Intermediate-statistics quantum bracket, coherent state, oscillator, and representation of angular momentum [su(2)] algebra

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(Received 4 December 2006; published 23 April 2007)

In this paper, we first discuss the general properties of an intermediate-statistics quantum bracket, $[u,v]_n = uv - e^{i2\pi/(n+1)}vu$, which corresponds to intermediate statistics in which the maximum occupation number of one quantum state is an arbitrary integer, n. A further study of the operator realization of intermediate statistics is given. We construct the intermediate-statistics coherent state. An intermediate-statistics oscillator is constructed, which returns to bosonic and fermionic oscillators respectively when $n\to\infty$ and $n=1$. The energy spectrum of such an intermediate-statistics oscillator is calculated. Finally, we discuss the intermediate-statistics representation of angular momentum [su(2)] algebra. Moreover, a further study of the operator realization of intermediate statistics is given in the Appendix.

DOI: 10.1103/PhysRevA.75.042111 PACS number(s): 11.10.−z, 03.70.+k

I. INTRODUCTION

Bosons and fermions, the only two kinds of particles that nature realizes, obey Bose-Einstein and Fermi-Dirac statistics, respectively. For bosons, the wave function is symmetric and the maximum occupation number is ∞; for fermions, the wave function is antisymmetric and the maximum occupation number is 1. There are two ways to generalize Bose-Einstein and Fermi-Dirac statistics: (1) One generalization is achieved by generalizing the symmetry property of the wave function. The wave function will change a phase factor when two identical particles exchange. The phase factor can be +1 (symmetric) or −1 (antisymmetric) related to bosons or fermions. Generalizing this result to an arbitrary phase factor $e^{i\theta}$, one obtains the concept of anyon [1]. The corresponding statistics is fractional statistics. Such a generalization has been applied to the fractional quantum Hall effect [2], high temperature superconductivity [3], supersymmetry [4], and quantum computing [5]. (2) Another generalization is based on counting the number of many-body quantum states, i.e., generalizing the Pauli exclusion principle [6–10]. A direct generalization is to allow more than one particle occupying one quantum state. Based on this idea, Gentile constructed a kind of statistics, called intermediate statistics or Gentile statistics, in which the maximum number of particles in any quantum state is neither 1 nor ∞, but equals a finite number n [6], and Bose-Einstein or Fermi-Dirac statistics becomes its limiting case when the maximum occupation number of one state equals ∞ or 1. Many authors discuss the properties of Gentile's intermediate statistics [11]. Reference [12] provides an operator realization of intermediate statistics, by introducing an intermediate-statistics quantum bracket (a generalized commutator). In this paper we denote this operation in a more operational form:

$$[u,v]_n \equiv uv - e^{i\theta_n}vu, \tag{1}$$

where $\theta_n = 2\pi/(n+1)$ and n is the maximum occupation number of one quantum state. The bracket, $[u,v]_n$, will return to commutativity and anticommutativity, respectively when $n\to\infty$ and $n=1$:

$$[u,v]_{n\to\infty} = [u,v], \quad [u,v]_{n=1} = \{u,v\}.$$

Just as the commutation relation of creation and annihilation operators in the Bose-Einstein case is commutative and in the Fermi-Dirac case is anticommutative, the relation of the creation operator $a^\dagger$ and the annihilation operator b in the intermediate-statistics case obeys an intermediate commutation relation between commutativity and anticommutativity, which, by use of the intermediate-statistics quantum bracket given in Eq. (1), can be expressed as [12]

$$[b,a^\dagger]_n = 1. \tag{2}$$

Note that creation operator $a^\dagger$ is not the Hermitian conjugate of the annihilation operator b unless $n\to\infty$ or $n=1$ [12]. This realization includes both the phase factor $e^{i\theta_n}$ and the maximum occupation number n; this implies that such a realization builds a bridge between the exchange symmetry of identical particles (in which the phase factor is extended to an arbitrary phase factor $e^{i\theta_n}$) and the generalized Pauli principle (in which the maximum occupation number is extended to an arbitrary integer n). When the value of n is given, the phase factor $e^{i\theta_n}$ and then the commutation relation of creation and annihilation operators is determined. In this scheme different kinds of intermediate statistics correspond to different commutation relations of creation and annihilation operators. The commutation relation of creation and annihilation operators of intermediate statistics is intermediate between commutativity (the Bose-Einstein case), and anticommutativity (the Fermi-Dirac case) [12]. In this operator realization of Gentile statistics, the state $|\nu\rangle_n$ satisfies $a^\dagger|n\rangle_n=0$ and $b|0\rangle_n=0$, where ν is the occupation number and subscript n represents that such an operator realization corresponds to Gentile statistics with n as its maxium occupation number [12].

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Although the elementary particles in nature are either bosons or fermions, the theory of intermediate statistics can be applied to describe composite-particle systems. Various composite-particle systems have been studied for many years [13], e.g., the Cooper pair in the theory of superconductivity, the Fermi gas superfluid [14], the exciton [15], etc. Some composite particles, composed of several fermions, may behave like bosons, obeying Bose-Einstein statistics, when they are far from each other. However, when they come closer together, the fermions in different composite bosons will begin to "feel" each other, and the statistics of the composite particles will somewhat deviate from Bose-Einstein statistics [16]. In this case intermediate statistics can be used as an effective tool for studying such systems.

