Violation of monogamy inequality for higher-dimensional objects

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Bipartite quantum entanglement for qutrits and higher-dimensional objects is considered. We analyze the possibility of violation of monogamy inequality, introduced by Coffman, Kundu, and Wootters, for some systems composed of such objects. An explicit counterexample with a three-qutrit totally antisymmetric state is presented. Since three-tangle has been confirmed to be a natural measure of entanglement for qubit systems, our result shows that the three-tangle is no longer a legitimate measure of entanglement for states with three qutrits or higher-dimensional objects.

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Quantification of quantum entanglement plays an important role not only in quantum information processing and quantum computation [1] but also in describing quantum phase transition in various interacting quantum many-body systems [2]. In the last 10 years a number of entanglement measures for qubit systems have been studied extensively, in which the well-known one with an elegant formula is concurrence derived analytically by Wootters [3], and the entanglement of formation (EOF) [4,5] is a monotonically increasing function of the concurrence. However, at least so far it is believed that there exists a drawback that they are confined into the qubit systems since the used spin-flip is only applicable to qubits [8]; because of which generally only a lower bound of concurrence can be achieved for states composed of qutrits or higher-dimensional objects. The seminal paper by Coffman, Kundu, and Wootters [6] provided a basis for the quantification of three-party entanglement by introducing the so-called residual tangle, and a general monogamy inequality in the case of n qubits has been proved [7].

Since the monogamy inequality has been established, whether it can be generalized to qutrits or higher-dimensional objects remains still open. In this Brief Report, we will first show that the monogamy inequality can be violated for some quantum composed of qutrits or higher-dimensional objects, and then offer an explicit example of an antisymmetric state to show this violation. Therefore the main idea here is to show that the monogamy inequality characterized by the concurrence cannot be generalized to a quantum state apart from qubits. This result gives a caveat when we are studying genuine multipartite entanglement for such states where the residual entanglement or three-tangle is defined.

For completeness we recall the original monogamy inequality. Consider a triple of spin-1/2 particles A, B, and C, and its density matrix is denoted by ρ_{ABC}, the distribution of entanglement among them is constrained by the following inequality:

$$C_{AB}^2 + C_{AC}^2 \leqslant C_{A(BC)}^2, \tag{1}$$

where C_{AB} and C_{AC} are the concurrences of the state ρ_{ABC} with traces taken over the particles C, B. $C_{A(BC)}$ is the concurrence of $\rho_{A(BC)}$ with the particles B and C regarded as a single object. In this case the particle A can be viewed as a focus such that the three-tangle can be defined as

$$\tau_{ABC} = C_{A(BC)}^2 - C_{AB}^2 - C_{AC}^2, \tag{2}$$

which is independent on the choice of the focus mainly because it is invariant under the permutations of the particles. The three-tangle has found wide applications in the research of genuine multiparticle entanglement [9] since it satisfies necessary conditions that a natural entanglement measure for pure state must require [10]. (1) τ_{ABC} is invariant under local unitary operations. (2) $\tau_{ABC} \geqslant 0$ for all pure states. (3) τ_{ABC} is an entanglement monotone [11], i.e., it does not increase on average under local quantum operations assisted with classical communication.

Now one may naturally ask what will happen as a general $M \times N \times Q$ system is concerned. It is in our great interest to know whether Eq. (1) remains valid since it determines justification of τ_{ABC} as an natural entanglement measure. In order to get the answer, we review some progresses on the quantification of entanglement of a bipartite $M \times N$ system. Actually only a few special classes of higher-dimensional states can give a closed-form expression of EOF [12,13]. Through analytical or numerical approaches generally only a lower bound of concurrence or EOF can be obtained [14–16]. The unavailability of exact entanglement makes us suspect the validity of corresponding monogamy inequality in Eq. (1) for higher-dimensional tripartite systems.

In what follows we show there exists a possibility that Eq. (1) does not necessarily hold always for qutrits or higher-dimensional objects. For a pure bipartite $M \times N$ system the squared concurrence can be expressed as $C^2 = \sum_{\alpha=1}^{M(M-1)} \sum_{\beta=1}^{N(N-1)} |C_{\alpha\beta}|^2$ [17], where $C_{\alpha\beta} = \langle \psi | L_\alpha \otimes L_\beta | \psi^* \rangle$, and L_α, L_β are generators of SO(M) and SO(N), respectively. For mixed state ρ of such system the squared concurrence is given as the convex roof,

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$$C^2(\rho) = \inf \sum_i p_i C^2(|\Phi_i\rangle), \quad \rho = \sum_i p_i |\Phi_i\rangle\langle\Phi_i|, \tag{3}$$

where $p_i \geqslant 0$ and ρ consists of all possible decompositions into pure states $|\Phi_i\rangle$. According to [14,16] we can obtain a lower bound $\mathbb{C}^2(\rho)$ for the mixed state as

$$\mathbb{C}^2(\rho) \equiv \left(\lambda_1 - \sum_{i>1} \lambda_i\right)^2 \leqslant \mathcal{C}^2(\rho), \tag{4}$$

where λ_i are the singular values of $\Sigma_{\alpha=1}^{M(M-1)} \Sigma_{\beta=1}^{N(N-1)} z_{\alpha\beta} A_{\alpha\beta}$ in decreasing order. For details of terms $z_{\alpha\beta}$ and $A_{\alpha\beta}$, see [14,16]. Note that the choice of the phase of $z_{\alpha\beta}$ is important for the tightness of the bound and in general it needs a complicated numerical optimization procedure for the bound. For the simplest 2×2 systems, Eq. (4) becomes equal and analytical. Moreover, the lower bound can also be written as an analytical expression

$$\mathbb{C}^2(\rho) = \sum_{j>i} \sum_{i=1}^{M-1} \mathcal{C}_{ij}^2(\rho), \tag{5}$$

for the more complicated $2\times M$ systems [18], where $\mathcal{C}_{ij}(\rho) = \max(0, \lambda_1^{ij} - \lambda_2^{ij} - \lambda_3^{ij} - \lambda_4^{ij})$ and λ^{ij} are the square roots of the four largest eigenvalues of the matrix $\rho^{1/2} S^{ij} \rho^* S^{ij} \rho^{1/2}$ [18]. It does not need numerical optimization for the bound of concurrence of the $2\times M$ systems.

