Stability and entanglement in optical-atomic amplification of trapped atoms: The role of atomic collisions

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Atomic collisions are included in an interacting system of optical fields and trapped atoms allowing field amplification. We study the effects of collisions on the system stability. Also a study of the degree of entanglement between atomic and optical fields is made. We found that, for an atomic field initially in a vacuum state and optical field in a coherent state, the degree of entanglement does not depend on the optical field intensity or phase. We show that in conditions of exponential instability the system presents at long times two distinct stationary degree of entanglement with collisions affecting only one of them.

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I. INTRODUCTION

Bose-Einstein condensation of trapped atomic gases [1–3] has produced a fantastic advance in atom optics. Particularly, the interaction of condensates with single mode quantized light fields has been a fascinating topic [4–17], allowing, for instance, light and matter wave amplification [12–14], optical control of atomic statistical properties [15,16], and potential applications in quantum information technology [17].

It is known that the trap environment can modify the properties of ultracold atoms, such as its critical temperature [18,19]. Then it is expected that trap environment also influences the interaction between ultracold atoms and optical fields. In Ref. [20], the trap environment effects on the condensate collective atomic-recoil laser (CARL) [21,22] was considered by expanding the matter-wave field in the trap matter-wave modes. Such a situation was called cavity atom optics (CAO), in analogy with the cavity quantum electrodynamics (CQE) where the spontaneous emission is modified by the presence of the cavity. The model obtained presents interesting properties such as two regimes of exponential instability [20] and statistical properties that depend on intensity and phase of the optical field [23]. However, neither atomic collisions was taken into account nor a detailed study of the degree of entanglement between atomic and optical fields was done. Entanglement, one of the most notable characteristics of quantum mechanics [24], has been object of intense study, both in systems involving light and matter [25,26] and in solids [27,28], due to its important role for quantum-information processing [29]. Therefore, inclusion of collisions and characterization of entanglement are important in order to turn the model more realistic and potentially useful for applications in the context of quantum information theory.

This paper deals with the extension of the model of atom-optical parametric amplifier considered in Refs. [20,23] by including collisions between atoms. A study of the changes due collisions on the thresholds of exponential instability as well as in growth rate of the fields amplitudes is presented. Also a characterization of the degree of entanglement between atomic and optical fields is done. We show that in conditions of exponential instability the system presents at long times two distinct stationary degree of entanglement with collisions affecting only one of them.

The article is organized as follows. In Sec. II we derive the effective Hamiltonian describing the system studied. In Sec. III we present an analysis of the system stability. In Sec. IV we consider the atom-photon degree of entanglement in the regime of field amplification. Finally, in Sec. V we present the conclusion.

II. MODEL

We consider a Schrödinger field of bosonic two-level atoms, with transition frequency ν, interacting via two-body collisions and coupled by electric-dipole interaction to two single-mode running wave optical fields of frequencies ω_1 and ω_2, treated as a quantum and a classical field, respectively. Both optical fields are assumed being far off-resonant from any electronic transition. Thus, although the atom's internal state remains unchanged, the center-of-mass motion may change due to the atomic recoil induced by two-photon virtual transitions. In the far off-resonance regime the excited state population is small, and therefore spontaneous emission as well as collisions between excited state atoms, and between ground state and excited states atoms, may be neglected. In this regime the excited state can be adiabatically eliminated and the ground state atomic field evolves coherently under the effective Hamiltonian

$$\hat{H} = \int d^3\mathbf{r}\hat{\Psi}^\dagger(\mathbf{r})\Bigg[\mathcal{H}_0 + \frac{U}{2}\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r}) + \hbar\left(\frac{g_1^* g_2}{\Delta}\hat{a}_1^\dagger a_2 e^{-i\mathbf{K}\cdot\mathbf{r}} + \frac{g_2^* g_1}{\Delta}a_2^*\hat{a}_1 e^{i\mathbf{K}\cdot\mathbf{r}}\right)\Bigg]\hat{\Psi}(\mathbf{r}) + \hbar\,\delta\hat{a}_1^\dagger\hat{a}_1, \quad (1)$$

where

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$$\mathcal{H}_0 = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}), \tag{2}$$