We will first give a general discussion of the intermediate-statistics quantum bracket, $[u,v]_n$, and give a further study of the operator realization of intermediate statistics, including the general properties of the bracket $[u,v]_n$, relations of creation and annihilation operators of intermediate statistics, and a different construction of the number operator of intermediate statistics.

Coherent states in our cases are just the eigenstates of the annihilation operator. Bosonic and fermionic coherent states have been widely discussed [17]. We will construct the eigenstates of the annihilation operator of intermediate statistics, the intermediate-statistics coherent states. The result shows that the construction of the intermediate-statistics coherent state is not unique.

Bosonic and fermionic oscillators play important roles in many physical theories. In this paper we construct a kind of intermediate-statistics oscillator which returns to bosonic oscillators when $n \to \infty$ and returns to fermionic oscillators when $n=1$, and calculate the energy spectrum of such a system.

Reference [12] shows that by use of only a single set of creation and annihilation operators of intermediate statistics, one can establish a kind of representation of the angular momentum [su(2)] algebra. Notice that a bosonic realization of the su(2) algebra needs two independent sets of bosonic operators (the Schwinger representation) [18]; otherwise, the realization with only one set of operators needs the help of some kinds of intermediate statistics (e.g., the Holstein-Primakoff representation) [19]. The representation of su(2) algebra has been discussed by many authors [20]. In this paper, we give a more general discussion of the intermediate-statistics representation of su(2) algebra and present some kinds of representations of su(2) algebra.

In this paper we (1) give a general discussion of the properties of the intermediate-statistics quantum bracket (Sec. II and Appendix A) and the operator relations of intermediate statistics (Appendix B), (2) construct a kind of intermediate-statistics coherent state (Sec. III), (3) construct an intermediate-statistics oscillator and calculate its spectrum (Sec. IV), and (4) present some realizations of angular momentum [su(2)] algebra based on creation and annihilation operators of intermediate statistics (Sec. V). The conclusions are summarized in Sec. VI.

II. PROPERTIES OF THE INTERMEDIATE-STATISTICS QUANTUM BRACKET

In this section, we will present some general results of the intermediate-statistics quantum bracket, $[u,v]_n$. In the following, u, v, w, and o denote operators, and λ denotes a c number.

The intermediate-statistics quantum bracket of an operator and itself is

$$[u,u]_n = (1-e^{i\theta_n})u^2, \tag{3}$$

and the intermediate-statistics quantum bracket of an operator and a c number is

$$[u,\lambda]_n = [\lambda,u]_n = (1-e^{i\theta_n})\lambda u. \tag{4}$$

Some basic properties of the intermediate-statistics quantum bracket are as follows:

$$[u \pm v,w]_n = [u,w]_n \pm [v,w]_n,$$

$$[w,u \pm v]_n = [w,u]_n \pm [w,v]_n,$$

$$[u,\lambda v]_n = [\lambda u,v]_n = \lambda[u,v]_n,$$

$$[u,v]_n = -e^{-i\theta_n}[v,u]_n - 2i \sin \theta_n vu. \tag{5}$$

Notice that in the case of commutation ($n \to \infty$) or anticommutation ($n=1$) the phase factor $e^{i\theta_n}$ is 1 or -1. The relations between the intermediate-statistics quantum bracket and the commutator and anticommutator are

$$[u,v]_n - [v,u]_n = (1+e^{i\theta_n})[u,v],$$

$$[u,v]_n + [v,u]_n = (1-e^{i\theta_n})\{u,v\}. \tag{6}$$

A general result of the intermediate-statistics quantum bracket of the product of an arbitrary number of operators is

$$[u_1 \cdots u_k, v_1 \cdots v_l]_n = \sum_{i=1}^{k} \sum_{j=1}^{l} u_1 \cdots u_{i-1} v_1 \cdots v_{j-1} [u_i,v_j] v_{j+1} \cdots v_l u_{i+1} \cdots u_k + (1-e^{i\theta_n}) v_1 \cdots v_l u_1 \cdots u_k. \tag{7}$$

Moreover, the properties of the twofold intermediate-statistics quantum bracket are

$$\begin{aligned}&[[u,v]_n,w]_n + [[w,u]_n,v]_n + [[v,w]_n,u]_n + [[v,u]_n,w]_n + [[w,v]_n,u]_n + [[u,w]_n,v]_n \\ &= (1-e^{i\theta_n})^2(uvw + wuv + vwu + vuw + wvu + uwv),\end{aligned} \tag{8}$$

$$[[u,v]_n,w]_n+[[w,u]_n,v]_n+[[v,w]_n,u]_n-[[v,u]_n,w]_n-[[w,v]_n,u]_n-[[u,w]_n,v]_n = (1-e^{i2\theta_n})(uvw+wuv+vwu-vuw-wvu-uwv). \quad (9)$$

Equation (9) is in fact a generalized Jacobi identity. Generalizing these relations to the case of k operators, we have

$$\sum_p [\cdots[[u_1,u_2]_n,u_3]_n, \ldots ,u_k]_n = (1-e^{i\theta_n})^{k-1}\sum_p u_1u_2\cdots u_k. \quad (10)$$

