# Quantum time-of-flight distribution for cold trapped atoms

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The time-of-flight distribution for a cloud of cold atoms falling freely under gravity is considered. We generalize the probability current density approach to calculate the quantum arrival time distribution for the mixed state describing the Maxwell-Boltzmann distribution of velocities for the falling atoms. We find an empirically testable difference between the time-of-flight distribution calculated using the quantum probability current and that obtained from a purely classical treatment which is usually employed in analyzing time-of-flight measurements. The classical time-of-flight distribution matches with the quantum distribution in the large mass and high temperature limits.

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## I. INTRODUCTION

In recent times laser cooling and trapping of atoms has become an area of active research [1]. The measurement of the initial temperature of the cloud of atoms is crucial for characterizing the properties of atom traps. The temperature of the cloud can be inferred from the velocity distribution of atoms in the cloud. A well-known technique of measuring this velocity distribution is the time-of-flight (TOF) method. Measurements of the TOF distribution have been employed to analyze various experimental data such as those involving ions and isotopes [2], and also in performing mass spectroscopy of biomolecules like DNA [3].

The theoretical treatment of the TOF distribution that can be obtained using, for instance, the Green's function method [4], however, turns out to produce perfect agreement with the TOF distribution obtained by using Newton's equations for ballistic motion of particles accelerated by the Earth's gravitational field [5]. Thus the interpretation of the results of the various TOF experiments [1–3] where classical trajectories are inferred from Newtonian mechanics [6] remains debatable, especially in the domain of small atomic masses and low temperatures where quantum mechanical effects should be significant.

Though there exists no unique prescription for the definition of time of flight and arrival time in quantum mechanics, experimentalists measure arrival times of elementary particles, atoms, and molecules using the TOF methods. In spite of the difficulties to give time an observable status in quantum mechanics, several logically consistent schemes for the treatment of the arrival time distribution have been formulated, such as those based on axiomatic appraches [7], opearator constructions [8], and trajectory models [9]. It is thus desirable that some of the conceptually sound theoretical formulations of the quantum mechanical arrival time distribution [10] be confronted with accurate experimental data. If such quantum mechanical approaches are employed for analyzing experiments using TOF measurements, it should not only enable one to determine the empirical viability of various competing arrival time models [10], but also possibly shed new light on the conventional interpretation of the results of these experiments.

In this paper we employ the probability current approach [11] towards obtaining the quantum time-of-flight distribution of cold trapped atoms. The probability current approach for computation of the mean arrival time of a quantum ensemble not only provides an unambiguous definition of arrival time at the quantum mechanical level [11–13], but also adresses the issue of obtaining the proper classical limit of the time of flight of massive quantum particles [14,15]. Here we derive the quantum arrival time distribution for the case of initially trapped atomic clouds that are subsequently allowed to fall freely under gravity [4,16]. We compute the mean time of flight for these atoms and compare it with the mean time of flight obtained through the classical time-of-flight analysis [5] that has frequently been employed for such experiments [1–3]. Our analysis predicts the mass and temperature range of the atomic clouds where the quantum mechanical treatment alters the arrival time distribution and the mean arrival time from that obtained through the classical analysis.

## II. CLASSICAL ANALYSIS OF TIME-OF-FLIGHT MEASUREMENTS

We begin with a brief description of the classical analysis of TOF measurements of trapped atoms. A probe laser, focused in the form of a sheet, is placed underneath the atomic cloud. When the trapping forces are turned off, the cold atom cloud falls through the laser probe under the influence of gravity. It is then possible to detect the fluorescence from the atoms as they reach the sheet. The fluorescence is measured as a function of time and the initial temperature of the cloud is determined by fitting the experimental result to the theoretically predicted TOF signal of the cloud [4,16]. A detailed derivation of the TOF signal recorded by the detector (that is,

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the number of atoms arriving at the probe laser as a function of time) was derived by Yavin *et al.* [5].

