equation at both low and high temperatures. At temperatures well above the Fermi temperature the Boltzmann approach used in paper I is accurate for any value of a, including the unitarity limit in which the cross section is proportional to the inverse of the energy in the relative motion. We shall demonstrate this in detail in the Appendix.

When $k_F|a|$ becomes comparable to or larger than unity, one enters the regime of strong coupling, in which perturbative approximation schemes such as the Boltzmann approach can no longer be trusted at temperatures comparable to or less than the Fermi temperature. Progress in understanding the viscosity of such strongly coupled Fermi systems must necessarily rely on an interplay between experiment and theory, since there is no small parameter available for a perturbation expansion that could yield firm theoretical predictions. For an atomic gas close to a Feshbach resonance we explored in paper II the influence of the medium on the scattering cross section, which in paper I was taken to be its value in vacuum. Due to Fermi blocking of the pair states into which the molecular state can decay, the lifetime of the resonant state was found to be significantly increased, leading to a corresponding increase in scattering rate (and hence a decrease in viscosity) close to the superfluid transition temperature T_c. For a uniform gas the calculated viscosity just

above T_c was found to be reduced by the factor 7.5 compared to the value obtained in paper I by use of the vacuum scattering matrix. For a trapped gas the difference was less pronounced: the thermal relaxation rate, which is closely related to the inverse viscosity, was found near T_c to be 3.6 times the value obtained using the vacuum scattering matrix.

The assumption underlying the approach taken in paper II was that the main effects of the interaction in the strong-coupling limit arose through a modification of the scattering cross section, while strong-coupling effects that lead to spectral broadening of single-particle excitations were not taken into account. Put in different terms, only the collision term in the kinetic equation was modified by taking into account the medium effects mentioned above, while the streaming terms were assumed to be unaffected by interactions. In the present paper we abandon this assumption and consider specifically the role of the pseudogap occurring in the spectral function of single-particle excitations [10]. The presence of the pseudogap in the normal phase influences the Bragg spectrum observed when an atom absorbs a photon from one laser beam and emits a photon into another, resulting in a change of the energy and momentum of the atom [11]. In the normal phase, at the unitarity limit, the pseudogap was found to cause a significant suppression of the low-frequency Bragg spectrum.

The use of a Boltzmann equation as in papers I and II implicitly assumes the existence of quasiparticles with a definite energy-momentum relationship. When the spectral functions broaden, the quasiparticles are less well defined, and it therefore becomes relevant to investigate the effect of this broadening on the transport properties of the gas. Ideally, one should derive a kinetic equation that takes all strong-coupling effects systematically into account, but due to the lack of a small parameter in the strong-coupling limit this would be far too ambitious an undertaking. Our aim here is more modest: to compare results for the viscosity in the presence and absence of spectral broadening in order to gain insight into its quantitative significance, and to use this together with the results of papers I and II to construct an approximate formula for α that allows for an explicit comparison with experiment.

A main result of the paper is the calculation in Sec. II of the shear viscosity from a Kubo formula, which allows one to take into account the presence of the pseudogap in the spectral density of states for single-particle excitations. Since our results are based on a ladder approximation to the self-energies, they cannot be expected to be quantitatively accurate, but as we shall see, our present results are quite close to those of paper II at low temperatures. At high temperatures, however, the Kubo expression gives results that lie significantly below the classical result obtained from the Boltzmann equation. Since the latter is essentially exact for all values of a in the classical limit, as we shall demonstrate in the Appendix, we construct an approximate formula for α which interpolates between the low-temperature result and the exact high-temperature limit. This interpolation formula is then used for comparison with experiment in Sec. III, where the decay rate is related to the viscosity integrated over an effective volume of the trapped gas. The resulting agreement with experiments [2,3] that were carried out at two very different frequencies indicates that the interpolation formula provides a qualitatively correct picture of the physics involved in the viscosity of a strongly interacting Fermi gas. The calculations also illustrate how information on the viscosity of strongly interacting Fermi gases can be extracted from measurements of the damping of collective modes.

II. THE SHEAR VISCOSITY

Consider the shear viscosity η of a uniform, two-component Fermi gas of atoms with mass m in the normal phase. There is no interaction between atoms in the same internal state whereas the interaction between atoms in the two different internal states is characterized by the s-wave scattering length a. Unitarity means that we take $k_F|a| \to \infty$. The shear viscosity relates the momentum current density Π_{xy} to the gradient in flow velocity $u_x(y)$ according to $\Pi_{xy} = -\eta \partial u_x/\partial y$.