In this equation and following, by Σ_p we mean the sum over all permutations of the operators and Σ_r over all cyclic permutations. Furthermore, we also have the following results:

$$\sum_p [u_1\cdots u_i,u_{i+1}\cdots u_k]_n = (1-e^{i\theta_n})\sum_p u_1\cdots u_k, \quad (11)$$

$$\sum_r [u_1\cdots u_i,u_{i+1}\cdots u_k]_n = (1-e^{i\theta_n})\sum_r u_1\cdots u_k. \quad (12)$$

It should be pointed out that, wherever the comma in the intermediate-statistics quantum brackets in Eqs. (11) and (12) appears, the right-hand sides are the same, i.e.,

$$\sum_p [u_1\cdots u_{k-1},u_k]_n = \sum_p [u_1\cdots u_{k-2},u_{k-1}u_k]_n = \cdots = \sum_p [u_1,u_2\cdots u_k]_n, \quad (13)$$

$$\sum_r [u_1\cdots u_{k-1},u_k]_n = \sum_r [u_1\cdots u_{k-2},u_{k-1}u_k]_n = \cdots = \sum_r [u_1,u_2\cdots u_k]_n. \quad (14)$$

More properties of the intermediate-statistics quantum brackets are listed in Appendix A.

III. INTERMEDIATE-STATISTICS COHERENT STATE

In this section we introduce the intermediate-statistics coherent state. We will show that the construction of the intermediate-statistics coherent state is not unique.

The concept of coherent states is applied to a wide class of objects [17]. The coherent state in this case is the eigenstate of the annihilation operator. The coherent states in the Bose case and in the Fermi case are very different due to the difference between the exchange symmetries of bosons and fermions. For constructing the fermionic coherent state, one needs to use the Grassmann number—anticommuting c numbers. The annihilation operator corresponding to intermediate statistics is of course different from the Bose and the Fermi cases; for constructing such a kind of coherent state we need to introduce the generalized Grassmann number.

Let $|\psi\rangle$ be the eigenstate of the annihilation operator b:

$$b|\psi\rangle = \psi|\psi\rangle, \quad (15)$$

where the state $|\psi\rangle$ is the intermediate-statistics coherent state. For constructing the coherent state $|\psi\rangle$, we assume

$$|\psi\rangle = M[|0\rangle_n + \delta(1,n)|1\rangle_n\psi + \delta(2,n)|2\rangle_n\psi^2 + \cdots \delta(n,n)|n\rangle_n\psi^n], \quad (16)$$

where M is the normalization constant, and $\delta(i,n)$ $(i=1,2,\ldots,n)$ are coefficients to be determined. For satisfying Eq. (15), one has to assume that ψ here is neither an ordinary commuting c number nor an anticommuting c number like that in the Fermi case. ψ must be a generalized Grassmann number satisfying

$$\psi^{n+1} = 0. \quad (17)$$

When $n=1$ we have $\psi^2=0$ and ψ returns to the Grassmann number. Like that in the Fermi case [17], $\psi|\nu\rangle_n \neq |\nu\rangle_n\psi$. Assuming that

$$\psi|\nu\rangle_n = \lambda(\nu,n)|\nu\rangle_n\psi, \quad (18)$$

one can check that the coefficients in Eq. (16) can be taken as

$$\delta(\nu,n) = \prod_{j=0}^{\nu-1}\frac{\lambda(j,n)}{\sqrt{\langle j+1\rangle_n}}, \quad (19)$$

where $\langle\nu\rangle_n=[1-e^{i2\pi\nu/(n+1)}]/[1-e^{i2\pi/(n+1)}]=\Sigma_{j=0}^{\nu-1}e^{i2\pi j/(n+1)}$, and the relations $a^\dagger|\nu\rangle_n=\sqrt{\langle\nu+1\rangle_n}|\nu+1\rangle_n$ and $b|\nu\rangle_n =\sqrt{\langle\nu\rangle_n}|\nu-1\rangle_n$ [12] have been used. Equation (19) gives the relations between $\delta(\nu,n)$ and $\lambda(\nu,n)$. The normalization constant, M, is determined by $\langle\psi|\psi\rangle=1$, where the adjoint state vector

$$\langle\psi| = M[\langle 0|_n + \delta(1,n)^*\bar{\psi}\langle 1|_n + \delta(2,n)^*\bar{\psi}^2\langle 2|_n + \cdots \delta(n,n)^*\bar{\psi}^n\langle n|_n]. \quad (20)$$

Then we have

$$M = \left[1+\sum_{m=1}^{n}(\bar{\psi}\psi)^m|\delta(m,n)|^2\right]^{-1/2}. \quad (21)$$

As in the Fermi case, $\bar{\psi}$ is independent of ψ and is not a proper mathematical adjoint [17].