The cloud of atoms consisting of noninteracting particles has a Maxwell-Boltzmann velocity distribution given (in one dimension) by

$$\Pi(v)dv = \left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left(-\frac{mv^2}{2kT}\right)dv, \tag{1}$$

where $T$ is the initial temperature of the cloud, and $m$ is the atomic mass. Using the Newton's equations for ballistic motion of a particle accelerated by the Earth's gravitational field (in the vertical $z$-direction), the velocity is obtained in terms of the time of flight as

$$v = \left(z + \frac{1}{2}gt^2\right)/t. \tag{2}$$

Substituting the above expression for $v$ from Eq. (2) in Eq. (1), one can obtain the time-of-flight distribution at an arbitrary distance $z$, given by

$$\Pi_C(t)dt = \left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left(-\frac{m\left(z+\frac{1}{2}gt^2\right)^2}{2kTt^2}\right) \times \frac{\left(-z+\frac{1}{2}gt^2\right)}{t^2}dt. \tag{3}$$

The corresponding classical mean time of flight or mean arrival time $\overline{\tau_C}$ for the atomic cloud calculated using $\Pi_C(t)$ as the time-of-flight distribution is given by

$$\overline{\tau_C} = \frac{\int_0^\infty \Pi_C(t)t\,dt}{\int_0^\infty \Pi_C(t)dt}. \tag{4}$$

For simplicity, here we restrict ourselves to the case of a point-sized cloud. The three-dimensional calculation was done by Yavin *et al.* [5] using a simple coordinate transformation and the same expression was obtained for the TOF distribution. These authors [5] have claimed perfect agreement of their results with a previous calculation [4] where the TOF distribution was derived using a sophisticated Green's function technique. We call this TOF distribution given by Eq. (3) the classical time-of-flight distribution, and Eq. (4) denotes the corresponding classical mean arrival time since the classical Newtonian equation is used from the outset to derive this TOF distribution.

### III. THE QUANTUM ARRIVAL TIME DISTRIBUTION THROUGH THE PROBABILITY CURRENT

Our aim here is to derive an expression for the time-of-flight distribution for the atomic cloud through the quantum probability current without using any classical ingredients. To that end, let the initial state of each of the atoms be represented by a one-dimensional Gaussian wave function of the form

$$\psi(z,0) = (2\pi\sigma_0{}^2)^{-1/4} \exp\left(\frac{imv}{\hbar}z\right)\exp\left(-\frac{z^2}{4\sigma_0{}^2}\right) \tag{5}$$

centered at $z=0$ and moving with a group velocity $v$. The Schrödinger time evolved wave function under the Hamiltonian $H=p^2/2m+mgz$ is given by

$$\psi(z,t) = (2\pi s_t^2)^{-1/4} \exp\left(\frac{-\left(z-vt+\frac{1}{2}gt^2\right)^2}{4s_t\sigma_0}\right) \times \exp\left[i\left(\frac{m}{\hbar}\right)\left((v-gt)(z-vt/2)-\frac{1}{6}g^2t^3\right)\right], \tag{6}$$

where $s_t=\sigma_0(1+i\hbar t/2m\sigma_0^2)$.

Considering the free fall of the atoms under gravity, the expression for the Schrödinger probability current density

$$J(z,t) \equiv \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial z} - \psi^*\frac{\partial\psi}{\partial z}\right) \tag{7}$$

for the time evolved state is calculated using the initial state given by Eq. (5) to be

$$J(z,t) = P(z,t)\left[(v-gt) + \frac{\hbar^2 t}{4m^2\sigma_0{}^2\sigma^2}\left(z-vt+\frac{1}{2}gt^2\right)\right], \tag{8}$$

where the expression for the position probability distribution is given by

$$P(z,t) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{\left(z-vt+\frac{1}{2}gt^2\right)^2}{2\sigma^2}\right) \tag{9}$$