The Landau-Boltzmann approach assumes well-defined quasiparticle excitations. However, with strong interactions present the spectral functions may become significantly broadened and the quasiparticles therefore ill defined. Close to T_c in the normal phase, the spectral weight is found to be suppressed near the Fermi surface resulting in a double-peak structure of the spectral function [10,11]. This suppression is referred to as the presence of a pseudogap. In order to investigate the importance of the pseudogap we turn to the Kubo formalism, which allows for a treatment of these effects.

The velocity field $u_x(y)$ gives rise to a perturbation

$$\hat{H}' = -m \int d^3r\, u_x(y) \hat{\mathbf{j}}_x(\mathbf{r}) = \frac{1}{i\omega} \frac{\partial u_x}{\partial y} \int d^3r\, \hat{\Pi}_{xy} \tag{2}$$

with $\hat{\mathbf{j}}$ being the current density operator and $\hat{\Pi}_{xy}$ the momentum-current density operator. To obtain the second equality in Eq. (2), we have used momentum conservation $m\partial \hat{j}_i/\partial t = -\partial \hat{\Pi}_{ij}/\partial r_j$ (i and $j = x,y,z$) taking $\hat{\mathbf{j}}(\mathbf{r},t) = \exp(-i\omega t)\hat{\mathbf{j}}(\mathbf{r})$. We can now calculate Π_{xy} induced by $\hat{H}'$ within linear response. The shear viscosity is then obtained by taking the limit $\omega \to 0$:

$$\eta = -\lim_{\omega \to 0} \mathrm{Im}\, \Xi(\omega)/\omega \tag{3}$$

with

$$\Xi(\omega) = -i \int d^3r\, dt\, e^{i\omega t} \theta(t) \langle [\hat{\Pi}_{xy}(\mathbf{r},t), \hat{\Pi}_{xy}(0,0)] \rangle. \tag{4}$$

The approximation for Ξ shown in Fig. 1 yields

$$\eta = -\frac{1}{15m^2} \int \frac{d^3p}{(2\pi)^3} p^4 \int \frac{d\epsilon}{2\pi} A^2(p,\epsilon) \frac{\partial f(\epsilon)}{\partial \epsilon}. \tag{5}$$

Here, $A(p,\epsilon) = -2\,\mathrm{Im}\, G_R(p,\epsilon)$ is the spectral function for the atoms with G_R being the retarded Green function and $f(\epsilon) = [\exp(\epsilon/T)+1]^{-1}$. A relativistic version of Eq. (5) has been used to calculate η for quark-gluon plasmas using a phenomenological ansatz for $A(p,\epsilon)$ [12]. Here, we use a microscopic theory for $A(p,\epsilon)$ as explained below.

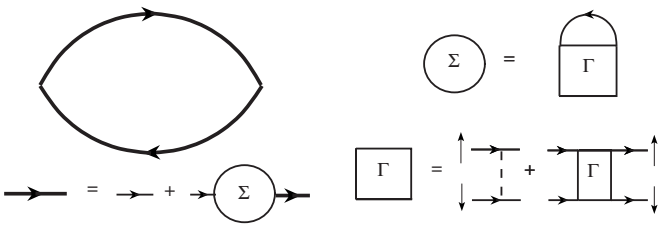

FIG. 1. Ξ and the atomic propagator G in the ladder approximation for a broad resonance. Σ is the self-energy and Γ is the scattering matrix.

To obtain $A(p,\epsilon)$, we calculate the Green function using a multichannel BEC-BCS crossover theory based on the ladder approximation for the thermodynamic potential and the atom self-energies. This is the minimal approach which includes the correct two-body physics leading to the Feshbach resonance [13–15]. The structure of the theory is shown in Fig. 1 and is described in detail in Refs. [11,15]. We take a broad resonance for which the multichannel theory becomes equivalent to the original single channel BEC-BCS crossover theory [16] for most observables. The spectral functions are found numerically to obey the sum rule $\int A(p,\omega)d\omega = 2\pi$ to a very good approximation. It should be noted, however, that the approximation leading to Eq. (5) is not conserving. To obtain a conserving approximation, we need to solve an integral equation for Ξ [17]. However, the present analysis is already heavy numerically since one needs to integrate over the atom self-energies which themselves are evaluated numerically; a conserving approximation is thus beyond the scope of the present paper.