There is not only one way to construct $\lambda(\nu,n)$. $\lambda(\nu,n)$, for example, can be taken as

$$\lambda(\nu,n) = e^{\pm i2\pi\nu/(n+1)}, \quad (22)$$

in this case

$$\delta(\nu,n) = \frac{e^{\pm i\pi\nu(\nu-1)/(n+1)}[1-e^{i2\pi/(n+1)}]^{\nu/2}}{\prod_{j=1}^{\nu}\sqrt{1-e^{i2\pi j/(n+1)}}}; \quad (23)$$

or

$$\lambda(\nu,n) = (-1)^{\nu}, \quad (24)$$

in this case

$$\delta(\nu,n) = \frac{(-1)^{(\nu-1)\nu/2}[1-e^{i2\pi/(n+1)}]^{\nu/2}}{\prod_{j=1}^{\nu}\sqrt{1-e^{i2\pi j/(n+1)}}}. \quad (25)$$

Especially, in the Fermi case, i.e., $n=1$, the coherent state can be constructed as $|\psi\rangle = M(|0\rangle + |1\rangle\psi)$. It can be checked directly that the above constructions will return to the Fermi case when $n=1$.

Moreover, we also have

$$\psi(b^{\dagger})^{\nu} = \frac{\lambda(\nu,n)}{\lambda(0,n)}(b^{\dagger})^{\nu}\psi, \quad \psi(a^{\dagger})^{\nu} = \frac{\lambda(\nu,n)}{\lambda(0,n)}(a^{\dagger})^{\nu}\psi,$$

$$b^{\nu}\psi = \frac{\lambda(\nu,n)}{\lambda(0,n)}\psi b^{\nu}, \quad a^{\nu}\psi = \frac{\lambda(\nu,n)}{\lambda(0,n)}\psi a^{\nu}. \quad (26)$$

IV. INTERMEDIATE-STATISTICS OSCILLATOR

The Hamiltonians for bosonic and fermionic oscillators can be expressed as

$$H_{\text{Bose}} = \frac{1}{2}(a_b^{\dagger}a_b + a_b a_b^{\dagger}) = a_b^{\dagger}a_b + \frac{1}{2}, \quad (27)$$

$$H_{\text{Fermi}} = \frac{1}{2}(a_f^{\dagger}a_f - a_f a_f^{\dagger}) = a_f^{\dagger}a_f - \frac{1}{2}, \quad (28)$$

where $a_b^{\dagger}$, a_b and $a_f^{\dagger}$, a_f are creation and annihilation operators of bosons and fermions, respectively, obeying $[a_b, a_b^{\dagger}]=1$ and $\{a_f, a_f^{\dagger}\}=1$. As a generalization of bosonic and fermionic oscillators, we can construct an intermediate-statistics oscillator using the creation and the annihilation operators of intermediate statistics $a^{\dagger}$ and b. The Hamiltonian and the spectrum of such an intermediate-statistics oscillator should return to the bosonic oscillator when $n\to\infty$ and return to the fermionic oscillator when $n=1$.

The Hamiltonian of the intermediate-statistics oscillator should be a quadratic form of creation and annihilation operators. It can be constructed in the following general form:

$$H = \frac{1}{4}[\alpha(n)a^{\dagger}b + \beta(n)ba^{\dagger} + \text{H.c.}]. \quad (29)$$

In the two limit cases, $n=\infty$ and $n=1$, one has $a=b$, and then the creation and the annihilation operators are Hermitian conjugate of each other [12]. The coefficients $\alpha(n)$ and $\beta(n)$ should satisfy $\text{Re}\alpha(\infty)=1$, $\text{Re}\beta(\infty)=1$, $\text{Re}\alpha(1)=1$, and $\text{Re}\beta(1)=-1$ for recovering the Bose case, Eq. (27), when $n=\infty$, and the Fermi case, Eq. (28), when $n=1$, respectively. The choice of $\alpha(n)$ and $\beta(n)$ is not unique. One of the simplest choices is $\alpha(n)=1$ and $\beta(n)=e^{-i2\pi/(n+1)}$, i.e.,

$$H = \frac{1}{4}[a^{\dagger}b + e^{-i2\pi/(n+1)}ba^{\dagger} + b^{\dagger}a + e^{i2\pi/(n+1)}ab^{\dagger}]. \quad (30)$$

In this case, we have

$$[H,a^{\dagger}] = \cos\frac{2N\pi}{n+1}a^{\dagger}, \quad [H,a] = -\cos\frac{2(N-1)\pi}{n+1}a,$$

$$[H,b^{\dagger}] = \cos\frac{2N\pi}{n+1}b^{\dagger}, \quad [H,b] = -\cos\frac{2(N-1)\pi}{n+1}b. \quad (31)$$

The spectrum should be discussed separately in four cases: $n=4t+1$, $n=4t+2$, $n=4t+3$, and $n=4t+4$, where $t\geq 0$ and $n\geq 1$.

The case of $n=4t+1$. The spectrum is

$$E_k^{(n)} = \cos^2\frac{\pi}{n+1} + \frac{1}{2}\csc\frac{\pi}{n+1}\sin\frac{(4k-n-1)\pi}{2(n+1)},$$

$$k=0,\ldots,\frac{1}{2}(n+1). \quad (32)$$

Except the highest energy level which is nondegenerate, all the energy levels are twofold degenerate. The total number of energy levels is $(n+3)/2$.