with $\sigma=\sigma_0(1+\hbar^2t^2/4m^2\sigma_0^4)^{1/2}$. The modulus of the probability current density $J(z,t)$ given by Eq. (8) provides the arrival time distribution for a pure wave packet falling under gravity. Note that the quantum probability current as defined by Eq. (7) is formally ambiguous up to a total divergence term [17]. However, $J(z,t)$ can be uniquely defined through relativistic wave equations which impart appropriate spin-dependent corrections to it that persist even in the nonrelativistic limit [12,18]. The ensuing arrival time distribution defined through the probability current thus contains a spin-dependent correction for particles with spin [13].

The atomic cloud is represented by an ensemble of particles in thermal equilibrium with a thermal distribution of initial velocities. Each particle has a wave function of the form (5), with a Maxwell-Boltzmann distribution of initial

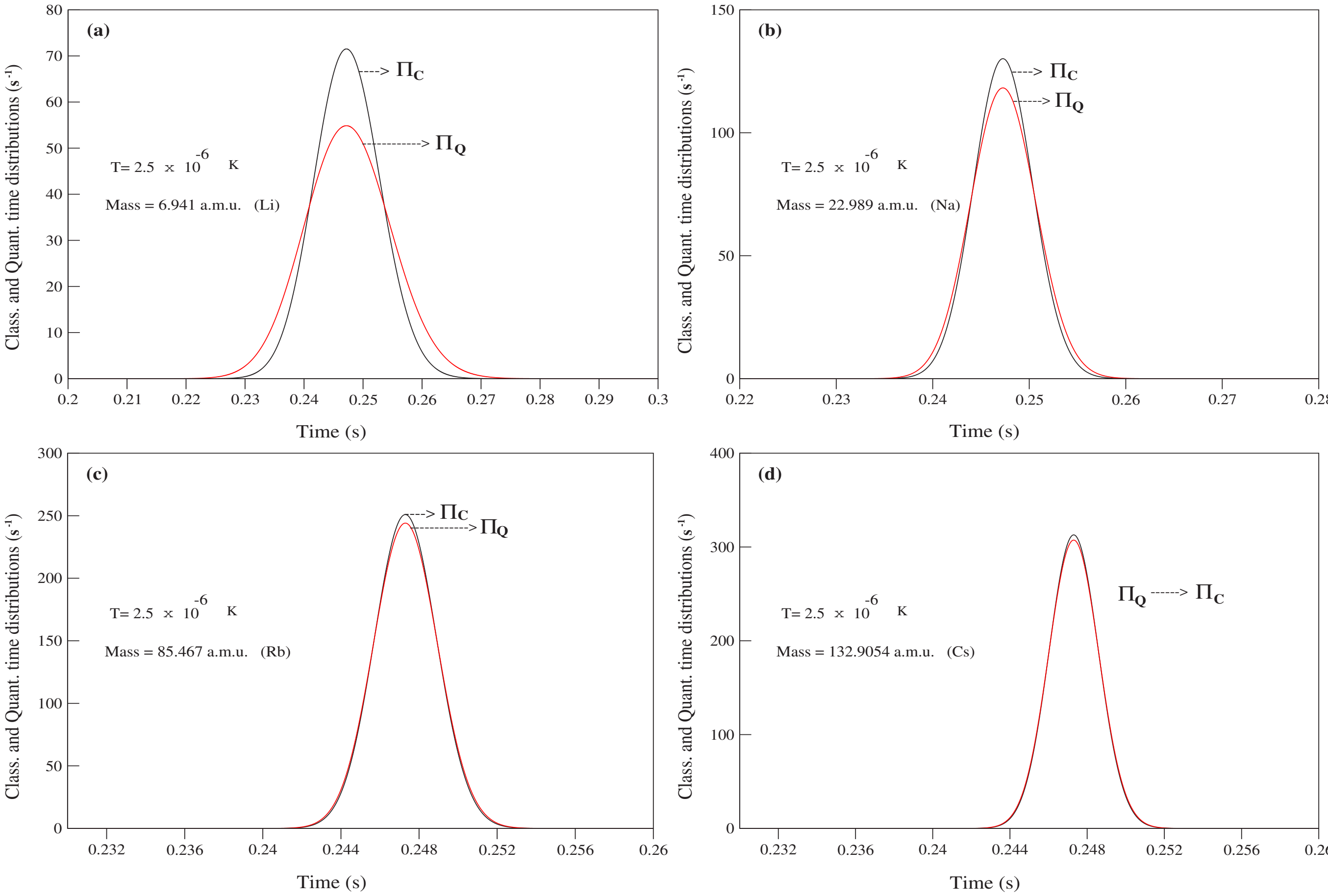


FIG. 1. (Color online) The classical $[\Pi_C(t)]$ and quantum $[\Pi_Q(t)]$ TOF distributions of the atomic cloud falling freely under gravity are plotted for varying mass of the atoms at a fixed temperature $T=2.5\times10^{-6}$ K with $\sigma_0=10^{-5}$ cm and $Z=-30$ cm.

velocities given by Eq. (1). Thus the initial thermal state of the atomic cloud we have described is a mixed state. We obtain the corresponding position probability distribution by averaging the pure state distribution (9) over a thermal distribution of initial velocities. The result is

$$P_T(z,t) = \left(\frac{m}{2\pi kT}\right)^{1/2}\int_{-\infty}^{\infty} P(z,t)\exp\left(-\frac{mv^2}{2kT}\right)dv = \frac{1}{(2\pi\sigma_T^{\,2})^{1/2}}\exp\left(-\frac{\left(z+\frac{1}{2}gt^2\right)^2}{2\sigma_T^{\,2}}\right), \tag{10}$$