It is instructive to compare the Kubo approach with the relaxation-time approximation to the Boltzmann equation, since these yield identical results at high temperatures. In the relaxation-time approximation, the collision integral of the Boltzmann equation given in Eq. (13) of Ref. [8] becomes $I[f] = \delta f/\tau$, where δf is the deviation of the distribution function from local equilibrium. The relaxation time $\tau(p)$ is obtained by setting the distribution functions for particles in states other than $\mathbf{p}$ equal to their equilibrium values

$$\frac{1}{\tau(p)} = \int \frac{d^3p_1}{(2\pi)^3} \int d\Omega \frac{d\sigma}{d\Omega} \frac{|\mathbf{p}-\mathbf{p}_1|}{m} \times \frac{f(\xi_{p_1})[1-f(\xi_{p'})][1-f(\xi_{p_1'})]}{1-f(\xi_p)}. \tag{6}$$

Here $d\sigma/d\Omega$ is the differential cross section for the scattering of two particles with incoming momenta $\mathbf{p}$ and $\mathbf{p}_1$ to outgoing momenta $\mathbf{p}'$ and $\mathbf{p}_1'$. The corresponding energies are $\xi_p = p^2/2m - \mu$, etc., and Ω is the solid angle of $(\mathbf{p}' - \mathbf{p}_1')/2$ with respect to $(\mathbf{p}-\mathbf{p}_1)/2$. It is straightforward to find δf from the Boltzmann equation and thus the momentum current density from $\Pi_{xy} = 2(2\pi)^{-3}\int d^3p (p_x p_y/m)\delta f$, where the factor of 2 is from the two internal states. We obtain for the shear viscosity

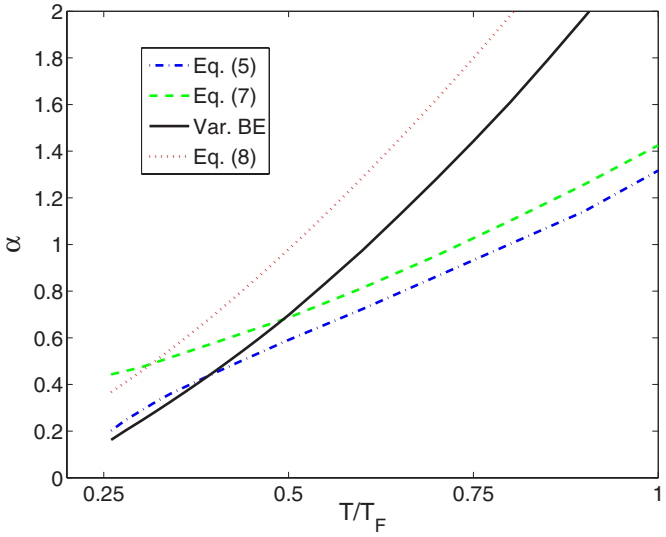

FIG. 2. (Color online) The viscosity in units of $n\hbar$ for $T \geqslant T_c$. The dashed-dotted line is the Kubo formula (5), the dashed line is the Boltzmann equation result in the relaxation-time approximation (7), the solid line is the variational solution of the Boltzmann equation with the medium cross section [9], and the dotted line is the high-T result (8).

$$\eta = -\frac{2}{15m^2}\int \frac{d^3p}{(2\pi)^3} p^4 \frac{\partial f}{\partial \xi_p}\tau(p). \tag{7}$$

The factor 1/15 arises from the angular average of $p_x^2 p_y^2$.

The Kubo formula (5) reduces to the relaxation-time approximation (7) in the limit of weak interaction with narrow spectral function peaks. To see this, put $A(p,\epsilon) = 2\Gamma_p/[(\epsilon - \xi_p)^2 + \Gamma_p^2]$ with $\Gamma_p = -\mathrm{Im}\,\Sigma(p,\xi_p)$ the imaginary part of the atom self-energy. Using $A^2 \to 2\pi\delta(\epsilon - \xi_p)/\Gamma_p$ for $\Gamma_p \to 0$, Eq. (5) reduces to Eq. (7) with $\tau^{-1}(p) = -2\,\mathrm{Im}\,\Sigma(p,\xi_p)$. It can furthermore be shown that the ladder approximation for $\mathrm{Im}\,\Sigma(p,\xi_p)$ yields Eq. (6) when medium effects are ignored [15]. The Kubo formula for the viscosity (5) thus reduces to Eq. (7) in the limit of weak interactions.