The energy of the ground state

$$E_0^{(n)} = \cos^2\frac{\pi}{n+1} - \frac{1}{2}\csc\frac{\pi}{n+1}.$$

The highest energy is

$$E_{(n+1)/2}^{(n)} = \cos^2\frac{\pi}{n+1} + \frac{1}{2}\csc\frac{\pi}{n+1}.$$

The case of $n=4t+2$. The spectrum is

$$E_k^{(n)} = \cos^2\frac{\pi}{n+1} + \frac{1}{2}\csc\frac{\pi}{n+1}\sin\frac{(2k-n)\pi}{2(n+1)}, \quad k=0,\ldots,n. \quad (33)$$

All energy levels in this case are nondegenerate. The total number of energy levels is $n+1$.

The energy of the ground state

$$E_0^{(n)} = \cos^2\frac{\pi}{n+1} - \frac{1}{2}\csc\frac{\pi}{n+1}\sin\frac{n\pi}{2(n+1)}.$$

The highest energy is

$$E_n^{(n)} = \cos^2\frac{\pi}{n+1} + \frac{1}{2}\csc\frac{\pi}{n+1}\sin\frac{n\pi}{2(n+1)}.$$

The case of $n=4t+3$. The spectrum is

$$E_k^{(n)} = \cos^2\frac{\pi}{n+1} + \frac{1}{2}\csc\frac{\pi}{n+1}\sin\frac{(4k-n+1)\pi}{2(n+1)},$$

$$k = 0, 1, \ldots, \frac{1}{2}(n-1). \tag{34}$$

All energy levels are twofold degenerate. The total number of energy levels is $(n+1)/2$.

The energy of the ground state

$$E_0^{(n)} = \cos^2 \frac{\pi}{n+1} - \frac{1}{2} \csc \frac{\pi}{n+1} \sin \frac{(n-1)\pi}{2(n+1)}.$$

The highest energy is

$$E_{(n-1)/2}^{(n)} = \cos^2 \frac{\pi}{n+1} + \frac{1}{2} \csc \frac{\pi}{n+1} \sin \frac{(n-1)\pi}{2(n+1)}.$$

The case of $n=4t+4$ is just the same as the case of $n=4t+2$.

The above result shows that, for an intermediate-statistics oscillator with a finite n, the total number of energy levels is finite and the energy levels are often degenerate.

The intermediate-statistics oscillator will return to bosonic and fermionic oscillators when $n \to \infty$ and $n=1$ as follows.

(a) $n=1$: *The Fermi case*. When $n=1$, the intermediate-statistics oscillator will return to a fermionic oscillator. $n=1$ corresponds to the case of $n=4t+1$. The spectrum, Eq. (32), reduces to

$$E_k^{(1)} = -\frac{1}{2} \cos k\pi, \quad k = 0, 1.$$

The total number of energy levels is $(n+3)/2=2$. The energy of the ground state

$$E_0^{(1)} = -\frac{1}{2},$$

and the highest energy is

$$E_1^{(1)} = \frac{1}{2}.$$

This is just a fermionic oscillator.

(b) $n \to \infty$: *The Bose case*. When $n=\infty$, the intermediate-statistics oscillator will return to a bosonic oscillator. In Ref. [10] we have shown that, in the theory of statistical mechanics, for recovering Bose-Einstein statistics from Gentile statistics, besides the condition $n \to \infty$, one also needs an additional condition $\lim_{n\to\infty, N\to\infty} N/n = 0$, where N is the total number of particles in the system. Concretely, $\lim_{n\to\infty, N\to\infty} N/n$ can also take other nonzero values, and different nonzero values of such a limit correspond to different kinds of intermediate statistics [21]; only $\lim_{n\to\infty, N\to\infty} N/n=0$ corresponds to the Bose case. In our cases, ν is the occupation number of one quantum state. Therefore, in the Bose case $\nu \ll n$. Thus, for recovering the bosonic oscillator, the condition $\nu \ll n$ is needed.

The spectrum of the Hamiltonian Eq. (30) can also be expressed as

$$E^{(n)}(\nu) = \frac{1}{2} \csc \frac{\pi}{n+1} \left[\sin \frac{(2\nu-1)\pi}{n+1} + \sin \frac{2\pi}{n+1} \cos \frac{\pi}{n+1} \right]. \tag{35}$$

When $\nu \ll n$, the spectrum

$$E^{(n)}(\nu)\big|_{\nu \ll n} \approx \frac{\pi\nu}{n+1} \cot \frac{\pi}{n+1} + \frac{1}{2} \cos \frac{2\pi}{n+1}.$$

Taking $n \to \infty$, we have

$$E^{(\infty)}(\nu) = \lim_{n\to\infty} E^{(n)}(\nu)\big|_{\nu \ll n} = \nu + \frac{1}{2}.$$

This is just the spectrum of a bosonic oscillator. That is to say, in the case of $\nu \ll n$, when $n \to \infty$ the intermediate-statistics oscillator will return to a bosonic oscillator. Such a result agrees with the conclusion drawn in Refs. [10,21]: Intermediate statistics cannot recover the Bose case by only taking $n \to \infty$; for recovering the Bose case, one also needs $\nu \ll n$.