where $\sigma_{T^2}=\sigma^2+(kT/m)t^2$. The peak of the position probability distribution $P_T(z,t)$ follows the classical trajectory and the effect of the mass and temperature dependences of the position probability occurs essentially because the spreading of the wave packet is different for different atomic mass and temperature of the cloud.

The corresponding probability current density for the mixed state at finite temperature $J_T(z,t)$ is also obtained by averaging the pure state current density given by Eq. (8) over the thermal distribution of initial velocities. The result is

$$J_T(z,t) = \left(\frac{m}{2\pi kT}\right)^{1/2}\int_{-\infty}^{\infty} J(z,t)\exp\left(-\frac{mv^2}{2kT}\right)dv = \frac{1}{(2\pi\sigma_T^{\,2})^{1/2}}\exp\left(-\frac{\left(z+\frac{1}{2}gt^2\right)^2}{2\sigma_T^{\,2}}\right) \times\left(\frac{\left(z+\frac{1}{2}gt^2\right)\left(\frac{kT}{m}+\frac{\hbar^2}{4m^2\sigma_0^{\,2}}\right)t}{\sigma_T^{\,2}}-gt\right). \tag{11}$$

Taking the modulus of the mixed state quantum probability current density as determining the quantum arrival time distribution [11], we obtain the arrival time probability distribution for the atomic cloud given by

$$\Pi_Q(t) = |J_T(z,t)|. \tag{12}$$

In this way we generalize the probability current density approach to calculate the quantum TOF distribution for the mixed state at finite temperature. It may be mentioned here that in the present calculation we neglect the small spin-dependent correction [13] that may appear in the probability current density, as mentioned above, and consequently the mean arrival time that we compute for any fermionic atoms.