In Fig. 2, we plot $\alpha(T/T_F)$ in Eq. (1) for a gas in the unitarity limit with $T \geqslant T_c$. For the numerical calculations, we have chosen parameters corresponding to a resonant interaction with $k_F|a| = 25 \gg 1$ and a negligible effective range. The critical temperature is found from the divergence of the scattering matrix at zero total momentum (the Thouless criterion) yielding $T_c \approx 0.26 T_F$ in good agreement with other BEC-BCS crossover results [13,18]. The Kubo result (5) approaches Eq. (7) for $T \gg T_F$. This is to be expected since medium effects are negligible in the classical limit. For $T \gg T_F$, $\eta \propto T^{3/2}$ which can be seen in the relaxation-time approximation from Eqs. (6) and (7) which give $\eta = nT\tau$ with $\tau \propto T^{1/2}$, resulting in $\alpha = 1.1(T/T_F)^{3/2}$. At low T, the difference between Eqs. (7) and (5) is significant; for $T = T_c$, Eq. (7) yields $\alpha = 0.4$ whereas Eq. (5) predicts $\alpha = 0.2$. This reduction is due to strong-coupling medium effects. The imaginary part $\mathrm{Im}\,\Sigma$ is large leading to a significant damping of the quasiparticles. Close to T_c a pseudogap opens up and the effective density of states at the Fermi surface is suppressed, leading to a reduction of the viscosity.

We also plot η obtained from a variational solution to the Boltzmann equation [8,9], which yields a lower bound on the viscosity, given by

$$\alpha = \frac{45\pi^{3/2}}{64\sqrt{2}}\left(\frac{T}{T_F}\right)^{3/2} = 2.77\left(\frac{T}{T_F}\right)^{3/2}. \tag{8}$$

This lower bound deviates by less than 2% from the exact result, as we shall demonstrate in the Appendix, where we investigate the leading correction to the lowest-order variational result for a general form of the cross section. In the unitarity limit the leading correction is found to increase the lower bound (8) by 3/190 or 1.58%. For comparison we consider in the Appendix also the weak-coupling limit where the cross section is a constant, independent of the relative momentum of the colliding particles. In this case the leading correction is found to increase the lower bound by 3/202 or 1.49% in precise agreement with the known result for hard-sphere molecules [19].

Since medium effects are negligible for $T \gg T_F$, we conclude that Eq. (8) is a very accurate expression for η at high temperatures in the unitarity limit. At low T, however, we saw by comparing Eqs. (5) and (7) that medium effects are important. Compared to the Kubo result, the variational solution includes medium effects only in the sense that the medium scattering rate $\tau^{-1}(p) = -2\,\mathrm{Im}\,\Sigma(p, \xi_p)$ is used in the Boltzmann equation implicitly assuming $\mathrm{Im}\,\Sigma \ll \epsilon_F$ whereas $\mathrm{Re}\,\Sigma$ is neglected [9]. On the other hand, the variational nature of the solution corresponds to approximately summing diagrams beyond the approximation in Fig. 1 leading to the Kubo result (5). Comparing the two approximations for η, we see that the variational solution of the Boltzmann equation obtained in Ref. [9] agrees reasonably well with the Kubo result for low T. Note that since in the unitarity limit $T_c/\mu(T_c) \approx 0.6$ is rather large, the T^{-2} divergence in η for $T \to 0$ due to Fermi blocking is not seen for $T \geqslant T_c$ in Fig. 2.

III. COMPARISON WITH EXPERIMENT

When conditions are hydrodynamic, the attenuation of a collective mode is related to the viscosity. We now use our calculated viscosity to interpret the measured [2,3] damping of the collective modes in an atomic gas trapped in a potential of the form

$$V(x,y,z) = \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2). \tag{9}$$

The attenuation Γ of a collective mode is defined in terms of the amplitude decay of the density oscillations given by one half the rate of change of mechanical energy, i.e.,

$$\Gamma = -\frac{\langle \dot{E}_{\mathrm{mech}} \rangle_t}{2\langle E_{\mathrm{mech}} \rangle_t}, \tag{10}$$

where $\langle \cdots \rangle_t$ indicates the time average over a period of the cycle [6]. The modes we examine are characterized by a velocity field $\mathbf{v} = (ax, by, cz)$. Following Ref. [6], we obtain

$$\Gamma = \frac{2(a^2 + b^2 + c^2 - ab - ac - bc)\int d^3r\, \eta(\mathbf{r})}{3m\int d^3r\, n(\mathbf{r})(a^2x^2 + b^2y^2 + c^2z^2)} \tag{11}$$

for the damping. Here $n(\mathbf{r})$ is the density of atoms in the trap. In the unitarity limit, Eq. (1) gives $\eta(\mathbf{r}) = n(\mathbf{r})\hbar\alpha[T/T_F(\mathbf{r})]$ with $T_F(\mathbf{r}) = [3\pi^2 n(\mathbf{r})]^{2/3}/2m$. We parametrize α for $T \geqslant T_c$ in the form

$$\alpha = \alpha_0 + \alpha_{3/2}(T/T_F)^{3/2}, \tag{12}$$

where $\alpha_0 = -0.2$ and $\alpha_{3/2} = 2.77$ are numbers chosen to fit our numerical results for η in Sec. II.

