Hyperfine structure and Zeeman splitting in two-fermion bound-state systems

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A relativistic wave equation for bound states of two fermions with arbitrary masses, which are exposed to a magnetic field, is derived from quantum electrodynamics. The interaction kernels are based upon the generalized invariant $\tilde{\mathcal{M}}$-matrices for interfermion and fermion-field interactions. As an application we calculate the energy corrections in a weak homogeneous **B** field to obtain the Zeeman splitting of the hyperfine structure and g factors in the lowest order [i.e., to $O(\alpha^4)$]. Landé g factors are presented for several of the first excited states of hydrogen, muonium, and muonic hydrogen.

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I. INTRODUCTION

The relativistic treatment of energy levels of two-fermion atomic systems (including atomic hydrogen, hydrogenlike ions, helium-3 ion, muonium, and muonic hydrogen), as well as their fine structure (FS) and hyperfine structure (HFS) in an external uniform magnetic field (Zeeman effect), is an important problem. The theoretical knowledge of energy spectra and transition frequencies provides a test of two-body bound-state QED [1]. One can then obtain information about the character of the coupling in the system, the gyromagnetic ratios of the bound particles, the magnetic moments [2], the mass ratio [2–5], and fundamental physical constants such as the Rydberg constant R_∞, and the fine structure constant α [6]. The Zeeman effect in the HFS can be used as a diagnostic tool for solar photospheric magnetic fields [7], fusion research and plasma physics, where the magnetic field is applied to control the shape and position of the plasma [8].

In the lowest-order approximation the linearly dependent part of the energy splitting for a two-fermion system placed in a weak static magnetic field **B** can be written as [1,9–11]

$$\Delta E^{ext}_{F,m_J,j_1,\ell,s_1,I} = (\mu_{B1} g_1 + \mu_{B2} g_2) B m_F, \tag{1}$$

where F, m_J, j_1, ℓ, s_1, I are quantum numbers, which characterize the system: s_1 and I are the spins of the first and second particle, respectively, ℓ and j_1 represent the orbital and total angular momentum quantum numbers of the first particle. The total angular momentum of the system is denoted by the quantum number $F=j_1+I, j_1+I-1, \ldots, |j_1-I|$. The projection of the total angular momentum on the **B** direction is $m_F=-F,-F+1,\ldots F-1,F$. The "Bohr magnetons" for the two particles are defined as $\mu_{B1}=Q_1\hbar/2m_1c$, and $\mu_{B2}=-Q_2\hbar/2m_2c$, where $Q_1,Q_2>0$). Usually, in our notation m_1 and m_2 correspond to the light and heavy particle, respectively. Assuming that the energy-level splitting (1) is much smaller then the HFS splitting, $\Delta E^{ext} \ll \Delta E_{HFS}$, the Landé ($g$-) factors g_1 and g_2 take the form [9–11]

$$g_1 = g_{j_1} \frac{F(F+1) + j_1(j_1+1) - I(I+1)}{2F(F+1)}, \tag{2}$$

where

$$g_{j_1} = 1 + (g_{s_1} - 1) \frac{j_1(j_1+1) + s_1(s_1+1) - \ell(\ell+1)}{2j_1(j_1+1)}, \tag{3}$$

and

$$g_2 = g_{s_2} \frac{F(F+1) - j_1(j_1+1) + I(I+1)}{2F(F+1)}. \tag{4}$$

Here g_{s_1} and g_{s_2} are the intrinsic spin magnetic moments of the constituent particles. According to the Dirac theory a free particle at rest has $g_s=2$. In QED g_s is corrected by the anomaly, which to lowest order is given by the Schwinger correction. For bound particles the intrinsic moment can be expressed as

$$g_{s_{1,2}} = 2 + \Delta g^{REL}_{s_{1,2}} + \Delta g^{QED}_{s_{12}}, \tag{5}$$

where the terms Δg^{REL}, Δg^{QED} represent the relativistic [9,12,13], and QED corrections, respectively (cf. the review [14]). There is also an additional higher-order contribution to Eq. (1), $\Delta g^{HFS}_{1,2} \mu_{B1,2} B m_F$, which is caused by the hyperfine structure [15].

The g factors (2) and (4) are not symmetrical, because they were obtained under the assumption that the orbital motion of the heavy particle can be neglected. In hydrogen the nucleus contributes a fraction of $m_1/(m_1+m_2) \approx 5\times10^{-4}$ to the orbital angular momentum, while for muonic hydrogen this fraction is ≈ 0.1. The relativistic and QED corrections in Eq. (5) can be comparable with the orbital angular momentum effects of the heavy particle. Recent high-precision measurements of the g factor in hydrogenlike systems have reached an accuracy of about 5×10^{-9} [16,17]. Thus, it is desirable to obtain a more general result for the g factor in order to overcome the shortcomings of Eqs. (2) and (4). It will be shown that this is particularly important for excited states.

In this work we present an analysis of the HFS of a two-fermion system in an external magnetic field based upon a

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reformulation of QED and the variational Hamiltonian formalism developed earlier [18–20]. A relativistic two-fermion wave equation for arbitrary fermion masses is, thus, derived from first principles. A solution of this equation permits, in principle, to obtain all QED energy corrections to any order of the coupling constant [18]. In the present paper we extend the method to derive the integral wave equation in momentum space for the case where a uniform weak magnetic field is present. We calculate the Zeeman splitting of the HFS energy levels to $O(\alpha^4)$ for all quantum states and unrestricted values for the fermion masses. We obtain a different result for the g factor, Eqs. (38)–(41), and demonstrate that it coincides with Eqs. (2) and (4) in the case of $m_2 \gg m_1$, as long as the intrinsic moment of m_1 is restricted to the Dirac value $g_s=2$.

The modification of the wave equations due to the external magnetic field is presented in Sec. II. In Sec. III we provide the classification of the quantum states, and a partial-wave decomposition of the momentum-space equations. Section IV contains expressions for the Zeeman energy splittings of the HFS levels, and the g-factor results. Numerical values for the Landé factors are compared with data from Eqs. (2) and (4) for various excited states of hydrogen, muonium, and muonic hydrogen. In most expressions we use natural units $\hbar=c=1$.

II. BOUND-STATE VARIATIONAL WAVE EQUATION

For two-fermion systems without external fields wave equations were derived in Refs. [18,19] on the basis of a modified QED Lagrangian [21,22]. With this Lagrangian a simple Fock-space trial state

$$|\psi_{trial}\rangle = \sum_{s_1 s_2} \int d^3\mathbf{p}_1 d^3\mathbf{p}_2 F_{s_1 s_2}(\mathbf{p}_1,\mathbf{p}_2) b^\dagger_{\mathbf{p}_1 s_1} D^\dagger_{\mathbf{p}_2 s_2}|0\rangle, \quad (6)$$

sufficed to obtain HFS levels correct to fourth order in the fine-structure constant. Here $b^\dagger_{\mathbf{q}_1 s_1}$ and $D^\dagger_{\mathbf{q}_2 s_2}$ are creation operators for a free fermion of mass m_1 and an (anti-)fermion of mass m_2, respectively, and $|0\rangle$ is the trial vacuum state such that $b_{\mathbf{q}_1 s_1}|0\rangle = D_{\mathbf{q}_2 s_2}|0\rangle = 0$.

As discussed in Sec. III below, the four adjustable functions $F_{s_1 s_2}$ must be chosen so that the trial state (6) is an eigenstate of the relativistic total angular momentum operator, its projection, and parity (as well as charge conjugation for the case $m_1=m_2$ such as positronium).