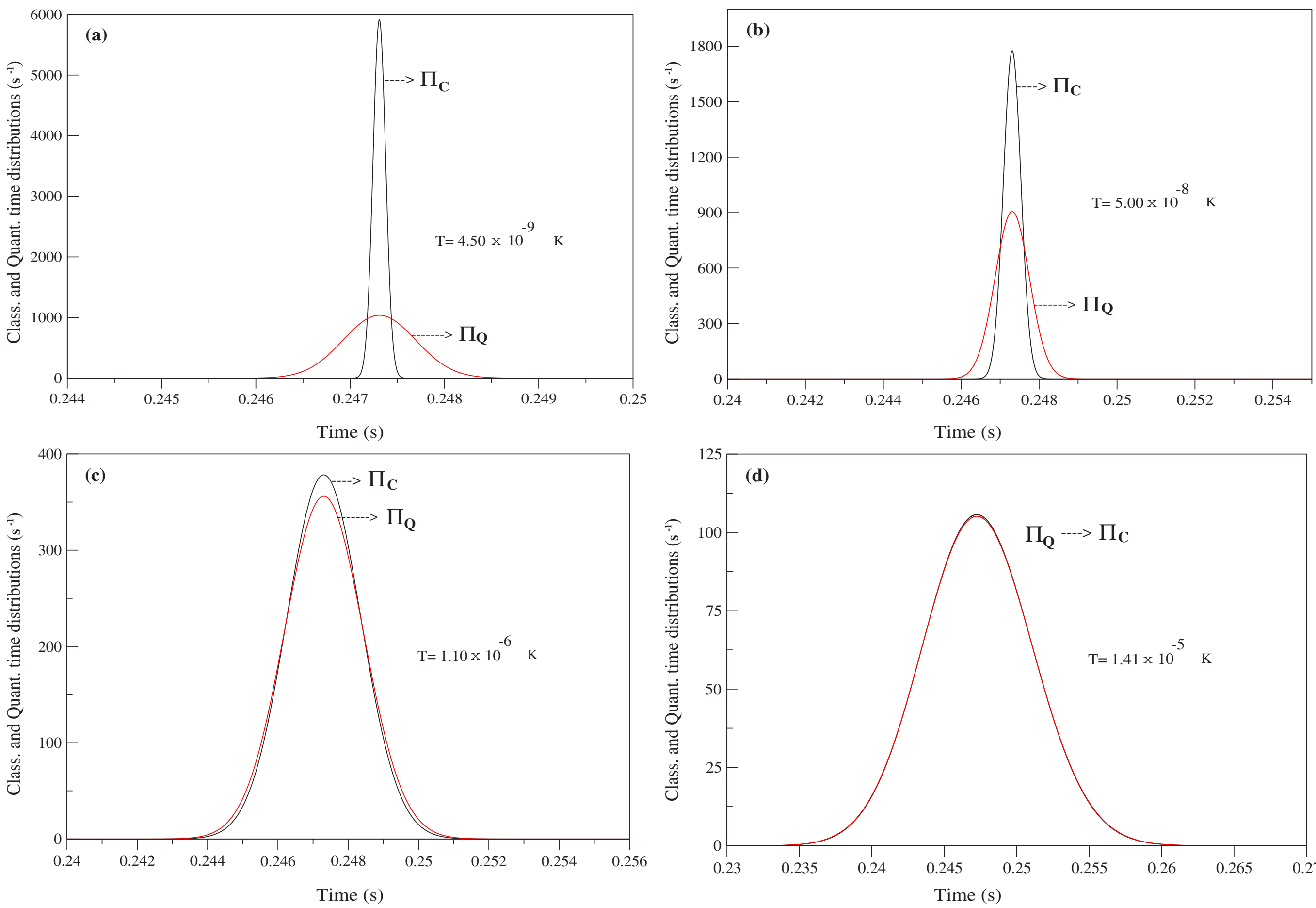


FIG. 2. (Color online) The classical $[\Pi_C(t)]$ and quantum $[\Pi_Q(t)]$ TOF distributions of the atomic cloud falling freely under gravity are plotted for varying temperatures at a fixed mass of Rb atom ($m$=85.4678 amu) with $\sigma_0=10^{-5}$ cm and $Z=-30$ cm.