The α_0 term yields a spatial integral of $n(\mathbf{r})$ in Eq. (11), whereas the $\alpha_{3/2}$ term gives a spatial integral of a constant since the viscosity in the classical limit is independent of density. The integration must be cut off near the edge of the cloud where the density is small and hence the mean free path long, resulting in the breakdown of the validity of hydrodynamics. We adopt the procedure described in Ref. [6] that the hydrodynamic description holds out to a distance given by the condition that an atom incident from outside the cloud has a probability of no more than $1/e$ of *not* suffering a collision. Since the density is well approximated by its classical value near the edge we obtain as in Ref. [6] a cutoff distance that depends weakly on the cross section σ. In the unitarity limit this cross section is $\sigma = C\lambda_T^2$, where $\lambda_T = \sqrt{2\pi/mT}$ is the thermal de Broglie wavelength and C is a numerical constant of order unity.

In the experiments [2,3], the trap is very elongated with $\omega_z \ll \omega_x, \omega_y$. The transverse frequencies ω_x and ω_y differ by 10–20%, but to leading order this ellipticity can be taken into account for the mode frequencies by considering a cylindrically symmetric trap

$$V(x,y,z) = \frac{m}{2}\omega_\perp^2(x^2 + y^2 + \lambda^2 z^2) \tag{13}$$

with $\omega_\perp = \sqrt{\omega_x \omega_y}$. To model the experiments, we therefore consider such a trap in the following with $\lambda \ll 1$, which results in a separation of the transverse and longitudinal motion. Hence, $a = b$ and $c = 0$ for the transverse mode, while $a = b = 0$ and $c \neq 0$ for the axial mode.

In the classical limit with η given by (8), Eq. (11) yields

$$\frac{\Gamma}{\omega_\perp} = 1.08\lambda^{2/3}N^{-1/3}\left(\frac{T}{T_F}\right)^2 f(\lambda, \tau_0), \tag{14}$$

where $f(\lambda, \tau_0)$, given in Ref. [6], is an angular average arising from the integration over volume, with $\tau_0 = \sigma n(0)\sqrt{T/2m}/2\lambda\omega_\perp$. The total number of trapped atoms is N. For the axial mode, one obtains a similar expression for Γ/ω_z with $\lambda^{2/3}$ replaced by $\lambda^{5/3}$, while the numerical constant in front is 3.1.

We shall now compare our calculated damping rates with experiments on trapped ^{6}Li atoms [2,3]. Measurements of the damping of the axial mode [2] yielded the value $\Gamma/\omega_z \approx 0.0045$ in the unitarity limit for a trap with $\lambda = 0.030$ and $N = 4 \times 10^5$. The temperature, however, was not known, and

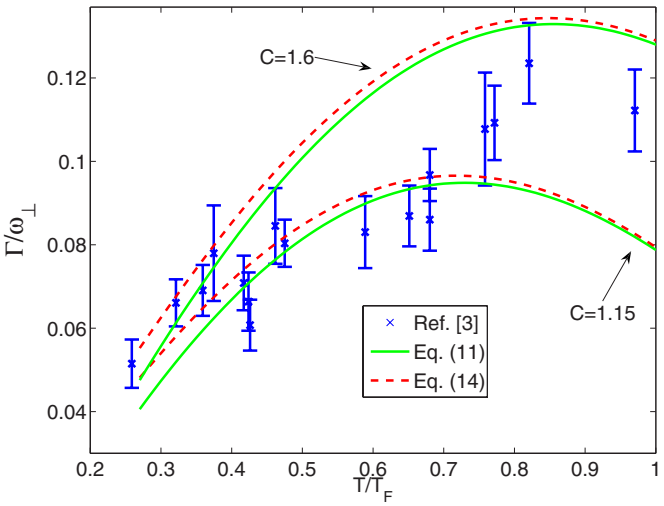

FIG. 3. (Color online) The damping of the transverse mode. The ×'s are experimental values from Ref. [3] and lines are theory.

we cannot therefore make a direct comparison to our calculated damping. To obtain an estimate, we use the classical result (14), which yields values in the range $0.004 \leqslant \Gamma/\omega_z \leqslant 0.007$ for $0.3 \leqslant T/T_F \leqslant 0.6$ $(C=1)$.

