Coherent acceleration of material wave packets in modulated optical fields

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We study the quantum dynamics of a material wave packet bouncing off a modulated atomic mirror in the presence of a gravitational field. We find the occurrence of coherent accelerated dynamics for atoms beyond the familiar regime of dynamical localization. The acceleration takes place for certain initial phase-space data and within specific windows of modulation strengths. The realization of the proposed acceleration scheme is within the range of present-day experimental possibilities.

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Enrico Fermi, in his seminal paper "On the origin of cosmic rays," conjectured that "cosmic rays are originated and accelerated primarily in the interstellar space of the galaxy by *collisions* against moving magnetic fields" [1]. Following this work, the possibility of accelerating particles bouncing off oscillating surfaces [2] was extensively studied in accelerator physics, leading to the development of two major models: The Fermi-Ulam accelerator, which deals with the bouncing of a particle off an oscillating surface in the presence of another fixed surface parallel to it; and the Fermi-Pustyl'nikov accelerator, where the particle bounces off an oscillating surface in the presence of gravity. In the case of the Fermi-Ulam accelerator [3,4] it was shown that the energy of the particle remains bounded and the unlimited acceleration proposed by Fermi is absent [5]. In the Fermi-Pustyl'nikov accelerator, by contrast, there exists disks of initial data within specific domains of phase space that result in trajectories speeding up to infinity.

In recent years, the efficient transfer of large momenta to laser-cooled atoms has become an important problem for a number of applications such as atom interferometry [7] and the development of matter-wave-based inertial sensors in quantum metrology [8]. Possible schemes of matter-wave acceleration have been proposed and studied. For example, a spatially periodic optical potential applied at discrete times to an atom was found to accelerate it in the presence of a gravitational field [9–11]. The δ-kicked accelerator operates for certain sets of initial data that originate in stable islands of phase space.

In this paper, we propose an experimentally realizable technique to accelerate a material wave packet in a coherent fashion. It consists of an atom-optics version of the Fermi-Pustyl'nikov accelerator [12] where a cloud of ultracold atoms falling in a gravitational field bounces off a spatially modulated atomic mirror. This scheme is different from previous accelerator schemes in the following ways: (i) The regions of phase space that support acceleration are located in the mixed phase space rather than in the islands of stability (or nonlinear resonances); (ii) the acceleration of the wave packet is coherent; (iii) it occurs only for certain windows of oscillation strengths [13].

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Our starting point is the analysis by Saif *et al.* [12] that establishes the dynamical localization of atoms in the Fermi-Pustyl'nikov accelerator (or Fermi accelerator for short) and shows a diffusive behavior both in the classical and in the quantum domains beyond the localization regime [14]. We now extend these results to identify conditions leading to the coherent acceleration of the atoms. We find clear signatures of that behavior both in an ensemble of classical particles and for a quantum wave packet. In quantum mechanics, however, the Heisenberg uncertainty principle restricts the phase-space size of the initial atomic wave packet which may result in coherent acceleration occurring on top of a diffusive background.

We consider a cloud of laser-cooled atoms that move along the vertical $\tilde{z}$ direction under the influence of gravity and bounce back off an atomic mirror [15]. This mirror is formed by a laser beam incident on a glass prism and undergoing total internal reflection, thereby creating an optical evanescent wave of intensity $I(\tilde{z})=I_0\exp(-2k\tilde{z})$ and characteristic decay length k^{-1} outside of the prism.

The laser intensity is modulated by an acousto-optic modulator as [14]

$$I(\tilde{z},\tilde{t}) = I_0 \exp(-2k\tilde{z} + \epsilon \sin \omega\tilde{t}), \tag{1}$$

where ω is the frequency and ϵ the amplitude of modulation. The laser frequency is tuned far from any atomic transition, so that there is no significant upper-state atomic population. The excited atomic level(s) can then be adiabatically eliminated, and the atoms behave for all practical purposes as scalar particles of mass m whose center-of-mass motion is governed by the one-dimensional Hamiltonian

$$\tilde{H} = \frac{\tilde{p}^2}{2m} + mg\tilde{z} + \frac{\hbar\Omega_{\text{eff}}}{4}e^{-2k\tilde{z}+\epsilon \sin \omega\tilde{t}}, \tag{2}$$

where $\tilde{p}$ is the atomic momentum along $\tilde{z}$ and g is the acceleration of gravity.