m is the atomic mass, $V(\mathbf{r})$ is the trap potential, $\Delta = \omega_2 - \nu$ is the detuning between atoms and the classical optical field, g_1 and g_2 are the optical fields coupling coefficients, $\mathbf{K} = \mathbf{k}_1 - \mathbf{k}_2$ is the difference between their wave vectors, and $\delta = \omega_1 - \omega_2$ is the detuning between them. The operator $\hat{a}_1$ is the photon annihilation operator for the quantized optical field, taken in the frame rotating at the classical field frequency ω_2. The optical field treated classically is assumed to remain undepleted, so that a_2 is simply a constant related to its intensity. Terms corresponding to the spatially independent light shift potential were neglected, so that the index of refraction of the atomic sample was assumed the same of the vacuum. Two-body collisions were included in the s-wave scattering limit by the second term inside brackets in the Hamiltonian (1), where

$$U = \frac{4\pi\hbar^2 a}{m}, \tag{3}$$

and a is the s-wave scattering length, which depending on the repulsive or attractive character of the interaction can assume positive or negative values, respectively. We consider only positive values of scattering length, which are suitable for the creation of large condensates, and correspond to a situation more consistent with the approximations in this paper.

We assume that the atomic field is initially a Bose-Einstein condensate with mean number of condensed atoms N, and that this condensate is well described by a number state so its initial state is described by

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{N!}}(\hat{c}_0^\dagger)^N|0\rangle, \tag{4}$$

where $|0\rangle$ is the vacuum state and

$$\hat{c}_0^\dagger = \int d^3\mathbf{r}\,\varphi_0(\mathbf{r})\hat{\Psi}^\dagger(\mathbf{r}) \tag{5}$$

is the creation operator for atoms in the condensate state $\varphi_0(\mathbf{r})$. Due to the presence of collisions the condensate wave function $\varphi_0(\mathbf{r})$ satisfies the Gross-Pitaevskii equation [30,31]

$$[\mathcal{H}_0 + NU|\varphi_0(\mathbf{r})|^2]\varphi_0(\mathbf{r}) = \mu\varphi_0(\mathbf{r}), \tag{6}$$

where μ is the chemical potential.

Now we expand the atomic field operator in terms of the trap eigenmodes $\{\varphi_n(\mathbf{r})\}$ according to

$$\hat{\Psi}(\mathbf{r}) = \varphi_0(\mathbf{r})\hat{c}_0 + \delta\hat{\psi}(\mathbf{r}), \tag{7}$$

where $\delta\hat{\psi}(\mathbf{r}) = \Sigma_{n\neq 0}^{\infty}\varphi_n(\mathbf{r})\hat{c}_n$ annihilates particles in the excited trap modes, $\hat{c}_n$ is the annihilation operator for atoms in mode n, and $\int d^3\mathbf{r}\varphi_m^*(\mathbf{r})\varphi_n(\mathbf{r}) = \delta_{mn}$. We are interested in the linear regime, valid for interaction times so that $\langle\int d^3\mathbf{r}\,\delta\hat{\psi}^\dagger\delta\hat{\psi}\rangle \ll \langle\hat{c}_0^\dagger\hat{c}_0\rangle$. Then we can maintain only quadratic terms in the fields operators and invoke the undepleted approximation, which permits to substitute the condensate mode by a c number evolving as

$$c_0(t) \approx \sqrt{N}e^{-i\mu t}. \tag{8}$$

Therefore, in the linear regime, by inserting expansion (7) and taking into account the Gross-Pitaevskii equation (6), the Hamiltonian (1) reduces to

$$\begin{aligned}\hat{H} = {}& \hbar\delta\hat{a}_1^\dagger\hat{a}_1 + \hbar\int d^3\mathbf{r}\,\delta\hat{\psi}^\dagger(\mathcal{H}_0 + 2NU)\delta\hat{\psi} \\ &+ \hbar\frac{U}{2}\left(Ne^{2i\mu t}\int d^3\mathbf{r}\,\varphi_0^{*2}\delta\hat{\psi}^2 + \text{H.c.}\right) \\ &+ \hbar\left[\frac{g_1^* g_2 a_2}{\Delta}\sqrt{N}\hat{a}_1^\dagger\int d^3\mathbf{r}(e^{i\mu t}\varphi_0^* e^{-i\mathbf{K}\cdot\mathbf{r}}\delta\hat{\psi} \right. \\ &\left. + e^{-i\mu t}\delta\hat{\psi}^\dagger e^{-i\mathbf{K}\cdot\mathbf{r}}\varphi_0) + \text{H.c.}\right],\end{aligned} \tag{9}$$