The Duke experiments [3] allow for a more direct comparison, since information on the temperature is available. In Fig. 3, we plot the observed damping rate of the lowest transverse collective mode. The temperature is determined by fitting the observed density profile to an ideal gas profile with a rescaled Thomas-Fermi radius $\tilde{R}_{\mathrm{TF}}^2 = \sqrt{1+\beta} R_{\mathrm{TF}}^2$, where $R_{\mathrm{TF}}^2 = 2(3\lambda N)^{1/3} a_\perp^2$ and $a_\perp^{-2} = m\omega_\perp$ [3]. Here, $(1+\beta)$ is the parameter used conventionally in the unitarity limit, equal to the ratio between the ground-state energy and that of a noninteracting Fermi gas. In the classical regime, the fitted temperature $\tilde{T}$ is related to the physical temperature by $\sqrt{1+\beta}\tilde{T} = T/T_F$. We have used this relation for all T (with $\beta=-0.5$), since the profile is approximately classical above the critical temperature [20].

We plot the calculated damping rate from Eq. (11) for two different values of C of order 1. The spatial integrals are performed with a density profile corresponding to an ideal gas with temperature $\tilde{T}$, Thomas-Fermi length $\tilde{R}_{\mathrm{TF}}$, $N=2\times 10^5$, and $\lambda=0.045$. We also plot the classical limit (14). As expected from Fig. 2, we see that the effects of the medium are small except for low T. Note that our theoretical curves are not reliable in the region where they predict a decreasing damping with increasing temperature, since this reflects the breakdown of hydrodynamic behavior: for larger T the cloud becomes less dense and the cross section σ also decreases as $1/T$, implying that the mean free path increases with temperature. To estimate the temperature above which the behavior ceases to be hydrodynamic, we compare the mean free path $l(0) \sim 1/n(0)\sigma$ in the center to the spatial extent of the cloud in the transverse direction. In the classical limit $l(0) \sim R_x R_y R_z / N\sigma$, where $R_i = (2\pi T/m\omega_i^2)^{1/2}$. The condition $l(0) \ll R_\perp$ implies that $T \ll \omega_\perp (\lambda N)^{1/2}$. For the experimental parameters, this means that the gas is hydrodynamic for $T \ll 5$ $\mu\mathrm{K} \sim 2T_F$ and there is a limited temperature range in which we can compare the theory to the measured damping rate, since our calculations only apply to the normal phase. The fact that the observed mode remains hydrodynamic with a small damping for $T \to 0$ indicates superfluidity [21].

The results for the damping yield the correct order of magnitude for two experiments measuring at two very different frequencies. This cannot be obtained simply by adjusting the parameter C since the results depend only weakly on C and we have used $C \sim 1$ in both cases.

IV. DISCUSSION AND CONCLUSIONS

The shear viscosity of a normal Fermi gas in the unitarity limit was analyzed as a function of temperature. For high temperatures where one can perform systematic calculations, we demonstrated that a variational solution to the Boltzmann equation yields a value of η which is accurate to better than two percent. At low temperatures, the role of the strong-coupling pseudogap effect was analyzed by calculating the viscosity within the Kubo formalism. We showed that the pseudogap reduces the viscosity significantly since it suppresses the density of states at the Fermi level. While we stress that our calculations of the viscosity are approximate in nature for $T_c \leqslant T \lesssim T_F$, it is interesting that the Kubo approach yields values that are close to those obtained from the Boltzmann equation (with a medium cross section) in this regime. This suggests that the main effects of the medium can be taken into account by using a medium cross section in a Boltzmann approach to the transport properties of the atomic cloud in the unitarity limit. Based on these high- and low-temperature results, we constructed an interpolation formula for the viscosity as a function of temperature at unitarity. This formula was used to analyze two experiments on the decay of collective modes in terms of viscous damping. In performing this analysis, it was crucial to introduce an explicit cutoff in the spatial integrations since hydrodynamics necessarily fails in the outer parts of the cloud, where the density is low. We concluded that the hydrodynamic approach of viscous damping accounts reasonably well for the experimental observations; this holds for both the longitudinal and the transverse modes for which the observed damping differs by an order of magnitude. It would be very valuable to be able to compare theory and experiment at higher temperatures, where the behavior should be collisionless. In this limit the mode frequencies should approach twice the oscillator frequencies and the damping rate become proportional to the inverse relaxation time rather than the viscosity.

ACKNOWLEDGMENTS

We are grateful to J. E. Thomas for very helpful correspondence. We also thank T.-L. Ho and C. J. Pethick for discussions.