A variational principle is invoked to obtain a momentum-space wave equation for the amplitudes [18]

$$\begin{aligned} 0 = & \sum_{s_1 s_2} \int d^3\mathbf{p}_1 d^3\mathbf{p}_2 (\omega_{p_1} + \Omega_{p_2} - E) F_{s_1 s_2}(\mathbf{p}_1,\mathbf{p}_2) \delta F^*_{s_1 s_2}(\mathbf{p}_1,\mathbf{p}_2) \\ & - \frac{m_1 m_2}{(2\pi)^3} \sum_{\sigma_1 \sigma_2 s_1 s_2} \int \frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{q}_1 d^3\mathbf{q}_2}{\sqrt{\omega_{p_1}\omega_{q_1}\Omega_{p_2}\Omega_{q_2}}} F_{\sigma_1\sigma_2}(\mathbf{q}_1,\mathbf{q}_2) \\ & \times (-i)\tilde{\mathcal{M}}_{s_1 s_2 \sigma_1 \sigma_2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}_1\cdot\mathbf{q}_2)\delta F^*_{s_1 s_2}(\mathbf{p}_1,\mathbf{p}_2), \end{aligned} \quad (7)$$

where $\omega^2_{p_1} = \mathbf{p}_1^2 + m_1^2$ and $\Omega^2_{p_1} = \mathbf{p}_1^2 + m_2^2$. The interaction is governed by the generalized invariant $\mathcal{M}$ matrix $\tilde{\mathcal{M}}_{s_1 s_2 \sigma_1 \sigma_2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}_1,\mathbf{q}_2)$. It has the form

$$\begin{aligned} \mathcal{M}^{(1)}_{s_1 s_2 \sigma_1 \sigma_2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}_1,\mathbf{q}_2) \equiv & \mathcal{M}^{ope}_{s_1 s_2 \sigma_1 \sigma_2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}_1,\mathbf{q}_2) \\ & + \mathcal{M}^{ext}_{s_1 s_2 \sigma_1 \sigma_2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}_1,\mathbf{q}_2), \end{aligned} \quad (8)$$

where $\mathcal{M}^{ope}_{s_1 s_2 \sigma_1 \sigma_2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}_1,\mathbf{q}_2)$ is the usual invariant matrix element, corresponding to the one-photon exchange Feynman diagram [19,20].

The element $\mathcal{M}^{ext}_{s_1 s_2 \sigma_1 \sigma_2}$ represents the interaction with a given external classical field A^{ext}_μ,

$$\mathcal{M}^{ext}_{s_1 s_2 \sigma_1 \sigma_2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}_1,\mathbf{q}_2) = i(2\pi)^{3/2} \left(\begin{array}{l} \dfrac{\sqrt{\Omega_{p_2}\Omega_{q_2}}}{m_2} A^{ext}_\mu(\mathbf{p}_1-\mathbf{q}_1)\bar{u}(\mathbf{p}_1,s_1)(-iQ_1)\gamma^\mu u(\mathbf{q}_1,\sigma_1)\delta_{s_2\sigma_2}\delta^3(\mathbf{p}_2-\mathbf{q}_2) \\ + \dfrac{\sqrt{\omega_{p_1}\omega_{q_1}}}{m_1} A^{ext}_\mu(\mathbf{q}_2-\mathbf{p}_2)\bar{V}(\mathbf{p}_2,\sigma_2)(-iQ_2)\gamma^\mu V(\mathbf{q}_2,s_2)\delta_{s_1\sigma_1}\delta^3(\mathbf{p}_1-\mathbf{q}_1) \end{array} \right). \quad (9)$$

The ansatz (6) cannot accommodate processes that include the emission or absorption of real, physical (as opposed to virtual) photons. Such radiative processes could be included by generalizing the trial state. Here we limit ourselves to the form (6), i.e., the effects of radiative decay or absorption of radiation are ignored in the present work.

In order to obtain the Landé factors we evaluate the $\mathcal{M}^{ext}_{s_1 s_2 \sigma_1 \sigma_2}$ matrix (9) in a stationary uniform magnetic field $\mathbf{B}=B\hat{\mathbf{z}}$. The vector potential can be chosen as

$$A^{ext}_1(\mathbf{x}) = -\frac{1}{2}yB, \quad A^{ext}_2(\mathbf{x}) = \frac{1}{2}xB, \quad A^{ext}_0(\mathbf{x}) = A^{ext}_3(\mathbf{x}) = 0. \quad (10)$$

The inverse Fourier transform of the nonzero components yields

$$A_1^{ext}(\mathbf{k}) = \frac{(2\pi)^{3/2} iB}{2} \delta(k_x) \frac{d\delta(k_y)}{dk_y} \delta(k_z), \tag{11}$$

$$A_2^{ext}(\mathbf{k}) = -\frac{(2\pi)^{3/2} iB}{2} \frac{d\delta(k_x)}{dk_x} \delta(k_y)\delta(k_z).$$

Using the semirelativistic expansion

$$\bar{u}(\mathbf{p}_1, s_1)\gamma^j u(\mathbf{q}_1, \sigma_1) = \frac{1}{2m_1 c} \varphi_{s_1}^\dagger (i[\vec{\sigma}_1 \times (\mathbf{p}_1 - \mathbf{q}_1)] + \mathbf{q}_1 + \mathbf{p}_1)_j \varphi_{\sigma_1}, \tag{12}$$

where $j=1,2$, $\varphi_1^\dagger=[1\ 0]$, $\varphi_2^\dagger=[0\ 1]$, and $(\omega_{p_1}\omega_{q_1})^{1/2} \simeq m_1$, and a similar expansion for antiparticle spinors we obtain

$$\mathcal{M}^{ext}_{s_1 s_2 \sigma_1 \sigma_2}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2) = \frac{(2\pi)^{3/2}}{2c} \begin{pmatrix} \frac{Q_1}{m_1} A_j^{ext}(\mathbf{p}_1 - \mathbf{q}_1) \varphi_{s_1}^\dagger (i[\vec{\sigma}_1 \times (\mathbf{p}_1 - \mathbf{q}_1)] + \mathbf{q}_1 + \mathbf{p}_1)_j \varphi_{\sigma_1} \delta_{s_2 \sigma_2} \delta^3(\mathbf{p}_2 - \mathbf{q}_2) \\ + \frac{Q_2}{m_2} A_j^{ext}(\mathbf{q}_2 - \mathbf{p}_2) \chi_{\sigma_2}^\dagger (i[\vec{\sigma}_2 \times (\mathbf{p}_2 - \mathbf{q}_2)] + \mathbf{q}_2 + \mathbf{p}_2)_j \chi_{s_2} \delta_{s_1 \sigma_1} \delta^3(\mathbf{p}_1 - \mathbf{q}_1) \end{pmatrix}, \tag{13}$$

where $\chi_1^\dagger=[0\ 1]$, $\chi_2^\dagger=-[1\ 0]$, and $j=1,2$. It is straightforward to show that

$$(\mathbf{q}_1)_j A_j^{ext}(\mathbf{p}_1 - \mathbf{q}_1) = -\frac{(2\pi)^{3/2} B}{2} \hat{L}_{1z}(\mathbf{q}_1) \delta^3(\mathbf{p}_1 - \mathbf{q}_1), \tag{14}$$

and

$$A_j^{ext}(\mathbf{p}_1 - \mathbf{q}_1) \varphi_{s_1}^\dagger (i[\vec{\sigma}_1 \times (\mathbf{p}_1 - \mathbf{q}_1)] + \mathbf{q}_1 + \mathbf{p}_1)_j \varphi_{\sigma_1} = -(2\pi)^{3/2} B [\varphi_{s_1}^\dagger \sigma_{1z} \varphi_{\sigma_1} + \delta_{s_1 \sigma_1} \hat{L}_{1z}(\mathbf{q}_1)] \delta^3(\mathbf{p}_1 - \mathbf{q}_1), \tag{15}$$

where $\hat{L}_{1z}(\mathbf{q}_1)$ is the z component of the angular momentum operator of the particle with mass m_1,

$$\hat{L}_{1z}(\mathbf{q}_1) = -i\left(q_{1x} \frac{\partial}{\partial q_{1y}} - q_{1y} \frac{\partial}{\partial q_{1x}} \right). \tag{16}$$