which in terms of the trap excited modes expansion becomes

$$\begin{aligned}\hat{H} = {}& \hbar\delta\hat{a}^\dagger\hat{a} + \hbar\sum_{n\neq 0}\omega_n\hat{c}_n^\dagger\hat{c}_n + \hbar\sum_{nl\neq 0}\frac{\kappa_{nl}}{2}(\hat{c}_n\hat{c}_l + \hat{c}_n^\dagger\hat{c}_l^\dagger) \\ &+ \hbar(\hat{a} + \hat{a}^\dagger)\sum_{n\neq 0}\chi_n(\hat{c}_n + \hat{c}_n^\dagger),\end{aligned} \tag{10}$$

where

$$\hbar\omega_n = \int d^3\mathbf{r}\,\varphi_n^*[\mathcal{H}_0 + 2NU|\varphi_0(\mathbf{r})|^2]\varphi_n \tag{11}$$

is the collision modified energy of the nth trap level, $\chi_n = \sqrt{N}A_{0n}|g_1||g_2||a_2|/|\Delta|$ is the effective coupling constant between the condensate and the quantum optical field, $\hat{a} = (g_1 g_2^* a_2^*\Delta/|g_1||g_2||a_2||\Delta|)\hat{a}_1$ is the optical field operator multiplied by a phase factor that is related to the classical optical field phase,

$$\kappa_{nl} = \frac{2NU}{\hbar}\int d^3\mathbf{r}\,\varphi_0^2\varphi_n^*\varphi_l^* \tag{12}$$

and

$$A_{0n} = \int d^3\mathbf{r}\,\varphi_n^* e^{-i\mathbf{K}\cdot\mathbf{r}}\varphi_0 \tag{13}$$

are the collision parameter and the element of matrix for the optical transition, respectively, both assumed being real numbers. The phase factors $e^{\pm i\mu t}$ were included in the operators $\hat{c}_n$ and $\hat{c}_n^\dagger$.

For simplicity, we assume that the overlap-integral in the collision parameter given by Eq. (12) has a significant contribution for a given $n=l=m$ and the matrix for optical transition A_{0n} is sharply peaked for $n=m$ [37]. In this case we can neglect all excited trap modes except the m mode in the Hamiltonian (10), and the following effective Hamiltonian is obtained:

$$\begin{aligned}\hat{H} = {}& \hbar\delta\hat{a}^\dagger\hat{a} + \hbar\omega_m\hat{c}_m^\dagger\hat{c}_m + \hbar\frac{\kappa_m}{2}(\hat{c}_m^2 + \hat{c}_m^{\dagger 2}) \\ &+ \hbar\chi_m(\hat{a}^\dagger\hat{c}_m^\dagger + \hat{a}^\dagger\hat{c}_m + \hat{c}_m^\dagger\hat{a} + \hat{c}_m\hat{a}).\end{aligned} \tag{14}$$

Terms such as $\hat{a}^\dagger \hat{c}_m^\dagger$ in Hamiltonian (14) correspond to the generation of correlated atom-photon pair. These terms are analogous to the nondegenerated optical parametric amplifier [32]. Inclusion of collisions introduces terms such as $\hat{c}_m^{\dagger 2}$, which are responsible by creation of an atomic pair, and are analogous to the degenerated optical parametric amplifier [32]. The Heisenberg equations of motion for the field operators obtained from Hamiltonian (14) result in the following 4×4 linear system of equations:

$$\frac{d}{dt}\begin{pmatrix}\hat{c}\\ \hat{c}^\dagger\\ \hat{a}\\ \hat{a}^\dagger\end{pmatrix} = i\begin{pmatrix}-1 & -\kappa & -\chi & -\chi\\ \kappa & 1 & \chi & \chi\\ -\chi & -\chi & -\delta & 0\\ \chi & \chi & 0 & \delta\end{pmatrix}\begin{pmatrix}\hat{c}\\ \hat{c}^\dagger\\ \hat{a}\\ \hat{a}^\dagger\end{pmatrix}, \tag{15}$$

where the index m was dropped in order to simplify notation, and we introduced the dimensionless parameters $t=\omega_m t$, $\delta=\delta/\omega_m$, $\kappa=\kappa_m/\omega_m$, and $\chi=\chi_m/\omega_m$.

III. STABILITY

The solution of the linear system (15) can be written as

$$\hat{x}_i(t) = \sum_{j=1}^{4} G_{ij}(t)\hat{x}_j(0), \tag{16}$$

where we defined $\hat{x}_1=\hat{c}$, $\hat{x}_2=\hat{c}^\dagger$, $\hat{x}_3=\hat{a}$, and $\hat{x}_4=a^\dagger$ for convenience, $G_{ij}(t)=\Sigma_{k=1}^4[\mathbf{U}]_{ik}[\mathbf{U}^{-1}]_{kj}e^{i\omega_k t}$, $[\mathbf{U}]_{ik}$ is the ith component of the kth eigenvector of the matrix on the right-hand side of Eq. (15), and ω_k are the system eigenfrequencies.

Stability analysis shows two regimes of exponential instability: (i) For $\delta>0$ and $\chi^2>\delta(1+\kappa)/4$ there are two purely real and two purely imaginary eigenfrequencies of the form $\{\omega_1=\Omega, \omega_2=-\Omega, \omega_3=i\Gamma, \omega_4=-i\Gamma\}$, where Ω and Γ are both real quantities. There is only one exponentially growing solution at the imaginary frequency ω_4 and the system is unstable. (ii) For $\delta<0$ and $\chi^2>(1-\kappa^2-\delta^2)^2/16|\delta|(1-\kappa)$ the eigenfrequencies are complex numbers of the form $\{\omega_1=\Omega+i\Gamma, \omega_2=-\Omega+i\Gamma, \omega_3=\Omega-i\Gamma, \omega_4=-\Omega-i\Gamma\}$. This case presents two exponentially growing solutions, ω_3 and ω_4, which grow at the same rate Γ, but rotate at equal and opposite frequencies $\pm\Omega$, producing a beating in the exponential growth of the fields intensities. Otherwise the eigenfrequencies are real and the system is stable.

In Fig. 1 we plot in the $\delta-\chi^2$ plane the values of parameters defining the threshold between stable and unstable solutions. Points inside region I correspond to one exponentially growing solution, whereas for points inside region II there are two counter-rotating exponentially growing solutions. The full line indicates the threshold in the absence of collisions whereas dashed and dotted lines shows the change due collisions, which have the effect of reducing the unstable regions. In addition, for small χ^2 the region II is centered around $\delta=-\sqrt{1-\kappa^2}$ and at asymptotically large detunings its threshold grows as $\chi^2>-\delta^3/16(1-\kappa)$.

In order to illustrate the effects of collisions on the fields intensities $I_i(t)=\langle \hat{x}_i^\dagger(t)\hat{x}_i(t)\rangle$, with $i=1,3$, we consider that the atomic mode begins in a vacuum state $|0\rangle$ whereas the light field is initially in a coherent state $|\alpha\rangle$. Then, with the help of Eq. (16), we found

$$I_i(t) = |G_{i2}(t)|^2 + |G_{i4}(t)|^2 + |G_{i3}(t)\alpha + G_{i4}(t)\alpha^*|^2. \tag{17}$$

Figures 2(a) and 2(b) show the logarithmic plot of the optical field intensity $I_3(t)$ as a function of time for parameters lying in the regions I and II of instability, respectively. The intensity of the atomic mode has a similar behavior. We see that collisions reduce the rate of exponential growing and change