We proceed by introducing the dimensionless position and momentum coordinates $z\equiv\tilde{z}\omega^2/g$ and $p\equiv\tilde{p}\omega/(mg)$, the scaled time $t\equiv\omega\tilde{t}$, the dimensionless intensity $V_0 \equiv\hbar\omega^2\Omega_{\text{eff}}/(4mg^2)$, the "steepness" $\kappa\equiv 2kg/\omega^2$, and the modulation strength $\lambda\equiv\omega^2\epsilon/(2kg)$ of the evanescent wave field, in terms of which the Hamiltonian takes the dimensionless form

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$$H = (\omega^2/mg^2)\widetilde{H} = \frac{p^2}{2} + z + V_0 \exp[-\kappa(z - \lambda \sin t)]. \quad (3)$$

When extended to an ensemble of noninteracting particles, the classical dynamics obeys the condition of incompressibility of the flow [3], and the phase-space distribution function $P(z,p,t)$ satisfies the Liouville equation

$$\left\{\frac{\partial}{\partial t} + p\frac{\partial}{\partial z} + \dot{p}\frac{\partial}{\partial p}\right\} P(z,p,t) = 0, \quad (4)$$

where $\dot{p} = -1 + \kappa V_0 \exp[-\kappa(z-\lambda \sin t)]$ is the force on a classical particle.

In the absence of mirror modulation, the atomic dynamics is integrable. For very weak modulations the incommensurate motion almost follows the integrable evolution and remains rigorously stable, as prescribed by the Kolmogorov-Arnold-Moser theorem. As the modulation increases, though, the classical system becomes chaotic.

In the quantum regime, the atomic evolution is determined by the corresponding Schrödinger equation

$$i\bar{k}\frac{\partial \psi}{\partial t} = \left(\frac{p^2}{2} + z + V_0 \exp[-\kappa(z - \lambda \sin t)]\right)\psi \quad (5)$$

where $\bar{k} \equiv \hbar\omega^3/(mg^2)$ is the dimensionless Planck constant, introduced consistently with the commutation relation $[z,p] = i(\omega^3/mg^2)\hbar \equiv i\bar{k}$ for the dimensionless variables z and p. We use Eqs. (4) and (5) to study the classical and quantum-mechanical evolution of an ensemble of atoms in the Fermi accelerator.

For very short decay lengths κ^{-1} and atoms initially far from the mirror surface, we may approximate the optical potential by an infinite potential barrier at the position $z = \lambda \sin \omega t$. In that limit the atoms behave like falling particles bouncing off a hard oscillating surface [12,16].

The classical version of the problem [6] demonstrates the existence of a set of initial conditions resulting in trajectories that accelerate without bound. Specifically, the classical evolution of the Fermi accelerator displays the onset of global diffusion above a critical modulation strength $\lambda_l = 0.24$ [4], while the quantum evolution remains localized until a larger value λ_u of the modulation [12,17,18]. Above that point both the classical and the quantum dynamics are diffusive. However, for specific sets of initial conditions that lie within phase-space disks of radius ρ, accelerating modes appear for values of the modulation strength λ within the windows [6]

$$s\pi \leq \lambda < \sqrt{1 + (s\pi)^2}, \quad (6)$$

where s can take integer and half-integer values for the sinusoidal modulation of the reflecting surface considered here.