Taking φ_{s_1} to be the eigenstates of the spin operator $\hat{S}_{1z}=\frac{1}{2}\hat{\sigma}_{1z}$, and using a similar procedure for the second particle, we obtain

$$\mathcal{M}^{ext}_{s_1 s_2 \sigma_1 \sigma_2}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2) = -\frac{(2\pi)^3 B}{2c} \begin{pmatrix} \frac{Q_1}{m_1} [2\varphi_{s_1}^\dagger \hat{S}_{1z} \varphi_{\sigma_1} + \delta_{s_1 \sigma_1} \hat{L}_{1z}(\mathbf{q}_1)] \delta_{s_2 \sigma_2} \delta^3(\mathbf{p}_1 - \mathbf{q}_1) \delta^3(\mathbf{p}_2 - \mathbf{q}_2) \\ - \frac{Q_2}{m_2} [2\chi_{\sigma_2}^\dagger \hat{S}_{2z} \chi_{s_2} + \delta_{\sigma_2 s_2} \hat{L}_{2z}(\mathbf{q}_2)] \delta_{s_1 \sigma_1} \delta^3(\mathbf{p}_2 - \mathbf{q}_2) \delta^3(\mathbf{p}_1 - \mathbf{q}_1) \end{pmatrix}, \tag{17}$$

or

$$\mathcal{M}^{ext}_{s_1 s_2 \sigma_1 \sigma_2}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2) = -(2\pi)^3 B \begin{pmatrix} \mu_{B1} [2\tilde{m}_{\sigma_1} + \hat{L}_{1z}(\mathbf{q}_1)] \\ - \mu_{B2} [2\tilde{m}_{\sigma_2} + \hat{L}_{2z}(\mathbf{q}_2)] \end{pmatrix} \delta_{s_2 \sigma_2} \delta_{s_1 \sigma_1} \delta^3(\mathbf{p}_1 - \mathbf{q}_1) \delta^3(\mathbf{p}_2 - \mathbf{q}_2), \tag{18}$$

where the spin projection quantum numbers $\tilde{m}_\sigma$ can take the values $\pm 1/2$. The quantities μ_{B1} and μ_{B2} are the "Bohr magnetons" defined in the previous section. As expected, a unit of spin interacts with a magnetic field twice as strongly as a unit of orbital angular momentum.

By going to the next order in the expansion of the invariant $\mathcal{M}$ matrix one can obtain self-energy corrections, which lead to divergent loop integrals that have to be cured by charge renormalization. The vertex term modifies the Dirac value of the magnetic moment by a factor $(1+k)$, where k is the anomaly (Schwinger correction). This factor can be included in our calculation by a replacement $2\tilde{m}_{\sigma_1}$ and $2\tilde{m}_{\sigma_2}$ in Eq. (18) by $g_{s_1}\tilde{m}_{\sigma_1}$ and $g_{s_2}\tilde{m}_{\sigma_2}$ respectively, where $g_{s_{1,2}}/2 = 1+k_{1,2}$. The anomaly is the lowest-order QED correction to the g factor $\Delta g_{s_{12}}^{QED}=2k_{1,2}$ in Eq. (5).

III. PARTIAL-WAVE DECOMPOSITION AND RADIAL WAVE EQUATIONS

The present work is an extension of Ref. [18], in which the partial-wave decomposition of the wave equation has been provided. The external magnetic field is treated as a first-order perturbation, which implies that the quantum labels for the eigenstates do not change. The restrictions on the magnetic field strength to justify a perturbative treatment of Eq. (18) are

$$B \lesssim \min\left[\frac{\alpha^4 m_r c^2}{\mu_{B1}}, \frac{\alpha^4 m_r c^2}{\mu_{B2}} \right], \tag{19}$$

where $\alpha=Q_1Q_2/4\pi$, and $m_r=m_1m_2/(m_1+m_2)$ is the reduced mass. A more explicit restriction on B will be presented in Sec. IV.

As outlined in Ref. [18] the trial state (6) is taken to be an eigenstate of total linear momentum $\hat{\mathbf{P}}$, total angular momentum squared $\hat{\mathbf{J}}^2$, its projection $\hat{J}_3$, and parity $\hat{\mathcal{P}}$. It is natural to work in the rest frame, where the total linear momentum vanishes. In this frame the adjustable functions take the form $F_{s_1s_2}(\mathbf{p}_1,\mathbf{p}_2)=\delta(\mathbf{p}_1+\mathbf{p}_2)F_{s_1s_2}(\mathbf{p}_1)$, where $F_{s_1s_2}(\mathbf{p}_1)$ (using $\mathbf{p}_1\equiv\mathbf{p}$) can be written as

$$F_{s_1s_2}(\mathbf{p})=\sum_{\ell_{s_1s_2}}\sum_{m_{s_1s_2}} f_{s_1s_2}^{\ell_{s_1s_2}m_{s_1s_2}}(p)Y_{\ell_{s_1s_2}}^{m_{s_1s_2}}(\hat{\mathbf{p}}), \tag{20}$$

and $Y_{\ell_{s_1s_2}}^{m_{s_1s_2}}(\hat{\mathbf{p}})$ are the usual spherical harmonics. Here and henceforth we will use the notation $p=|\mathbf{p}|$, etc., while four vectors will be written as p^μ. The orbital indices $\ell_{s_1s_2}$ and $m_{s_1s_2}$ and the radial functions $f_{s_1s_2}^{\ell_{s_1s_2}m_{s_1s_2}}(p)$ depend on the spin variables s_1 and s_2. In the rest frame, the operators $\hat{L}_{1z}(\mathbf{q})$ and $\hat{L}_{2z}(\mathbf{q})$ can be expressed in terms of the orbital angular momentum operator, $\hat{L}_z(\mathbf{q})$, of the relative motion,

$$\hat{L}_{1z}(\mathbf{q})=\frac{m_2}{m_1+m_2}\hat{L}_z(\mathbf{q}),\quad \hat{L}_{2z}(-\mathbf{q})=\frac{m_1}{m_1+m_2}\hat{L}_z(\mathbf{q}). \tag{21}$$

The substitution of the partial-wave expansion (20) into the rest-frame form of ansatz (6) leads to two categories of relations among the adjustable functions $F_{s_1s_2}(\mathbf{p})$.