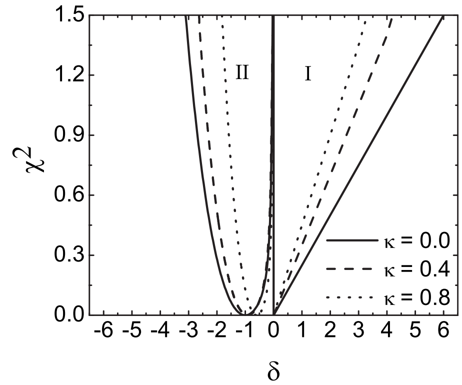

FIG. 1. Instability domains. Points inside region I correspond to one exponentially growing solution, whereas for points inside region II there are two counter-rotating exponentially growing solutions. The full line defines the threshold in the absence of collisions (κ=0.0). Change in the threshold due to collisions correspond to the dashed (κ=0.4) and dotted lines (κ=0.8). The parameters are dimensionless.

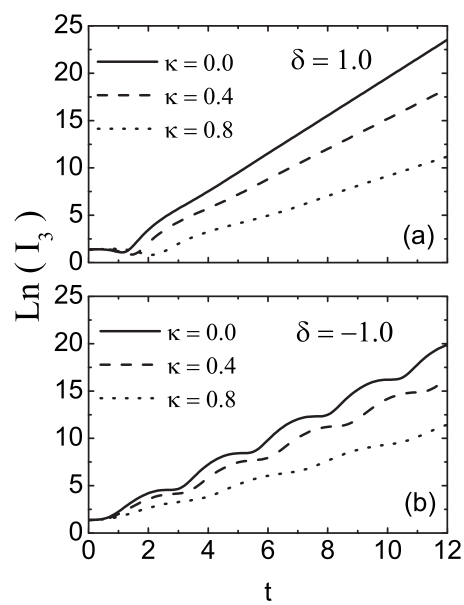

FIG. 2. Logarithmic plot of the light field intensity as function of time. (a) Parameters lying over the region I of the instability domains. (b) Parameters lying over the region II of the instability domains. Full line corresponds to absence of collisions (κ=0.0) while dashed and dotted lines correspond to inclusion of collisions with κ=0.4 and κ=0.8, respectively. We set χ=1 and α=2. The parameters are dimensionless.

the beating frequency of oscillation of the field intensity. The change in the density distribution of atoms in the condensate mode due to collisions reduces the scattering of photons into the optical field, as well as the scattering of atoms by the optical field into the atomic mode, in opposition to field amplification.

IV. ENTANGLEMENT

Now we turn to analyze the atom-photon degree of entanglement for the exponential growing regimes of fields amplitudes. It is known that for a two-component system in a pure state the degree of entanglement can be characterized by the entropy or purity of one of the system components [34,35]. Such entanglement measures require the calculation of the time-dependent quantum state for the system, which may not be an easy task. Instead, we consider a recently proposed entanglement coefficient in terms of cross-covariances of the fields operators defined by [36]

$$Y = \left[\frac{|\overline{\hat{a}\hat{c}^{\dagger}}|^2 + |\overline{\hat{a}\hat{c}}|^2}{2\left(\overline{\hat{a}^{\dagger}\hat{a}} + \frac{1}{2}\right)\left(\overline{\hat{c}^{\dagger}\hat{c}} + \frac{1}{2}\right)} \right]^{1/2}, \tag{18}$$

with the notation $\overline{x_i x_j} = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle$ for the unsymetrized centered second-order moments of the operators. Since we know the time dependent solutions for the fields amplitudes the parameter Y is easily calculated for the system considered in this paper.

The coefficient Y satisfies the inequality $0 \leq Y < 1$, where at the maximum value of Y the system is maximally entangled and was introduced taking into account that for two systems with operators $\hat{A}$ and $\hat{B}$, if any of the cross-covariances with these operators differs from zero, then the modes are entangled. For the excited trap mode starting in a vacuum state and the light field in a coherent state, we obtain

$$Y = \left[\frac{|G_{31}(t)G_{11}^{*}(t) + G_{33}(t)G_{13}^{*}(t)|^2 + |G_{31}(t)G_{12}(t) + G_{33}(t)G_{14}(t)|^2}{2\left(|G_{32}(t)|^2 + |G_{34}(t)|^2 + \frac{1}{2}\right)\left(|G_{12}(t)|^2 + |G_{14}(t)|^2 + \frac{1}{2}\right)} \right]^{1/2}. \tag{19}$$