V. INTERMEDIATE-STATISTICS REPRESENTATION OF ANGULAR MOMENTUM [su(2)] ALGEBRA

It is shown in Ref. [12] that one cannot obtain a bosonic realization of the angular momentum [su(2)] algebra by a single set of bosonic creation and annihilation operators. Two kinds of representations of angular momentum operators are the Schwinger representation [18] and the Holstein-Primakoff representation [19], which are successful in describing magnetism in various quantum systems [22]. The Schwinger representation needs two sets of independent boson operators, a_1 and a_2: $J_+ = a_1^\dagger a_2$, $J_- = a_2^\dagger a_1$, and $J_z = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$. The Holstein-Primakoff representation, $J_+ = \sqrt{2j-N}a$, $J_- = a^\dagger \sqrt{2j-N}$, and $J_z = j-N$, where $N=0,1,\ldots,2j$, though only using one set of creation and annihilation operators and a and $a^\dagger$ satisfying the bosonic commutation relations, is not a real bosonic realization. This is because in the Holstein-Primakoff case the occupation number N is restricted to be no more than $2j$, but in Bose-Einstein statistics N can take any integer. The result given in Ref. [12] shows that the angular momentum algebra can be represented in terms of a single set of creation and annihilation operators in the case of intermediate statistics.

There is a correspondence between the angular momentum and intermediate statistics, which allows us to construct a representation in terms of intermediate statistics. More concretely, when the maximum occupation number is n, there are $n+1$ states: $|0\rangle_n, |1\rangle_n \ldots, |n\rangle_n$; correspondingly, when the magnitude of the angular momentum is $j=n/2$, there are also $2j+1=n+1$ states: $|-j\rangle$, $|-j+1\rangle, \ldots, |j\rangle$. In Ref. [12], some special cases of the realization of angular momentum [su(2)] algebra corresponding to intermediate statistics, $j=1/2$, 1, 3/2, 2, 5/2, are discussed. In this paper, we give a systematic analysis for the intermediate-statistics realization of angular momentum algebra.

The angular momentum operators, J_+, J_-, and J_z, satisfy

$$[J_+, J_-] = 2J_z, \tag{36}$$

$$[J_z, J_\pm] = \pm J_\pm. \tag{37}$$

As in Ref. [12], we represent J_z by the particle number operator:

$$J_z = N - \frac{n}{2}. \tag{38}$$

Thus we have

$$J_z|\nu\rangle_n = \left(N - \frac{n}{2}\right)|\nu\rangle_n = \left(\nu - \frac{n}{2}\right)|\nu\rangle_n. \tag{39}$$

For a given n, the range of value of ν, the occupation number of one single quantum state in intermediate statistics, is $0 \leqslant \nu \leqslant n$, so the values of J_z are $-n/2, -n/2+1, \ldots, n/2$, a total of $n+1$, corresponding to the $n+1$ components of the angular momentum $j=n/2$.

For satisfying Eqs. (36) and (37), notice that $[N, J_\pm] = \pm J_\pm$, we can choose J_+ and J_- satisfying

$$J_+|\nu\rangle_n = c_+(\nu)|\nu+1\rangle_n,$$

$$J_-|\nu\rangle_n = c_-(\nu)|\nu-1\rangle_n, \tag{40}$$

so that

$$[J_z, J_\pm]|\nu\rangle_n = \left[c_\pm(\nu)\left(\nu \pm 1 - \frac{n}{2}\right) - c_\pm(\nu)\left(\nu - \frac{n}{2}\right)\right]|\nu \pm 1\rangle_n$$

$$= \pm J_\pm|\nu \pm 1\rangle_n, \tag{41}$$

and then Eq. (37) is automatically satisfied. The choice of $J_\pm$ satisfying Eq. (40) is not unique. In the following we will consider some kinds of constructions.

One possible construction is

$$J_+ = \sum_l \lambda_l^* A^l a^\dagger,$$

$$J_- = \sum_q \lambda_q a (A^\dagger)^q, \tag{42}$$

where the operator A satisfies

$$[A, N] = [A^\dagger, N] = 0. \tag{43}$$

Obviously, such a choice satisfies Eq. (37). Equation (43) implies that the operator A can be chosen as

$$A = N, \quad A = a^\dagger b \quad \text{or } b^\dagger a, \quad A = a^\dagger a = b^\dagger b,$$

$$A = aa^\dagger = bb^\dagger, \quad \text{etc.} \tag{44}$$

The coefficients can be determined by Eq. (36). For example, when $A = a^\dagger b$, Eq. (36) gives

$$\sum_{lq} \lambda_l^* \lambda_q [|\langle\nu\rangle_n|(\langle\nu\rangle_n^*)^q \langle\nu\rangle_n^l - |\langle\nu+1\rangle_n|(\langle\nu+1\rangle_n^*)^q \langle\nu+1\rangle_n^l]$$

$$= 2\nu - n. \tag{45}$$

This is a set of $n+1$ equations. In principle, one can obtain a realization of angular momentum [su(2)] algebra by solving the coefficients λ_l from this set of equations. One can prove that such a set of equations always possess solutions. Equation (45) is a set of linear equations of $\lambda_l^* \lambda_q$. However, once one wants to solve λ_l, he will encounter high-order algebraic equations. When the order of the equation is high, it is difficult to solve. Nevertheless, in a concrete problem, in which n is given, one can always obtain the solution, and then obtain the realization of the angular momentum [su (2)] algebra.