(i) *The spin-mixed (quasisinglet and quasitriplet) states*. In this case we have $\ell_{s_1s_2}\equiv\ell=J$, and the general solution under the condition of well-defined $\hat{\mathbf{P}}$, $\hat{\mathbf{J}}^2$, $\hat{J}_3$, and $\hat{\mathcal{P}}$ can be expressed with the help of Dirac Γ matrices as [18]

$$F_{s_1s_2}(\mathbf{p})=\bar{u}_{\mathbf{p}s_1}\Gamma_{m_{s_1s_2}}^{J(sg)}(\hat{\mathbf{p}})V_{-\mathbf{p}s_2}f_J^{(sg)}(p) + \bar{u}_{\mathbf{p}s_1}\Gamma_{m_{s_1s_2}}^{J(tr)}(\hat{\mathbf{p}})V_{-\mathbf{p}s_2}f_J^{(tr)}(p). \tag{22}$$

Here $f_J^{(sg)}(p)$ and $f_J^{(tr)}(p)$ are radial functions to be determined. They represent the contributions of spin-singlet and spin-triplet states, i.e., the total spin is not conserved in general.

(ii) *The ℓ-mixed triplet states*. These states occur for $\ell_{s_1s_2}\equiv\ell=J\mp1$. Their radial decomposition can be written as

$$F_{s_1s_2}(\mathbf{p})=\bar{u}_{\mathbf{p}s_1}\Gamma_{m_{s_1s_2}}^{J-1}(\hat{\mathbf{p}})V_{-\mathbf{p}s_2}f_{J-1}(p) + \bar{u}_{\mathbf{p}s_1}\Gamma_{m_{s_1s_2}}^{J+1}(\hat{\mathbf{p}})V_{-\mathbf{p}s_2}f_{J+1}(p). \tag{23}$$

The system in these states is characterized by J, m_J, and $\mathcal{P}=(-1)^J$, and ℓ is not a good quantum number. The two radial functions $f_{J-1}(p)$ and $f_{J+1}(p)$ correspond to the cases $\ell=J-1$ and $\ell=J+1$. Mixing of this type occurs only for principal quantum number $n\geq3$.

From the variational method we obtain a system of coupled radial equations expressed in matrix form as

$$(\omega_p+\Omega_p-E)\mathbb{F}(p)=\frac{m_1m_2}{(2\pi)^3}\int\frac{q^2dq}{\sqrt{\omega_p\omega_q\Omega_p\Omega_q}}\mathbb{K}(p,q)\mathbb{F}(q), \tag{24}$$

where $\omega_p^2=\mathbf{p}^2+m_1^2$ and $\Omega_p^2=\mathbf{p}^2+m_2^2$, and $q=|\mathbf{q}|$ as already mentioned. Here $\mathbb{F}(p)$ and $\mathbb{K}(p,q)$ are matrices composed of radial functions and kernels, respectively. The kernel matrix $\mathbb{K}=\mathbb{K}^{ope}+\mathbb{K}^{ext}$ is made up of one-photon-exchange and external-field parts. Explicit expressions for $\mathbb{K}^{ope}$ can be found in Ref. [18], while the external-field contributions are calculated in this work.

For the spin-mixed states the two-component Fock-space amplitude is given as

$$\mathbb{F}(p)=\begin{bmatrix} f_J^{(sg)}(p) \\ f_J^{(tr)}(p)\end{bmatrix}. \tag{25}$$

The equations imply a mixing of spin and radial variables, and the radial equations are usually coupled. We apply a unitary transformation with rotation angle β to the spin part of function (22) to diagonalize the kernel matrix. The diagonalization can be carried out for arbitrary p and q [cf. Eq. (A7) in the Appendix], and defines a new quasispin basis,

$$|s_1,s_2,\ell,\tilde{s},J,m_J\rangle=C_1|s_1,s_2,\ell,S=0,J,m_J\rangle + C_2|s_1,s_2,\ell,S=1,J,m_J\rangle, \tag{26}$$

where $\ell=J,S$ is the total spin of the system, and $\tilde{s}=0$ for quasisinglet and $\tilde{s}=1$ for quasitriplet states. The coefficients used to express the new basis states in terms of the previously defined singlet and triplet states are found to be $C_1=\sqrt{(1+\xi)/2}$, $C_2=-\sqrt{(1-\xi)/2}$, for the quasi-singlet states, and $C_1=\sqrt{(1-\xi)/2}$, $C_2=\sqrt{(1+\xi)/2}$ for the quasitriplet states. Here the rotation angle β has been replaced for convenience according to $\tan2\beta=\sqrt{1-\xi^2}/\xi$.

The quasisinglet and quasitriplet states are both characterized by the same quantum numbers J, m_J, and $\mathcal{P}=(-1)^{J+1}$, and they mix the states given in the LS coupling representation. The states are labeled for convenience not by the quasispin z projection $t_3=\mp1/2$, but rather by $\tilde{s}=t_3+1/2$, which takes on the values of 0,1. In the Appendix the kernels for spin-mixed states are given explicitly in order to solve for the angle β, i.e., to determine the ξ values.

In the limit $m_2\gg m_1$ the total angular momenta of the first and the second particles are $j_1=\ell_1\pm1/2$, $j_2=s_2=1/2$, where $\ell_1=\ell$. In this case j_1 can be used as a good quantum number, and the role of the indices $\tilde{s}_s,\tilde{s}_t$ are played by $j_1=\ell_1+1/2$ and $j_1=\ell_1-1/2$, respectively. In this case the coefficients C_1 and C_2 reduce to C-G coefficients,

$$C_{1,2}=(-1)^{1/2+1/2+\ell_1+j_1}\sqrt{(2S+1)(2j_1+1)}\begin{Bmatrix} 1/2 & 1/2 & S \\ \ell_1 & \ell_1 & j_1\end{Bmatrix}. \tag{27}$$

Note that the one-body limit corresponds to the $j_1 j_2$ coupling representation, which cannot be used in the general case of arbitrary masses since j_1 and j_2 are not independent (they are related through the common angular momentum ℓ). For positronium the quasistates become true singlet ($C_2=0$) and triplet ($C_1=0$) states with different charge conjugation quantum numbers.

We now proceed to calculate the kernels $\mathcal{K}^{ext}_{mn}(p,q)$ associated with the classical external field A^{ext}_{μ}. Using Eq. (9) for $\mathcal{M}^{ext}_{s_1 s_2 \sigma_1 \sigma_2}$ taken in the rest frame, we obtain

$$\mathcal{K}^{ext}_{mn}(p,q) = -\frac{(\pi/2)^{3/2}}{N(m_1 m_2)^2}\int d^3\hat{\mathbf{p}} d^3\hat{\mathbf{q}} \times \mathrm{Tr}\begin{pmatrix} Q_1\sqrt{\Omega_q \Omega_q} A^{ext}_{\mu}(\mathbf{p}-\mathbf{q})(\gamma^\lambda q_\lambda + m_1)\gamma^\mu(\gamma^\lambda q_\lambda + m_1)\Gamma^n(\hat{\mathbf{q}})(\gamma^\lambda \tilde{q}_\lambda - m_2)\Gamma'^m(\hat{\mathbf{p}}) \\ -Q_2\sqrt{\omega_p \omega_p} A^{ext}_{\mu}(\mathbf{q}-\mathbf{p})(\gamma^\lambda q_\lambda + m_1)\Gamma^n(\hat{\mathbf{q}})(\gamma^\lambda \tilde{q}_\lambda - m_2)\gamma^\mu(\gamma^\lambda \tilde{q}_\lambda - m_2)\Gamma'^m(\hat{\mathbf{p}}) \end{pmatrix}, \tag{28}$$