The quantum mean arrival time $\overline{\tau_Q}$ of the atomic cloud, which is an observable quantity, is given by

$$\overline{\tau_Q} = \frac{\int_0^\infty \Pi_Q(t) t dt}{\int_0^\infty \Pi_Q(t) dt}. \qquad (13)$$

One may note that though the integral in the numerator of Eq. (13) may diverge in general, several techniques have been employed in the literature ensuring rapid fall off for the probability distributions asymptotically [19], so that convergent results are obtained for the integrated arrival time. However, in the present case it turns out from Eqs. (11) and (12) that $\Pi_Q(t)$ falls off exponentially at large times, and there is no problem of convergence in Eq. (13).

If we impose now the classical limit for the quantum TOF distribution given by Eq. (12), then one can check that under the large mass and high temperature limits, and when $\sigma_0 \ll (kT/m)t^2$, one can take $\sigma_T{}^2 \approx (kT/m)t^2$. Thus $\Pi_Q(t) = \Pi_C(t)$, i.e., the two distributions match in the limit of large mass and high temperature. The probability current method of computing the quantum arrival time distribution furnishes an effective way of approaching the classical limit of the distribution by smoothly varying the parameters such as mass and temperature of the quantum distribution [14,15]. Note also that the mass dependence of arrival time distribution given by Eq. (12) and consequently, the observable mean arrival time given by Eq. (13), signifies the quantum mechanical violation of the gravitational weak equivalence principle [15]. Thus TOF measurements [1–4,16] offer a practical possibility for experimental demonstration of the equivalence principle violation at the quantum level [20].

## IV. NUMERICAL RESULTS

We perform a numerical study of the quantum arrival time distribution of the falling atomic cloud by varying its mass and temperature separately. We first plot $\Pi_C(t)$ and $\Pi_Q(t)$ for a fixed temperature of $2.5 \times 10^{-6}$ K in Fig. 1. It is seen that the classical and the quantum distributions are clearly different for clouds of small atomic mass such as Li and Na. However, as one increases the atomic mass, one sees that $\Pi_C(t)$ and $\Pi_Q(t)$ begin to overlap for heavy atoms such as Rb for this value of temperature. The temperature variation of the arrival time distributions are displayed in Fig. 2 where one sees that even for heavy Rb atoms, the two distributions are quite distinct in the low (nano kelvin) temperature range. It would be interesting if our prediction in the low tempera-

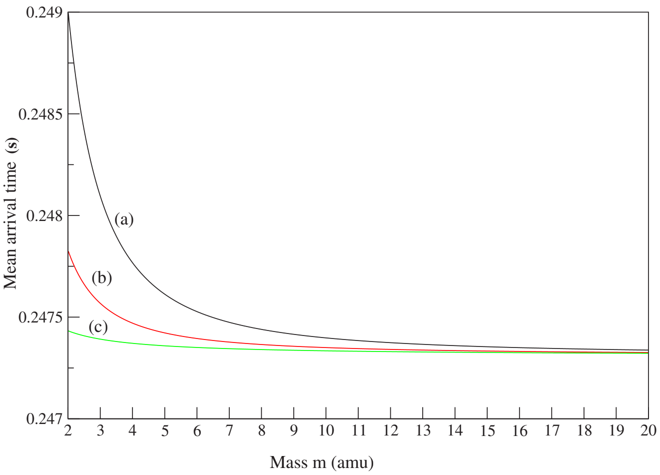


FIG. 3. (Color online) The mass variation of the mean arrival times $\overline{\tau_Q}$ and $\overline{\tau_C}$ are shown in the figure for a fixed value of temperature ($T=1.41\times10^{-6}$ K) with $Z=-30$ cm. The quantum mean arrival time $\overline{\tau_Q}$ is plotted for two different values of $\sigma_0$. 2(a) $\overline{\tau_Q}$ for $\sigma_0=10^{-5}$ cm, 2(b) $\overline{\tau_Q}$ for $\sigma_0=2\times10^{-5}$ cm. 2(c) The mean arrival time $\overline{\tau_C}$ calculated through the classical TOF distribution $\Pi_C(t)$ with $T=1.41\times10^{-6}$ K and $Z=-30$ cm.

ture and lower atomic mass region where the quantum time-of-flight distribution sharply differs from the classical TOF distribution could be verified in actual experiments.