APPENDIX: THE HIGH-TEMPERATURE LIMIT

In this Appendix we derive the leading correction to the lowest-order variational result for the shear viscosity of a classical gas. As is well known (see, e.g., paper I for details),

the properties of the collision integral in the Boltzmann equation allow one to derive a lower bound on the viscosity of the form

$$\eta \geqslant \frac{(U,X)^2}{(U,HU)}, \tag{A1}$$

where $(\ldots,\ldots)$ denotes a suitably defined scalar product. Here X denotes the inhomogeneous term in the Boltzmann equation while U is the trial function and the positive, semidefinite integral operator H represents the collision term. Since we are only interested in determining the relative correction to the high-temperature viscosity arising from an improved trial function we use units such that $2m=k_BT=1$ in order to simplify the presentation. It is convenient to consider the equivalent upper bound on the inverse viscosity, and we shall therefore seek to minimize the functional F given by

$$F = \frac{(U,HU)}{(U,X)^2} \tag{A2}$$

by varying the trial function U. The lower bound on the viscosity given in Eq. (8) corresponds to the choice $U=X$.

In order to improve this bound we derive the minimum value of F for a trial function U given by a variable linear combination of the functions U_1 and U_2,

$$U = \gamma U_1 + c(1-\gamma)U_2, \tag{A3}$$

where γ is a parameter to be varied, while c is a constant. We choose c such that (U,X) is independent of γ,

$$c = \frac{(U_1,X)}{(U_2,X)}, \tag{A4}$$

with which we obtain $(U,X)=(U_1,X)$. The numerator in Eq. (A2) is a quadratic form in γ, which is readily minimized resulting in the value $F_{\min}$. We are interested in the relative difference between $F_{\min}$ and the value $F_{\gamma=1}$ of F for $\gamma=1$. Consequently we define δ according to

$$\delta = 1 - \frac{F_{\min}}{F_{\gamma=1}}, \tag{A5}$$

where $F_{\gamma=1}=(U_1,HU_1)/(U_1,X)^2$. The integral operator H is symmetric and therefore $(U_1,HU_2)=(U_2,HU_1)$. We define the quantities h_{12} and h_{22} by

$$h_{12} = c\frac{(U_1,HU_2)}{(U_1,HU_1)} \quad \text{and} \quad h_{22} = c^2\frac{(U_2,HU_2)}{(U_1,HU_1)}, \tag{A6}$$

in terms of which δ assumes the form

$$\delta = \frac{(1-h_{12})^2}{1+h_{22}-2h_{12}}. \tag{A7}$$

Note that both h_{12} and h_{22} are independent of any constant multiplying U_1 or U_2. We take $U_1(\mathbf{p})=(p_z^2-p^2/3)$ (corresponding to $U_1 \propto X$) and $U_2(\mathbf{p})=(p_z^2-p^2/3)p^2$. We also define

$$\Delta_i = \frac{1}{2}[U_i(\mathbf{p}) + U_i(\mathbf{p}_1) - U_i(\mathbf{p}') - U_i(\mathbf{p}_1')] \tag{A8}$$

for $i=1,2$. It is convenient to introduce relative and center-of-mass momentum variables according to

$$\mathbf{p} = \mathbf{q} + \frac{\mathbf{Q}}{2}, \quad \mathbf{p}_1 = -\mathbf{q} + \frac{\mathbf{Q}}{2}; \quad \mathbf{p}' = \mathbf{q}' + \frac{\mathbf{Q}}{2}, \quad \mathbf{p}_1' = -\mathbf{q}' + \frac{\mathbf{Q}}{2}. \tag{A9}$$

Since the scattering is elastic, we have $q^2=q'^2$. We obtain

$$\Delta_1 = q_z^2 - q_z'^2 \tag{A10}$$

and

$$\Delta_2 = (q_z^2 - q_z'^2)\left(q^2 + \frac{Q^2}{4}\right) + \mathbf{q}\cdot\mathbf{Q}\left(q_z Q_z - \frac{\mathbf{q}\cdot\mathbf{Q}}{3}\right) - \mathbf{q}'\cdot\mathbf{Q}\left(q_z' Q_z - \frac{\mathbf{q}'\cdot\mathbf{Q}}{3}\right). \tag{A11}$$

In order to determine h_{12} and h_{22} we first calculate the angular averages $\langle\Delta_1^2\rangle$, $\langle\Delta_1\Delta_2\rangle$ and $\langle\Delta_2^2\rangle$ by integrating over the directions of each of the vectors $\mathbf{q}$, $\mathbf{q}'$, and $\mathbf{Q}$. We get

$$\langle\Delta_1^2\rangle = \frac{8}{45}q^4 \tag{A12}$$

together with

$$\langle\Delta_1\Delta_2\rangle = \frac{8}{45}q^4\left(q^2 + \frac{7}{12}Q^2\right) \tag{A13}$$

and

$$\langle\Delta_2^2\rangle = \frac{8}{45}q^4\left(q^2+\frac{Q^2}{4}\right)^2 + \frac{16}{135}Q^2q^4\left(q^2+\frac{Q^2}{4}\right) + 2q^4Q^4\left(\frac{2}{75} - \frac{4}{405}\right). \tag{A14}$$