where N-normalization factor, $q=(\omega_p,\mathbf{q})$, and $\tilde{q}=(\Omega_q,-\mathbf{q})$. The Γ matrices correspond to the various $J^{\mathcal{P}}$ states. The evaluation of these kernels would allow one to obtain all relativistic corrections to the g factor (5); however, this is a formidable task. To determine the lowest-order effect it is sufficient to use the nonrelativistic limit ($q^2/m^2 \ll 1$). In this case the kernels (28) take the form

$$\mathcal{K}^{ext}_{mn}(p,q) = -\frac{(\pi/2)^{3/2}}{N}\int d^3\hat{\mathbf{p}} d^3\hat{\mathbf{q}} \times \mathrm{Tr}\begin{pmatrix} Q_1 A^{ext}_{\mu}(\mathbf{p}-\mathbf{q})(\gamma^0 + I)\gamma^\mu(\gamma^0 + I)\Gamma^n(\hat{\mathbf{q}})(\gamma^0 - I)\Gamma'^m(\hat{\mathbf{p}}) \\ -Q_2 A^{ext}_{\mu}(\mathbf{q}-\mathbf{p})(\gamma^0 + I)\Gamma^n(\hat{\mathbf{q}})(\gamma^0 - I)\gamma^\mu(\gamma^0 - I)\Gamma'^m(\hat{\mathbf{p}}) \end{pmatrix}. \tag{29}$$

These are evaluated for a stationary uniform magnetic field (10). The results are given separately for the following two types of states.

(i) *The spin-mixed states* $[\ell=J, J\geq 1, \mathcal{P}=(-1)^{J+1}]$. In contrast to $\mathbb{K}^{(ope)}(p,q)$ the kernel matrix $\mathbb{K}^{(ext)}(p,q)$ is not diagonal in the basis of the quasisinglet $|sg_q\rangle$ and quasitriplet $|tr_q\rangle$ states, and can be written as

$$\mathcal{K}^{(ext)}_{11}(p,q) = -\frac{(2\pi)^3}{2c}\begin{pmatrix} \frac{Q_1}{m_1}\left[\left(1-\frac{1-\xi}{2J(J+1)}\right)\frac{m_2}{m_1+m_2} + \frac{g_{s_1}}{2}\left(\frac{1-\xi}{2J(J+1)} - 2\frac{|m_1-m_2|}{m_1+m_2}\xi\right)\right] \\ -\frac{Q_2}{m_2}\left[\left(1-\frac{1-\xi}{2J(J+1)}\right)\frac{m_1}{m_1+m_2} + \frac{g_{s_2}}{2}\left(\frac{1-\xi}{2J(J+1)} - 2\frac{|m_1-m_2|}{m_1+m_2}\xi\right)\right] \end{pmatrix} Bm_J, \tag{30}$$

$$\mathcal{K}^{(ext)}_{22}(p,q) = -\frac{(2\pi)^3}{2c}\begin{pmatrix} \frac{Q_1}{2m_1 c}\left[\left(1-\frac{1+\xi}{2J(J+1)}\right)\frac{m_2}{m_1+m_2} + \frac{g_{s_1}}{2}\left(\frac{1+\xi}{2J(J+1)} + 2\frac{|m_1-m_2|}{m_1+m_2}\xi\right)\right] \\ -\frac{Q_2}{2m_2 c}\left[\left(1-\frac{1+\xi}{2J(J+1)}\right)\frac{m_1}{m_1+m_2} + \frac{g_{s_2}}{2}\left(\frac{1+\xi}{2J(J+1)} + 2\frac{|m_1-m_2|}{m_1+m_2}\xi\right)\right] \end{pmatrix} Bm_J, \tag{31}$$

$$\mathcal{K}^{(ext)}_{12}(p,q) = \mathcal{K}^{(ext)}_{21}(p,q) = -\frac{(2\pi)^3}{2c}\begin{pmatrix} \frac{Q_1}{m_1}\left[\frac{\xi}{\sqrt{J(J+1)}}\frac{g_{s_1}}{2} + 2\left(\frac{m_1-m_2}{m_1+m_2}\right)^2\xi^2\left(1-\frac{g_{s_1}}{2}\right)\right] \\ -\frac{Q_2}{m_2}\left[\frac{\xi}{\sqrt{J(J+1)}}\frac{g_{s_2}}{2} + 2\left(\frac{m_1-m_2}{m_1+m_2}\right)^2\xi^2\left(1-\frac{g_{s_2}}{2}\right)\right] \end{pmatrix} Bm_J. \tag{32}$$

Thus, it couples the system (24).

(ii) *The pure triplet and ℓ-mixed states* $[\ell=J\mp 1, J\geq 1, \mathcal{P}=(-1)^J]$. The system (24) cannot be decoupled for these states, and the matrix $\mathbb{K}^{(ope)}(p,q)$ is not diagonal [19]. The magnetic part of the kernel is, however, diagonal,

$$\mathbb{K}^{(ext)}(p,q) = -\frac{(2\pi)^3}{2c}\left(\frac{Q_1}{m_1} - \frac{Q_2}{m_2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} Bm_J. \tag{33}$$

All kernels $\mathbb{K}^{(ext)}$ vanish in the case of equal masses and opposite charges ($Q_1=Q_2$), as occurs in the positronium case, where magnetic effects appear only in $O(B^2)$ [23].

IV. HFS TO $O(\alpha^4)$ ORDER IN A MAGNETIC FIELD

To obtain results for energy levels to $O(\alpha^4)$ we solve the radial equations (24) perturbatively using hydrogenlike radial functions [nonrelativistic Schrödinger form $f^{Sch}_{n,J,m_J}(p)$] in momentum space [9]. The energy eigenvalues can be calculated from the matrix equation, which follows from Eq. (24)

$$E\int p^2dp\mathbb{F}^\dagger(p)\mathbb{F}(p) = \int p^2dp(\omega_p+\Omega_p)\mathbb{F}^\dagger(p)\mathbb{F}(p) - \frac{m_1m_2}{(2\pi)^3}\int_0^\infty \frac{p^2dp}{\sqrt{\omega_p\Omega_p}}\int_0^\infty \frac{q^2dq}{\sqrt{\omega_q\Omega_q}}\mathbb{F}^\dagger(p)\mathbb{K}(p,q)\mathbb{F}(q). \tag{34}$$

If the system (24) has been decoupled, or the contribution of nondiagonal elements of the $\mathbb{K}(p,q)$ matrix with given radial functions in (34) is zero, Eq. (34) immediately gives the perturbative solution for the energy levels. As shown in Ref. [19], the contribution of the nondiagonal elements $\mathcal{K}^{ope}_{12}$ and $\mathcal{K}^{ope}_{21}$ in Eq. (34) to order $O(\alpha^4)$ is zero for the ℓ-mixing states. Thus, in the present scheme the energy corrections for the ℓ-mixing states can be calculated independently for $\ell=J-1$ and $\ell=J+1$ states. As a result, all triplet states with $\ell=J\mp1$ can be treated as pure states. In the case of spin-mixed states the kernel matrix $\mathbb{K}^{ope}$ has been diagonalized in the basis of quasistates (26), however, the magnetic part of the interaction gives rise to the nondiagonal terms (32). Since we are solving the system (34) perturbatively, we can use a new basis $|ext\rangle=C_1'|sg_q\rangle+C_2'|tr_q\rangle$, which mixes the quasistates with arbitrary constants C_1' and C_2'. This leads to a two-level problem with the solution $E_{n,J,m_J}=(H_{11}+H_{22})/2\pm([(H_{11}-H_{22})/2]^2+H_{12}H_{21})^{1/2}$, where $H_{11}=H_{11}^{ope}+H_{11}^{ext}$, $H_{22}=H_{22}^{ope}+H_{22}^{ext}$, and $H_{12}=H_{21}=H_{12}^{ext}=H_{21}^{ext}$. In our case $|H_{11}-H_{22}|\gg H_{12}H_{21}$, because the difference $|H_{11}-H_{22}|$ is of the order of the fine structure, which dominates over the hyperfine splitting and the magnetic perturbation H_{12}. Therefore, we can approximate $E_{n,J,m_J}\approx H_{11},H_{22}$.