The variation with mass of the quantum and classical mean arrival times at a particular detector location for the ensemble of falling atoms is depicted in Fig. 3. One can see that in the limit of large mass the mean arrival time $\overline{\tau_Q}$ asymptotically approaches the classical result. One can also investigate the variation of the mean arrival times ($\overline{\tau_Q}$ and $\overline{\tau_C}$) with varying temperature of the cloud, and obtain similar results, as expected from the temperature variation of the classical and quantum arrival time distributions plotted in Fig. 2. The mass and temperature dependences of the quantum arrival time distribution and consequently the quantum mean arrival time arise essentially due to the spread of the wave packet for the atoms. A smaller value of the initial width $\sigma_0$ for the wave packet results in its faster spread. The amount of departure of the quantum distribution from its classical counterpart is thus contingent on the magnitude of the ensemble spread [since $\Pi_C(t)$ is independent of $\sigma_0$]. This is clearly depicted in Fig. 4 where the classical [$\Pi_C(t)$] and quantum [$\Pi_Q(t)$] TOF distributions are plotted for three different values of $\sigma_0$ at a fixed mass and temperature.

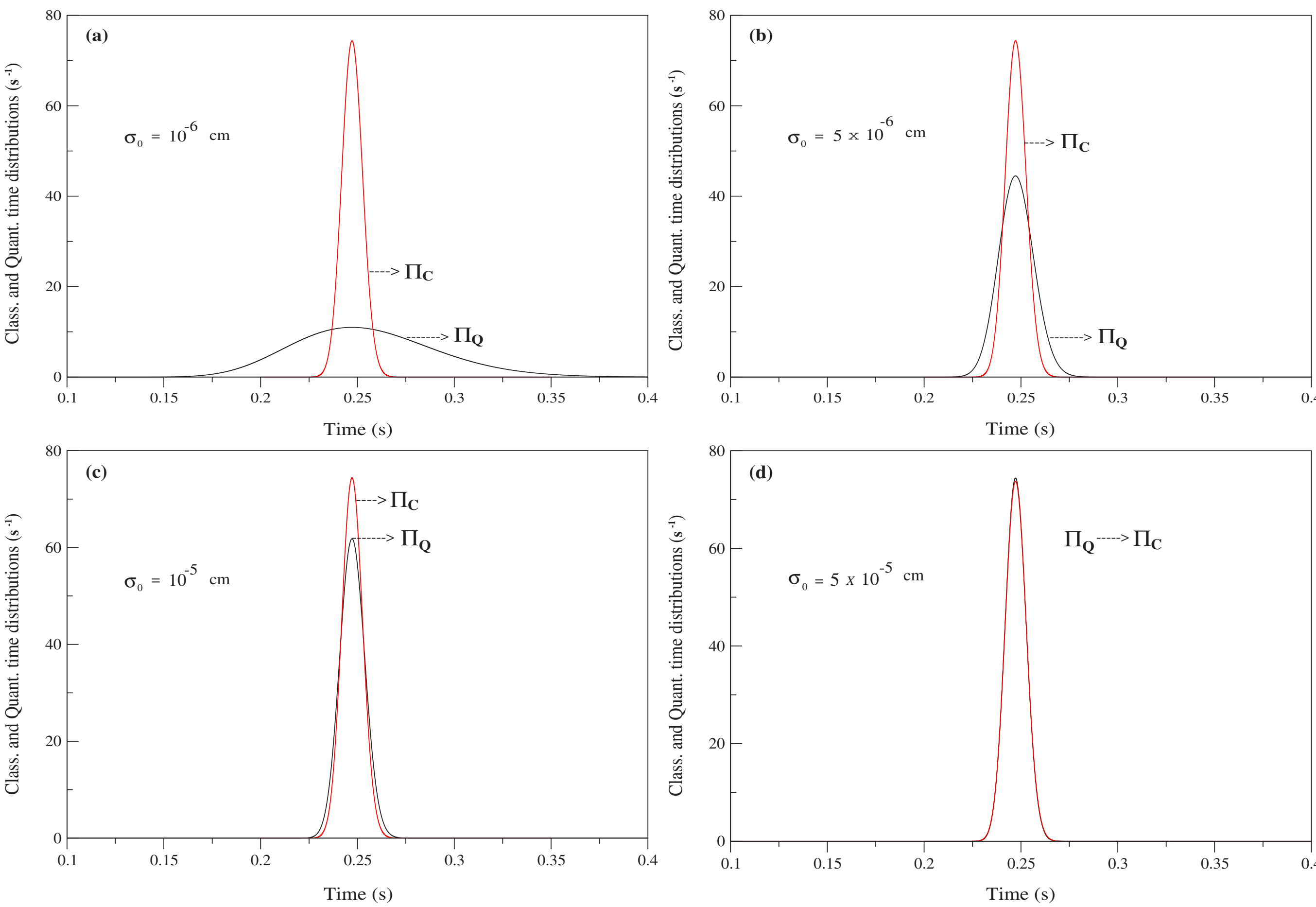


FIG. 4. (Color online) The classical [$\Pi_C(t)$] and quantum [$\Pi_Q(t)$] TOF distributions of the atomic cloud falling freely under gravity are plotted for four different values of $\sigma_0$ at a fixed mass of Be atom ($m=9.01$ amu) and at a fixed temperature $T=3.0\times10^{-6}$ K with $Z=-30$ cm.

## V. SUMMARY AND CONCLUSIONS

To summarize, in this work we have considered the analysis of the time-of-flight measurements of falling cold atomic clouds. The inference of the temperature of the cloud in various experiments [1–3] is usually performed through a classical analysis in which the results obtained are the same as through the solution of Newton's equations for ballistic motion of particles falling under gravity [5]. Here we emphasize the relevance of employing a quantum mechanical arrival time distribution for the analysis of such experiments. We