In the experimental setup [14] cesium atoms of mass $m = 2.2 \times 10^{-25}$ kg bouncing off an atomic mirror. The modulation frequency may take a value from 0 to 2 MHz, therefore, for a frequency $\omega = 5.84$ kHz of the external field, we find an effective Planck's constant $\bar{k} = \hbar\omega^3/mg^2 = 1$. Furthermore it is possible to change the amplitude of the modulation, ϵ, from 0.1 to 0.82, because, by using our suggested parameters, the first acceleration window can be realized in experiment by tuning ϵ from 0.54 to 0.64, provided the decay length of the exponentially decaying field is fixed at $k_L^{-1} \sim 0.55$ μm, which is still an experimentally accessible value [19].

In order to compare the classical and quantum atomic dynamics within these windows in the present situation we calculate the width of the momentum distribution, $\Delta p \equiv \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$, as a function of the modulation strength $\lambda > \lambda_u$. In the classical case, we consider an ensemble of particles with a Gaussian initial phase-space distribution $P(z,p,0)$ centered in a region of phase space that supports unbounded acceleration, and record the dispersion in mo-

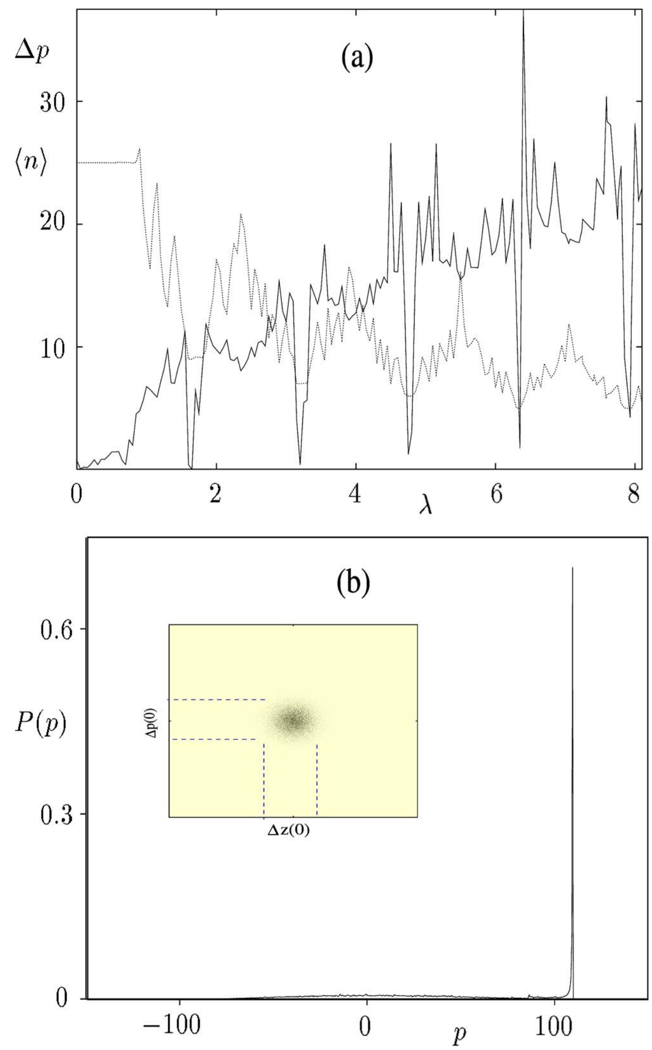

FIG. 1. (Color online) (a) Standard deviation of the momentum Δp (solid line) and average number of bounces $\langle n \rangle$ (dotted gray line) are displayed after an evolution time $t = 300$ as a function of the modulation strength λ. The initial ensemble of 10 000 particles is a narrowly peaked Gaussian distribution originating from the area of phase space that supports accelerated trajectories. The initial distributions in coordinate space are centered at $\bar{z} = 0$ and $\bar{p} = 2\pi^2$ with $\Delta p(0) = \Delta z(0) = 0.1$, as shown in inset in (b). The momentum distribution at $t = 300$ is illustrated in (b) for $\lambda = 3\pi/2$. The broad, barely visible background results from the tails of the initial Gaussian distribution outside the area of phase space that supports accelerated trajectories.