The results are presented in the form

$$\Delta E_{n,J,m_J}=E_{n,J,m_J}-(m_1+m_2)+\frac{(Z\alpha)^2m_r}{2n^2} = \Delta E_{n,J}(\alpha^4)+\Delta E^{ext}_{J,m_J}, \tag{35}$$

where $Q_2=ZQ_1$. The energy corrections $\Delta E_{n,J}(\alpha^4)$ due to the kernels $\mathbb{K}^{(ope)}(p,q)$ were obtained previously [19]. The corrections $\Delta E_{n,J}(\alpha^4)$ contain spin-spin interactions that lead to the HFS, which is illustrated in Fig. 1 for the low-lying excited states. A detailed analysis of the HFS to $O(\alpha^4)$ is provided in Ref. [19]. We note that the HFS of the $1S_{1/2}$ and $2S_{1/2}$ states is obtained in agreement with the known Fermi splittings [9], i.e., $\Delta E_{HFS}(1S_{1/2})=(Z\alpha)^4m_r(8m_r/3M)$, and $\Delta E_{HFS}(2S_{1/2})=(Z\alpha)^4m_r(m_r/3M)$, where $M=m_1+m_2$. The HFS of states with $\ell>0$, however, is more complicated [19]. In standard spectroscopic notation it has the form

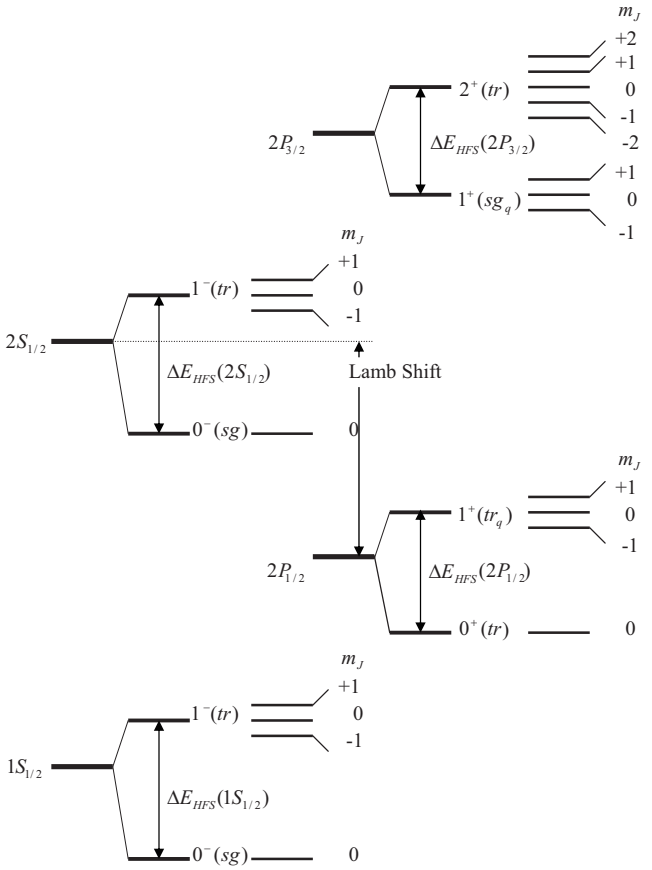

FIG. 1. Zeeman splitting of HFS for two-fermion bound state systems.

$$\Delta E_{HFS}(n,\ell,s_s) \equiv \Delta E_{n,J=\ell+1}-\Delta E_{n,J=\ell,s_s} = \frac{(Z\alpha)^4m_r}{n^3}\frac{1}{2\ell+1}\left(\frac{2\ell+1-\xi^{-1}}{4\ell(\ell+1)}+\frac{2m_r}{M}\frac{1}{2\ell+3}\right), \tag{36}$$

$$\Delta E_{HFS}(n,\ell,s_t) \equiv \Delta E_{n,J=\ell,s_t}-\Delta E_{n,J=\ell-1} = \frac{(Z\alpha)^4m_r}{n^3}\frac{1}{2\ell+1}\left(\frac{2\ell+1-\xi^{-1}}{4\ell(\ell+1)}+\frac{2m_r}{M}\frac{1}{2\ell-1}\right), \tag{37}$$

where the quantity ξ is defined by Eq. (A8), but with the quantum number J replaced by ℓ. The formulas (36) and (37) are valid for all quantum numbers n, ℓ and for any mass values m_1,m_2. The weak external field further splits the energy levels. Equations (36) and (37) give excellent agreement with experiment for the HFS [19].

The energy corrections $\Delta E^{ext}_{J,m_J}$ remove the degeneracy with respect to the m_J quantum number. The solution of Eq. (34) in the above-made approximation can be written in the form of Eq. (1) for all states.

For all *pure states* ($\ell = J \mp 1$) we obtain the following results:
for $\ell = J-1$:

$$g_{1,2} = 1 - \frac{m_{1,2}}{m_1+m_2}\frac{J-1}{J} + \left(\frac{g_{s_{1,2}}}{2}-1\right)\frac{1}{J}, \tag{38}$$

for $\ell = J+1$:

$$g_{1,2} = 1 - \frac{m_{1,2}}{m_1+m_2}\frac{J+2}{J+1} - \left(\frac{g_{s_{1,2}}}{2}-1\right)\frac{1}{J+1}. \tag{39}$$

For *spin–mixed* states $\ell = J \neq 0$ the solution of Eq. (34), as mentioned, reduces to a standard two-energy level problem. The diagonal elements of the kernel matrix give the first-order Zeeman splitting [in $O(B)$] in the quasispin representation (26), which was used to derive the HFS energies (36) and (37). Note that the nondiagonal elements give a contribution to higher-order Zeeman splitting corrections.

To first order in the magnetic field strength we obtain the Landé factors to be

$$g_1 = \frac{m_2}{m_1+m_2}\left(1 - \frac{1 \pm \xi}{2J(J+1)}\right) + \frac{g_{s_1}}{2}\left(\frac{1 \pm \xi}{2J(J+1)} \pm 2\frac{|m_1-m_2|}{m_1+m_2}\xi\right), \tag{40}$$

$$g_2 = \frac{m_1}{m_1+m_2}\left(1 - \frac{1 \mp \xi}{2J(J+1)}\right) + \frac{g_{s_2}}{2}\left(\frac{1 \mp \xi}{2J(J+1)} \mp 2\frac{|m_1-m_2|}{m_1+m_2}\xi\right), \tag{41}$$

where the upper sign is taken for sg_q and the lower sign for tr_q states, respectively. Our expressions (40) and (41) are symmetrical with respect to the masses of the two particles. Obviously all these first-order Zeeman corrections $\Delta E^{ext}_{J,m_J}$ vanish for the positronium case ($m_1 = m_2 = m_e$, $Z=1$), as expected. The intrinsic factors $g_{s_{1,2}}$ associated with the spins of the individual particles can include QED corrections.