Note that the realization given in Eq. (42) is not unique. There are still other choices, for example,

$$J_+ = \sum_l \lambda_l^* (A^l a^\dagger + B^l b^\dagger),$$

$$J_- = \sum_q \lambda_q [a(A^\dagger)^q + b(B^\dagger)^q], \tag{46}$$

where

$$[B, N] = [B^\dagger, N] = 0. \tag{47}$$

The coefficients λ_i can be obtained by the same procedure given above.

VI. CONCLUSIONS

In this paper, we discuss the properties of the intermediate-statistics quantum bracket, $[u,v]_n = uv - e^{i\theta_n} vu$, which is introduced in Ref. [12]. The operation $[u,v]_n$ will return to commutativity and anticommutativity when $n \to \infty$ or $n=1$. The physical meaning of n is clear. It denotes the maximum occupation number of statistics: $n \to \infty$ and $n=1$ correspond to Bose and Fermi cases, and the other values of n correspond to intermediate statistics. That is to say, if the commutator reflects the properties of a bosonic system, and the anticommutator reflects the properties of a fermionic system, the intermediate-statistics quantum bracket, $[u,v]_n$, corresponds to the system obeying intermediate statistics.

An operator realization of intermediate statistics is given in Ref. [12]. In this paper, we give a more detailed discussion of the operator realization. Especially, some operator relations corresponding to intermediate statistics are provided.

The coherent state is an important concept in physics. The bosonic and fermionic coherent states have been discussed in many literatures [17]. In this paper we construct the intermediate-statistics coherent state. The analysis shows that the construction of the intermediate-statistics coherent state is not unique. In this paper, we provide two constructions.

The Fermi oscillator and the Bose oscillator are very important models in quantum mechanics. Based on the operator realization of intermediate statistics, we construct a kind of intermediate-statistics oscillator which returns to the bosonic oscillator when $n \to \infty$ $(\nu \ll n)$, and returns to the fermionic oscillator when $n=1$. Its energy spectrum is calculated.

In this paper, we provide a more general discussion of the intermediate-statistics representation of angular momentum algebra. Our result shows that one can construct more than one representation of angular momentum algebra by a single set of creation and annihilation operators. Note that one cannot obtain a representation of angular momentum algebra by a single set of bosonic operators.

ACKNOWLEDGMENTS

The authors W.-S. D. and M. X. are very indebted to Dr. G. Zeitrauman for his encouragement. This work was supported in part by NSF of China, under Project No. 10605013 and No. 10675088.

APPENDIX A: PROPERTIES OF THE INTERMEDIATE-STATISTICS QUANTUM BRACKET

Equation (7) gives a very general result of the intermediate-statistics quantum bracket. In practice the following special results are often useful:

$$[u_1 \cdots u_k, v]_n = \sum_{i=1}^{k} u_1 \cdots u_{i-1}[u_i, v]u_{i+1} \cdots u_k + (1 - e^{i\theta_n})v u_1 \cdots u_k, \tag{A1}$$

$$[v, u_1 \cdots u_k]_n = \sum_{i=1}^{k} u_1 \cdots u_{i-1}[v, u_i]u_{i+1} \cdots u_k + (1 - e^{i\theta_n})u_1 \cdots u_k v. \tag{A2}$$

When $u_1 = \cdots = u_k = u$ and $v_1 = \cdots = v_l = v$, Eq. (7) becomes

$$[u^k, v^l]_n = \sum_{i=1}^{k} \sum_{j=1}^{l} u^{i-1} v^{j-1}[u, v]v^{l-j}u^{k-i} + (1 - e^{i\theta_n})v^l u^k. \tag{A3}$$

We also have

$$[u^k, v^m]_n = \frac{1}{(1 - e^{i\theta_n})^{k+m-2}} \times [\underbrace{[\cdots[u,u]_n \cdots, u\ ]_n}_{k-1}, \underbrace{[\cdots[v,v]_n \cdots, v]_n}_{m-1}]_n$$

and

$$[u^k, v]_n + [u^{k-1}v, u]_n = [u^{k-1}, u]_n v + [u^{k-1}, v]_n u,$$

$$[v, u^k]_n + [u, vu^{k-1}]_n = v[u, u^{k-1}]_n + u[v, u^{k-1}]_n.$$

Some useful special cases of Eqs. (A1) and (A2) are

$$[uv, w]_n = u[v, w] + [u, w]v + (1 - e^{i\theta_n})wuv,$$

$$[w, uv]_n = [w, u]v + u[w, v] + (1 - e^{i\theta_n})uvw, \tag{A4}$$

$$[uvw, o]_n = [u, o]vw + u[v, o]w + uv[w, o] + (1 - e^{i\theta_n})ouvw,$$

$$[o, uvw]_n = [o, u]vw + u[o, v]w + uv[o, w] + (1 - e^{i\theta_n})uvwo, \tag{A5}$$

and

$$[uv, wo]_n = \frac{1}{1 - e^{i2\theta_n}}([[u, v]_n, [w, o]_n]_n + e^{i\theta_n}[[v, u]_n, [w, o]_n]_n + e^{i\theta_n}[[u, v]_n, [o, w]_n]_n + e^{i2\theta_n}[[v, u]_n, [o, w]_n]_n),$$

$$[uv, wo]_n = u[v, w]_n o + e^{i\theta_n}uw[v, o] + e^{i\theta_n}[u, w]ov + e^{i\theta_n}w[u, o]v. \tag{A6}$$

Equations (7)–(12) are very general results. We can also obtain the corresponding properties for commutators or anticommutators by taking $n \to \infty$ or $n = 1$, which are often useful.