use the probability current density approach towards obtaining the arrival time or the TOF distribution. Our definition of the quantum arrival time distribution and the observable mean arrival time in terms of the modulus of the probability current density is particularly motivated from the equation of continuity, and other physical considerations discussed in the literature [9,11–13,21]. Further, we generalize the probability current density approach to calculate the quantum arrival time distribution for a mixed state describing the Maxwell-Boltzmann distribution of velocities for the falling atomic clouds in the relevant experiments [1,4,16]. We compute the TOF distribution and mean arrival time through this scheme and compare our results with those obtained through a classical analysis [6] for various atomic masses.

The obtained quantum arrival time distribution matches with the classical TOF distribution in the high temperature and the large mass limits, hence furnishing another example of the smooth emergence of the classical limit [14] of the quantum arrival time in the framework of the probability current approach. However, a clear distinction between the quantum and the classical distributions is exhibited for either small atomic mass or low temperature of the cloud. This results from differential wave packet spreading depending upon the mass velocity, and width of the wave packet for the atoms. Our scheme thus provides a method for experimental verification of the probability current density approach for calculating the arrival time distribution, in addition to an earlier proposed method using the spin rotator as a quantum clock [22]. Finally, we wish to emphasize that more investigations of modern experiments employing time-of-flight techniques should be performed using various quantum mechanical schemes [10]. Such studies have the potential to empirically resolve ambiguities inherent in the theoretical formulations of the quantum arrival time distribution using cold trapped atom experimental techniques, and may also shed new light on the inference of the experimental data.

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