In the case when $m_2 \gg m_1$ our general results agree with the result from Eqs. (2) and (4) in which the orbital motion of the heavy particle is ignored. It is only in this limit [as discussed below Eq. (27)], that the total angular momenta of the individual particles are not related through the common angular momentum ℓ, and can be written as $j_1 = \ell \pm 1/2$, and $j_2 = 1/2$. In j_1-j_2 coupling, the eigenstates are taken to be the eigenstates of the operators $\hat{\mathbf{j}}_1^2 = (\hat{\mathbf{L}} + \hat{\mathbf{s}}_1)^2$, $\hat{\mathbf{j}}_2^2 = \hat{\mathbf{s}}_2^2$, $\hat{\mathbf{J}}^2$, and $\hat{J}_z$, and are designated as $|j_1 j_2 J m_J\rangle$ in contrast to the spin-mixed $|L\tilde{s}Jm_J\rangle$ and pure states $|LSJm_J\rangle$, which diagonalize the expectation value of the Hamiltonian to order $O(\alpha^4)$. To facilitate the comparison we make the following replacement of quantum numbers: $F \to J$, $J \to j_1$, $L \to \ell_1 = \ell$, $S \to s_1$, $I \to s_2$. It follows that for all pure states $\ell = J \mp 1$, formulas (38), (39), and (2)–(4) give the same result, namely,

$$g_1 = 1 + \left(\frac{g_{s_1}}{2}-1\right)\frac{1}{J}, \quad g_2 = \frac{g_{s_2}}{2}\frac{1}{J}, \tag{42}$$

or
for $\ell = j_1 - 1/2 = J-1$ and

$$g_1 = 1 - \left(\frac{g_{s_1}}{2}-1\right)\frac{1}{J+1}, \quad g_2 = -\frac{g_{s_2}}{2}\frac{1}{J+1}, \tag{43}$$

or
for $\ell = j_1 + 1/2 = J+1$.

In the limit $m_2 \gg m_1$ the energy levels of spin-mixed states $\Delta E^{ext(sg_q)}_{J,m_J}$ and $\Delta E^{ext(tr_q)}_{J,m_J}$ reduce to $\Delta E^{ext}_{j_1=\ell+1/2,J,m_J}$ and $\Delta E^{ext}_{j_1=\ell-1/2,J,m_J}$, respectively, and the Landé factors given by Eqs. (40) and (41) take the form

$$g_1 = \frac{2J+3}{2J+1} + \left(\frac{g_{s_1}}{2}-1\right)\frac{1}{J}, \quad g_2 = -\frac{1}{J+1} - \left(\frac{g_{s_2}}{2}-1\right)\frac{1}{J+1}, \tag{44}$$

or
for $\ell = j_1 + 1/2 = J(tr_q)$, and

$$g_1 = \frac{2J-1}{2J+1} - \left(\frac{g_{s_1}}{2}-1\right)\frac{1}{J+1}, \quad g_2 = \frac{1}{J} + \left(\frac{g_{s_2}}{2}-1\right)\frac{1}{J} \tag{45}$$

or
for $\ell = j_1 - 1/2 = J(sg_q)$. Here the decoupling angle β is given by $\xi \approx 1/(2\ell+1)$ in the $m_2 \gg m_1$ case.

Formula (4) gives a similar result for the second particle, but for the lighter particle Eq. (2) yields

$$g_1 = \frac{2J+3}{2J+1} + \left(\frac{g_{s_1}}{2}-1\right)\frac{2J+3}{(2J+1)(J+1)} \tag{46}$$

for (sg_q) states, and

$$g_1 = \frac{2J-1}{2J+1} - \left(\frac{g_{s_1}}{2}-1\right)\frac{2J-1}{J(2J+1)} \tag{47}$$

for (tr_q) states. This result agrees with Eqs. (44) and (45) only in the particular case of $g_{s_1} = 2$. Note that most theoretical and experimental results are concerned with $nS_{1/2}$ ($J=1$) states for which the "mass ratio" correction in Eq. (38) disappears. Thus our results will be most useful for $\ell > 0$ states.

In Tables I–III we present results of our calculations of the g factors for the first excited states in hydrogen, muonium, and muonic hydrogen respectively. Only states with nonzero total angular momentum are included. Equation (40) and (41) are used for the spin-mixed states $P_{1/2(J=1)}$, $P_{3/2(J=1)}$, $D_{3/2(J=2)}$, $D_{5/2(J=2)}$. Equation (38) is used for the pure state $P_{3/2(J=2)}$.

Our calculations (given to five digits after the decimal point) are to be compared with the ($m_2 \to \infty$) results (2) and (4). Upper values for each g factor have taken into account the following anomalous magnetic moment values: $g_e/2 = 1.00118$, $g_p/2 = 1.792847$, and $g_\mu/2 = 1.001166$, [1,9,14]. The intrinsic proton anomaly reflects the fact that it is not a

TABLE I. g factors for the electron (g_1) and proton (g_2), respectively in excited atomic hydrogen states. Results from the present calculation, Eqs. (38) and (40) for electrons, are compared with Eq. (2) in the top half of the table. For protons the bottom half displays the present results from Eqs. (38) and (41) in comparison with Eq. (4). Each row contains in the upper part the Landé factor where the intrinsic g_s value is corrected for the anomaly (see text), while the numbers below are based upon the Dirac value $g_s=2$.

pe^-	$P_{1/2(J=1)}$	$P_{3/2(J=1)}$	$P_{3/2(J=2)}$	$D_{3/2(J=2)}$	$D_{5/2(J=2)}$
g_1 using Eqs. (38) and (40)	0.33237	1.66740	1.00032	0.59912	1.40008
	0.33296	1.66622	0.99973	0.59951	1.39949
g_1 using Eq. (2)	0.33294	1.66765	1.00059	0.59965	1.39945
	1/3	5/3	1	3/5	7/5
g_2 using Eqs. (38) and (41)	1.79321	−0.89597	0.89670	0.89691	−0.59711
	1.00036	−0.49955	0.50027	0.50049	−0.33283
g_2 using Eq. (4)	1.79285	−0.89642	0.89642	0.89642	−0.59762
	1	−1/2	1/2	1/2	−1/3

fundamental particle, while in the case of electrons and muons the lowest-order radiative correction was included. The lower values in each row were calculated with $g_{s_{1,2}}=2$. We used the following values for the mass ratios: $m_p/m_e \approx 1836.15267$ and $m_\mu/m_e \approx 206.76828$ [1,9,14].

For the case of muonium we find that the deviations between the present results and those obtained from the one-body limit are in the few-percent range. The muon as the heavier of the two particles acquires a systematically increased Landé factor, while the values are always lowered for the electron.

For muonic hydrogen the effects are more pronounced, and range from 3% to 25% for the states shown in Table III. Only those results that take the anomalous magnetic moment of the proton into account should be considered as physically relevant. The systematics are similar to those shown in Table II for muonium, with the largest decrease in the Landé factor observed for the muon in the $P_{1/2(J=1)}$ state (−25%), while the largest increase (19%) for the proton g value occurs in the $D_{5/2(J=2)}$ state.