The final integrations over the magnitude of the total and relative momentum are given by

$$\int_0^\infty dq\, q^2 q\sigma(q) \int_0^\infty dQ\, Q^2 e^{-2q^2-Q^2/2}\langle\cdots\rangle, \tag{A15}$$

where the exponential factors arise from the product of the equilibrium distribution functions $f(\xi_p)f(\xi_{p_1})$ and $\sigma(q)$ is the q-dependent cross section. The additional factor of q in the integrand of Eq. (A15) is due to the relative velocity occurring in the collision integral. Putting these results together and using that $c=\Gamma(7/2)/\Gamma(9/2)=2/7$ we finally get

$$h_{12} = \frac{2}{7}\left(\frac{I_6}{I_4} + \frac{7}{4}\right) \tag{A16}$$

and

$$h_{22} = \frac{4}{49}\left(\frac{I_8}{I_4} + \frac{7}{2}\frac{I_6}{I_4} + \frac{301}{48}\right), \tag{A17}$$

where we have defined the integrals I_n by

$$I_n = \int_0^\infty dq\, q^{n+3}\sigma(q)e^{-2q^2}. \tag{A18}$$

In the weak-coupling limit σ is independent of q and we obtain $I_8/I_4=5$ and $I_6/I_4=2$, resulting in $h_{12}=15/14$ and $h_{22}=877/588$. When inserted into Eq. (A7) these values yield $\delta=3/205$ in precise agreement with [19], since $(1-\delta)^{-1}=1+3/202$. As shown in Ref. [19] the convergence is very fast, the next-order term yielding a further correction of only a tenth of a percent. For resonant scattering $\sigma\propto q^{-2}$, in which case $I_8/I_4=3$ and $I_6/I_4=3/2$, resulting in $h_{12}=13/14$ and $h_{22}=697/588$, which yields $\delta=3/193$ or, correspondingly, $(1-\delta)^{-1}=1+3/190$. We can thus safely conclude that the expression (8) is accurate to better than 2% at all values of a, including the unitarity limit $a\to\infty$.

[1] K. M. O'Hara *et al.*, Science **298**, 2179 (2002).
[2] M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. H. Denschlag, and R. Grimm, Phys. Rev. Lett. **92**, 203201 (2004).
[3] J. Kinast, A. Turlapov, and J. E. Thomas, Phys. Rev. Lett. **94**, 170404 (2005).
[4] Z. Xu, J. Phys. G **30**, S927 (2004).
[5] D. A. Teaney, J. Phys. G **30**, S1247 (2004).
[6] G. M. Kavoulakis, C. J. Pethick, and H. Smith, Phys. Rev. A **57**, 2938 (1998).
[7] B. A. Gelman, E. V. Shuryak, and I. Zahed, Phys. Rev. A **72**, 043601 (2005).
[8] P. Massignan, G. M. Bruun, and H. Smith, Phys. Rev. A **71**, 033607 (2005).
[9] G. M. Bruun and H. Smith, Phys. Rev. A **72**, 043605 (2005).
[10] A. Perali, P. Pieri, G. C. Strinati, and C. Castellani, Phys. Rev. B **66**, 024510 (2002).
[11] G. M. Bruun and G. Baym, Phys. Rev. A **74**, 033623 (2006).
[12] A. Peshier and W. Cassing, Phys. Rev. Lett. **94**, 172301 (2005).
[13] J. N. Milstein, S. J. J. M. F. Kokkelmans, and M. J. Holland, Phys. Rev. A **66**, 043604 (2002).
[14] Y. Ohashi and A. Griffin, Phys. Rev. A **67**, 033603 (2003).
[15] G. M. Bruun and C. J. Pethick, Phys. Rev. Lett. **92**, 140404 (2004); G. M. Bruun, Phys. Rev. A **70**, 053602 (2004).
[16] See M. Randeria, in *Bose-Einstein Condensation*, edited by A. Griffin *et al.* (Cambridge University Press, Cambridge, 1995), and references therein.
[17] G. Baym and L. P. Kadanoff, Phys. Rev. **124**, 287 (1961).
[18] A. Perali, P. Pieri, L. Pisani, and G. C. Strinati, Phys. Rev. Lett. **92**, 220404 (2004).
[19] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, Cambridge, 1939), p. 169.
[20] J. E. Thomas (private communication).
[21] J. Kinast, S. L. Hemmer, M. E. Gehm, A. Turlapov, and J. E. Thomas, Phys. Rev. Lett. **92**, 150402 (2004).