Commutator case:

$$[u_1 \cdots u_k, v_1 \cdots v_l] = \sum_{i=1}^{k} \sum_{j=1}^{l} u_1 \cdots u_{i-1} v_1 \cdots v_{j-1}[u_i, v_j]v_{j+1} \cdots v_l u_{i+1} \cdots u_k,$$

and

$$\sum_{i,j,k=1}^{3} |\varepsilon_{ijk}|[[u_i, u_j], u_k] = 0, \tag{A7}$$

$$\sum_{i,j,k=1}^{3} \varepsilon_{ijk}[[u_i, u_j], u_k] = 0. \tag{A8}$$

Equation (A8) is just the Jacobi identity. Moreover,

$$\sum_{p} ([\cdots[u_1, u_2] \cdots, u_{k-1}], u_k) = 0,$$

and

$$\sum_{p} [u_1 \cdots u_i, u_{i+1} \cdots u_k] = 0,$$

$$\sum_{r} [u_1 \cdots u_i, u_{i+1} \cdots u_k] = 0.$$

Anticommutator case:

$$\{u_1 \cdots u_k, v_1 \cdots v_l\} = \sum_{i=1}^{k} \sum_{j=1}^{l} u_1 \cdots u_{i-1} v_1 \cdots v_{j-1}[u_i, v_j]v_{j+1} \cdots v_l u_{i+1} \cdots u_k + 2v_1 \cdots v_l u_1 \cdots u_k,$$

$$\sum_{i,j,k=1}^{3} |\varepsilon_{ijk}|\{\{u_i, u_j\}, u_k\} = 4 \sum_{l,h,m=1}^{3} |\varepsilon_{lhm}|u_l u_h u_m,$$

$$\sum_{i,j,k=1}^{3} \varepsilon_{ijk}\{\{u_i, u_j\}, u_k\} = 0,$$

$$\sum_{p} \{\{\cdots\{u_1, u_2\} \cdots, u_{k-1}\}, u_k\} = 2^{k-1} \sum_{p} u_1 \cdots u_k,$$

and

$$\sum_{p} \{u_1 \cdots u_i, u_{i+1} \cdots u_k\} = 2 \sum_{p} u_1 \cdots u_k,$$

$$\sum_{r} \{u_1 \cdots u_i, u_{i+1} \cdots u_k\} = 2 \sum_{r} u_1 \cdots u_k.$$

APPENDIX B: OPERATOR RELATIONS OF INTERMEDIATE STATISTICS

Reference [12] presents an operator realization of intermediate statistics. Concretely, Ref. [12] constructs a set of creation, annihilation, and number operators for intermediate statistics, Bose and Fermi cases becoming its two limiting cases. In this appendix we will first construct a realization for the number operator of intermediate statistics. Then, using the properties of intermediate-statistics quantum brackets given in the above section, we will give some operator relations for creation, annihilation, and number operators of intermediate statistics.

For completeness, we rewrite the basic operator relations of intermediate statistics given in Ref. [12]:

$$[b,a^\dagger]_n = 1, \quad [N,a^\dagger] = a^\dagger, \quad [N,b] = -b, \tag{B1}$$

where $a^\dagger$, b, and N are creation, annihilation, and number operators in intermediate statistics, respectively.

Reference [12] gives a construction for the number operator N. However, the construction of N is not unique; it can also be constructed as

$$N = \frac{n+1}{2\pi} \arcsin\left[\frac{i}{2}(a^\dagger b - b^\dagger a + ab^\dagger - ba^\dagger)\right]. \tag{B2}$$

It can be checked directly that this construction satisfies Eq. (B1).

Moreover, we give some operator relations of $a^\dagger$, b, and N:

$$[N,a^\dagger]_n = [(1-e^{i\theta_n})N + e^{i\theta_n}]a^\dagger,$$

$$[a^\dagger,N]_n = [(1-e^{i\theta_n})N - 1]a^\dagger,$$

$$[N,b]_n = [(1-e^{i\theta_n})N - e^{i\theta_n}]b, \quad [b,N]_n = [(1-e^{i\theta_n})N + 1]b,$$

$$[Nb,a^\dagger b] = e^{i2\pi(N-1)/(n+1)}Nb.$$

We also have

$$[(a^\dagger)^k b,a^\dagger]_n = [b(a^\dagger)^k,a^\dagger]_n = (a^\dagger)^k,$$

$$[b,a^\dagger b^k]_n = [b,b^k a^\dagger]_n = b^k,$$

$$[a^\dagger b^2,a^\dagger]_n = [b,(a^\dagger)^2 b]_n = (1+e^{i\theta_n})a^\dagger b.$$

Like the fact that intermediate statistics returns to Bose-Einstein and Fermi-Dirac statistics when $n\to\infty$ and $n=1$, the intermediate-statistics quantum bracket will return to commutator and anticommutator in these two limiting cases. Therefore, when $n\to\infty$ and $n=1$ the above results lead to the operator relations of creation and annihilation operators in Bose-Einstein and Fermi-Dirac cases.

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