Equation (19) shows that the degree of entanglement do not depend on the optical field intensity or phase. Furthermore, in the regime of exponential instability the parameter Y attains at long times a stationary value thats is dependent on the signal of the detuning δ. To see that, in Fig. 3 the parameter Y is plotted as a function of time for several values of detunings in the absence of collisions. We observe that for $\delta > 0$ (one exponentially growing solution) the system attains at long times the maximum degree of entanglement, while for $\delta < 0$ (two exponentially growing counter-rotating solutions) the long time degree of entanglement attains an oscillating stationary value far below the maximum one.

The effects of collisions are presented in Figs. 4(a) and 4(b) which show the entanglement coefficient as a function

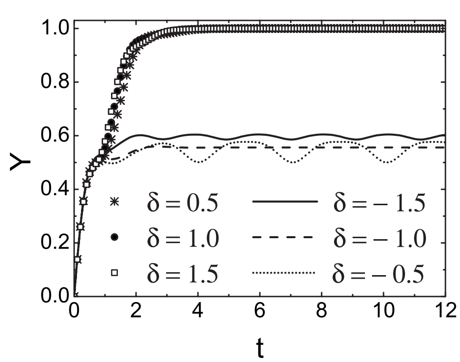

FIG. 3. Plot of the entanglement coefficient as a function of time in absence of collisions for several values of detuning δ. We set $\chi = 1$. The parameters are dimensionless.

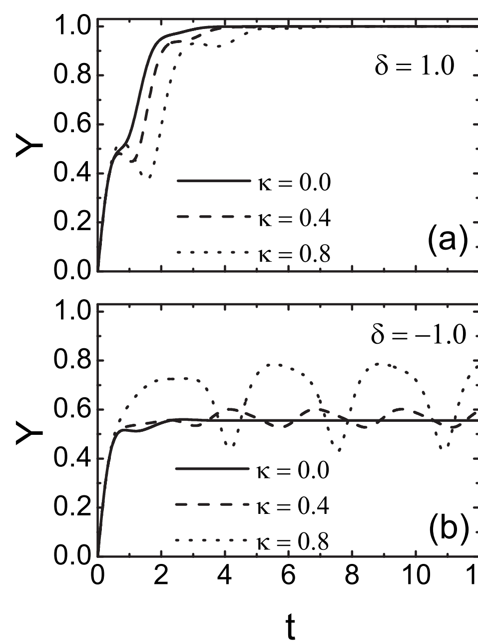

FIG. 4. Plot of the entanglement coefficient as a function of time. (a) Parameters lying over the region I of the instability domains. (b) Parameters lying over the region II of the instability domains. Full line corresponds to absence of collisions ($\kappa = 0.0$) while dashed and dotted lines correspond to inclusion of collisions with $\kappa = 0.4$ and $\kappa = 0.8$, respectively. We set $\chi = 1$. The parameters are dimensionless.

of time for values of parameters lying in regions of one and two exponential growing solutions, respectively, and considering different values for the collision parameter κ. We see in Fig. 4(a) that collisions do not affect at long times the degree of entanglement for $\delta>0$, which always tend to the maximum value. However, for $\delta<0$ [Fig. 4(b)] collisions strongly affects the long time degree of entanglement by changing the amplitude of oscillations of its stationary value.

V. CONCLUSION

In conclusion, in this work we have included atomic collisions in a model of an atom-optical parametric amplifier of trapped atoms. Analyzing the system stability in the regime of field amplification, we found that atomic collisions reduce the growth rate of the fields amplitudes. For an atomic field initially in a vacuum state and optical field in a coherent state, we have verified that the degree of entanglement between atomic and optical fields does not depend on the optical field intensity or phase. Furthermore, in conditions of field amplification the degree of entanglement attains at long time a stationary value dependent on the regime of exponential instability, being maximum only in the case of one exponentially growing solution. Finally, atomic collisions only affect the long time degree of entanglement in the case of two exponentially growing solutions.

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