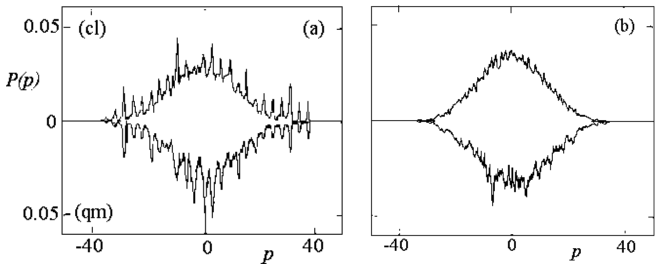

FIG. 2. Mirror images of the classical and quantum-mechanical momentum distributions $P(p)$ plotted for $\lambda=1.7$ (a) and 2.4 (b), after a propagation time $t=500$. The spikes in the momentum distribution for $\lambda=1.7$ are a signature of coherent accelerated dynamics. The initial width of the momentum distribution is $\Delta p=0.5$.

mentum of that ensemble after a fixed propagation time. The corresponding quantum problem is treated by directly solving the Schrödinger equation for an initial Gaussian wave packet of zero mean momentum.

Classically, the width in the momentum space Δp remains small and almost constant for very small modulation strengths, which indicates the absence of diffusive dynamics, but starts to increase linearly [12] as a function of the modulation strength for larger values of λ. As λ is further increased, though, we find that the diffusion of the ensemble is sharply reduced for modulation strengths within the acceleration windows of Eq. (6) [see Fig. 1(a)]. Following each window, the momentum dispersion grows again approximately linearly with λ. We interpret the sharply reduced value of the dispersion as resulting from the nondispersive, coherent acceleration of the atomic sample above the atomic mirror which is illustrated in Fig. 1(b) and absent otherwise.

In the quantum case the Heisenberg uncertainty principle imposes a limit on the smallest size of the initial wave packet. Thus in order to form an initial wave packet that resides entirely within regions of phase space leading to a coherent and dispersionless acceleration, the appropriate value of the effective Planck constant is to be evaluated, for example, by controlling the frequency ω [20]. For broad wave packets, the situation that we consider in this paper, the coherent acceleration manifests itself instead, both in the classical and in the quantum cases, as regular spikes in the marginal probability distributions $P(p,t)=\int dx\, P(x,p,t)$ and $P(x,t)=\int dp\, P(x,p,t)$, which are absent otherwise.

Figure 2 illustrates the marginal probability distribution $P(p,t)$ in momentum space for $\lambda=1.7$ (a) and 2.4 (b), both in the classical and in the quantum domains. In this example, the initial area of the particle phase-space distribution is taken to be large compared to the size of the phase-space regions leading to purely unbounded dispersionless acceleration. The sharp spikes in the time-evolved momentum distribution $P(p,t)$ appear when the modulation strength satisfies the condition of Eq. (6), and gradually disappear as it exits these windows. These spikes are therefore a signature of the coherent accelerated dynamics. In contrast, the portions of the initial probability distribution originating from the regions of the phase space that do not support accelerated dynamics undergo diffusive dynamics.

We can gain some additional understanding of the diffu-

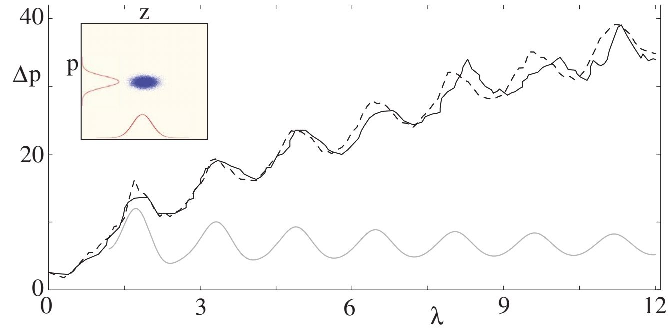

FIG. 3. (Color online) Standard deviation of momentum Δp as a function of λ for an atomic de Broglie wave (thick line) and an ensemble of particles (dashed line) initially in a Gaussian distribution after a scaled propagation time $t=500$. The initial probability distributions, shown in the inset, have $\Delta p=0.5$ and $\Delta z=1$. The gray line shows scaled diffusion coefficient D_λ/D_0 as a function of λ (arbitrary units).