For atomic hydrogen the effect is smallest due to the small e/p mass ratio. Given that atomic spectroscopy is far more advanced in hydrogen than in muonic atoms one should not neglect these corrections. For the two above-mentioned states that are most affected we observe about 0.1% deviations in the electron and proton Landé factors, respectively.

As mentioned above, our results are applicable only in low magnetic fields, such that the hyperfine energy splitting exceeds the Zeeman splitting, namely,

$$B \ll \frac{\Delta E_{HFS}(n,\ell)}{\mu_B^* g}. \tag{48}$$

Thus, formula (48), for $2P_{3/2}$ states, requires that $B \ll 300$ G for muonium and $B \ll 100$ G for hydrogen.

V. CONCLUSION

We have used the Hamiltonian variational method in reformulated QED to derive relativistic stationary-state equations for two-fermion systems in an external magnetic field. These equations can include interactions to any order of the coupling constant, at least in principle. The classification of the states follows naturally from the conserved quantum numbers that appear in the trial state (6). For given total angular momentum J there are, in general, coupled equations, both for mixed-spin states, and for triplet mixed-ℓ states [cf. Eq. (24)]. We present explicit forms for the kernels (momentum-space potentials) for the case of a constant, weak external magnetic field.

We solved the radial equations perturbatively to obtain the Zeeman splitting of the HFS to order $O(\alpha^4)$, and calculated the g factors for the system of two bound fermions. Our

TABLE II. Same as in Table I, but for muonium. The Landé factor for the electron is g_1, and for the muon it is g_2.

μ^+e^-	$P_{1/2(J=1)}$	$P_{3/2(J=1)}$	$P_{3/2(J=2)}$	$D_{3/2(J=2)}$	$D_{5/2(J=2)}$
g_1 using Eqs. (38) and (40)	0.329451	1.66392	0.99818	0.59527	1.39610
	0.33004	1.66274	0.99759	0.59566	1.39551
g_1 using Eq. (2)	0.33294	1.66765	1.0006	0.59965	1.39945
	1/3	5/3	1	3/5	7/5
g_2 using Eqs. (38)–(41)	1.00434	−0.49657	0.50299	0.50491	−0.32923
	1.00320	−0.49598	0.50241	0.50433	−0.32884
g_2 using Eq. (4)	1.00117	−0.50058	0.50058	0.50058	−0.33372
	1	−1/2	1/2	1/2	−1/3

TABLE III. Same as in Table I, but for muonic hydrogen. The Landé factor for the muon is g_1, and for the proton it is g_2.

$p^+\mu^-$	$P_{1/2(J=1)}$	$P_{3/2(J=1)}$	$P_{3/2(J=2)}$	$D_{3/2(J=2)}$	$D_{5/2(J=2)}$
g_1 using Eqs. (38) and (40)	0.26707	1.58232	0.95019	0.50961	1.30580
	0.26765	1.58116	0.94960	0.51000	1.30521
g_1 using Eq. (2)	0.33295	1.66764	1.00058	0.59965	1.39946
	1/3	5/3	1	3/5	7/5
g_2 using Eqs. (38) and (41)	1.85425	−0.80664	0.94682	0.98584	−0.50225
	1.06317	−0.41198	0.55040	0.58982	−0.23836
g_2 using Eq. (4)	1.79285	−0.89642	0.89642	0.89642	−0.59762
	1	−1/2	1/2	1/2	−1/3

results are applicable to all states (i.e., for all quantum numbers) and any fermion masses. In the limit $m_2 \gg m_1$ our formulas reproduce the well-known g-factor result. For the spin-mixed states, however, Eq. (2) is found to be not exact if the intrinsic magnetic moment is different from the Dirac value $g_{s_1}=2$.

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APPENDIX: ONE-PHOTON EXCHANGE KERNELS FOR THE SPIN-MIXED STATES TO ORDER α^4

We use the notation $z=(p^2+q^2)/2pq$, and $\mathrm{Q}_J(z)$ is the Legendre function of the second kind [24]. The contributions of the various terms to the kernel are as follows [$\ell=J$ ($J \geq 1$), $\mathcal{P}=(-1)^{J+1}$]:

(i) orbital term:

$$\mathcal{K}_{11}^{(orb)}(p,q)=\mathcal{K}_{22}^{(orb)}(p,q)=\frac{2\pi Q_1 Q_2}{pq}\mathrm{Q}_J(z)+\frac{\pi Q_1 Q_2}{2m_1 m_2}\left[\left(\frac{m_1}{m_2}+\frac{m_2}{m_1}-(J-1)\right)\times\left(\frac{p}{q}+\frac{q}{p}\right)Q_J(z)+2(J+1)Q_{J+1}(z)\right], \tag{A1}$$

(ii) spin-orbit interaction:

$$\mathcal{K}_{11}^{(s\text{-}o)}(p,q)=0, \tag{A2}$$

$$\mathcal{K}_{12}^{(s\text{-}o)}(p,q)=-\frac{\pi Q_1 Q_2}{2m_1 m_2}\left|\frac{m_1}{m_2}-\frac{m_2}{m_1}\right|\frac{2\sqrt{J(J+1)}}{2J+1}[Q_{J+1}(z)-Q_{J-1}(z)], \tag{A3}$$

$$\mathcal{K}_{22}^{(s\text{-}o)}(p,q)=-\frac{\pi Q_1 Q_2}{2m_1 m_2}\left(\frac{m_1}{m_2}+\frac{m_2}{m_1}+4\right)\frac{1}{2J+1}[Q_{J+1}(z)-Q_{J-1}(z)], \tag{A4}$$

(iii) spin-spin interaction:

$$\mathcal{K}_{11}^{(s\text{-}s)}(p,q)=0, \mathcal{K}_{22}^{(s\text{-}s)}(p,q)=\frac{\pi Q_1 Q_2}{m_1 m_2}\frac{1}{2J+1}[Q_{J+1}(z)-Q_{J-1}(z)]. \tag{A5}$$

The diagonalization condition

$$\tan 2\beta[\mathcal{K}_{22}(p,q)-\mathcal{K}_{11}(p,q)]=2\mathcal{K}_{12}(p,q). \tag{A6}$$

determines the parameters β and ξ:

$$\tan 2\beta=2\left|\frac{m_1-m_2}{m_1+m_2}\right|\sqrt{J(J+J)}, \tag{A7}$$

and

$$\xi=\left[4\left(\frac{m_1-m_2}{m_1+m_2}\right)^2 J(J+1)+1\right]^{-1/2}. \tag{A8}$$

Therefore, we obtain the diagonalized kernels for the quasistates

$$\mathcal{K}^{(sg_q)},\mathcal{K}^{(tr_q)}=\mathcal{K}_{11}^{(orb)}+\frac{\xi\pm 1}{\sqrt{1-\xi^2}}\mathcal{K}_{12}^{(s\text{-}o)}=\frac{2\pi Q_1 Q_2}{pq}Q_J(z)+\frac{\pi Q_1 Q_2}{2m_1 m_2}\left[\left(\frac{m_1}{m_2}+\frac{m_2}{m_1}-(J-1)\right)\times\left(\frac{p}{q}+\frac{q}{p}\right)Q_J(z)+2(J+1)Q_{J+1}(z)\right]-\frac{\pi Q_1 Q_2}{2m_1 m_2}\frac{\xi\pm 1}{\xi(2J+1)}[Q_{J+1}(z)-Q_{J-1}(z)]. \tag{A9}$$

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