sive behavior from the close mathematical analogy between the system at hand and the kicked rotor model [12]. It has been established mathematically [21] that for large modulation strengths the diffusive behavior of classical systems described by the standard map displays a modulated growing behavior. For large λ, the diffusion coefficient is given by

$$D_\lambda = D_0\left(\frac{1}{2} - J_2(K) - J_1^2(K) + J_2^2(K) + J_3^2(K)\right), \qquad (7)$$

where $K=4\lambda$ and $D_0=K^2/2=8\lambda^2$ [21] and J_1, J_2, and J_3 are first-, second-, and third-order Bessel functions. Recent experiments by Kanem *et al.* [22] also report such a behavior for the δ-kicked accelerator in the case of large modulations.

A comparison between the classical behavior and the quantum momentum dispersion as a function of λ is illustrated in Fig. 3, while the scaled diffusion coefficient D_λ/D_0 is shown in arbitrary units. It is interesting to note that the dispersion exhibits maxima for oscillation strengths $\lambda_m = [s\pi+\sqrt{1+(s\pi)^2}]/2$, which reside at the center of the acceleration windows [2], indicating that those trajectories that do

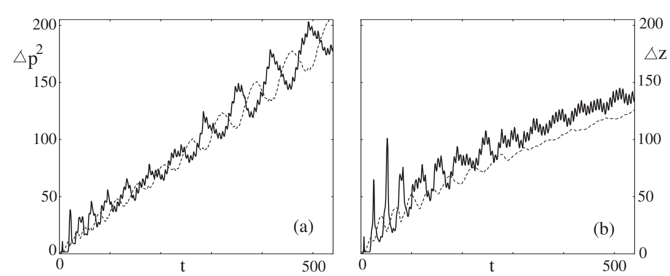

FIG. 4. (a) Variance in the momentum space (dark lines) and standard deviation in the coordinate space (gray lines) as a function of time for $\lambda=1.7$. The coherent acceleration results in a breathing of the atomic wave packet, as evidenced by the out-of-phase oscillations of the variances. (b) Dynamics for $\lambda=2.4$, a modulation strength that does not result in coherent acceleration. Note the absence of breathing in that case. The two values $\lambda=1.7$ and 2.4 correspond to the first maximum and first minimum in Fig. 3. Same parameters as in Fig. 3.

not correspond to the phase-space area supporting accelerated dynamics display maximum dispersion instead.

From the numerical results of Fig. 2, we conjecture that the spikes are well described by a sequence of Gaussian distributions separated by a distance π, in both momentum space and coordinate space. We can therefore express the complete time-evolved wave packet composed of a series of sharply peaked Gaussian distributions superposed to a broad background due to diffusive dynamics, such that

$$P(p) = \mathcal{N} e^{-p^2/4\Delta p^2} \sum_{n=-\infty}^{\infty} e^{-(p-n\pi)^2/4\epsilon^2}, \qquad (8)$$

where $\epsilon \ll \Delta p$, and $\mathcal{N}$ is a normalization constant.

Further insight into the quantum acceleration of the atomic wave packet is obtained by studying its temporal evolution. We find that within the window of acceleration the atomic wave packet displays a linear growth in the momentum variance and the standard deviation in coordinate space as a function of time. Figure 4 illustrates that, for modulation strengths within the acceleration window, the growth in the momentum variance displays oscillations of increasing periodicity whereas the standard deviation in coordinate space follows with a phase difference of 180°. The out-of-phase oscillatory evolutions of Δp^2 and Δz indicate a breathing of the wave packet and are a signature of the coherence in accelerated dynamics as it disappears outside. As a final point we note that outside the acceleration window the linear growth in the momentum variance, a consequence of normal diffusion, translates into a t^{α} law, with $\alpha < 1$, which is a consequence of anomalous diffusion.

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