Let us consider the tripartite $2\times M\times N$ system ABC. Choose the particle A with two dimensions as a focus, it then follows from Eq. (4) that the lower bound of the squared concurrences in AB and AC satisfies

$$\mathbb{C}_{AB}^2 \leqslant \mathcal{C}_{AB}^2, \quad \mathbb{C}_{AC}^2 \leqslant \mathcal{C}_{AC}^2. \tag{6}$$

Combining the two inequalities in Eq. (6) gives

$$\mathbb{C}_{AB}^2 + \mathbb{C}_{AC}^2 \leqslant \mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2. \tag{7}$$

On the other hand, the authors in [16] have proven that the sum of the lower bound in $\mathcal{C}_{AB}^2$ and $\mathcal{C}_{AC}^2$ is not greater than the squared concurrence between A and BC

$$\mathbb{C}_{AB}^2 + \mathbb{C}_{AC}^2 \leqslant \mathcal{C}_{A(BC)}^2. \tag{8}$$

From the observation of Eqs. (7) and (8), it is possible for us to find some states making Eq. (8) equal but Eq. (7) strict inequality, thus resulting in the violation of the monogamy inequality

$$\mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2 \geqslant \mathcal{C}_{A(BC)}^2. \tag{9}$$

Note that Eq. (7) can only be an equality for the $2\times 2\times 2$ systems, making the monogamy inequality in Eq. (1) hold for each state in such a system. In practice it is a formidable task to find the state satisfying Eq. (9) because of the requirement of a complicated convex roof for the concurrence of a higher-dimensional mixed state. Fortunately we find a state in the following, permitting us to easily calculate the concurrence of the mixed state.

Finally we present the explicit example that the monogamy inequality dose not work. Consider the pure totally antisymmetric state on a three-qutrit system ABC,

$$|\Psi\rangle = \frac{1}{\sqrt{6}}(|123\rangle - |132\rangle + |231\rangle - |213\rangle + |312\rangle - |321\rangle). \tag{10}$$

It is obvious that antisymmetric subspace $V \in H_A \otimes H_B$, $H_A \otimes H_C$, and $H_B \otimes H_C$ is spanned by the vectors

$$|x\rangle_{ij} \equiv \frac{1}{\sqrt{2}}(|23\rangle - |32\rangle),$$

$$|y\rangle_{ij} \equiv \frac{1}{\sqrt{2}}(|31\rangle - |13\rangle),$$

$$|z\rangle_{ij} \equiv \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle), \tag{11}$$

where $\{ij\} \in \{AB, AC, BC\}$. If A is chosen as a focus, then

$$\rho_{AB} = \frac{1}{3}(|x\rangle_{AB}\langle x| + |y\rangle_{AB}\langle y| + |z\rangle_{AB}\langle z|),$$

$$\rho_{AC} = \frac{1}{3}(|x\rangle_{AC}\langle x| + |y\rangle_{AC}\langle y| + |z\rangle_{AC}\langle z|),$$

$$\rho_{A(BC)} = \frac{1}{3}(|x\rangle_{BC}\langle x| + |y\rangle_{BC}\langle y| + |z\rangle_{BC}\langle z|). \tag{12}$$

Since $\rho_{A(BC)}$ is pure, it is readily to check $C_{A(BC)}^2 = 4/3$. For mixed states ρ_{AB} and ρ_{BC} we have to make an infimum for their concurrences. Generally, it is difficult, however, the system (10) is a special case. For arbitrary pure states $|\Phi\rangle_{AB} = c_1|x\rangle + c_2|y\rangle + c_3|z\rangle$ with $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$, their reduced density matrix $\rho_A \equiv \mathrm{Tr}_B|\Phi\rangle_{AB}\langle\Phi|$ has the same spectrum $\{1/2, 1/2, 0\}$ [19], implying any two antisymmetric states $|\Phi\rangle_{AB}$ can be transformed into each other by local unitary transformations. As a result, $C^2(|\Phi\rangle_{AB}) = 1$. While ρ_{AB} in Eq. (9) can be decomposed into

$$\rho_{AB} = \sum_i p_i |\Phi_i\rangle_{AB}\langle\Phi_i|. \tag{13}$$

Why the system (10) is special lies in $C^2(|\Phi_i\rangle_{AB}) = 1$ for each $|\Phi_i\rangle_{AB}$ such that $\mathcal{C}_{AB}^2 = \Sigma_i p_i = 1$. Analogously, $\mathcal{C}_{AC}^2 = \Sigma_i p_i = 1$. Therefore we obtain

$$\mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2 = 2 \geqslant \frac{4}{3} = \mathcal{C}_{A(BC)}^2, \tag{14}$$

which is not superseded by the monogamy inequality in Eq. (1). In the similar way, it is confirmed that $\mathcal{C}_{BA}^2 + \mathcal{C}_{BC}^2 \geqslant \mathcal{C}_{B(AC)}^2$ and $\mathcal{C}_{CA}^2 + \mathcal{C}_{CB}^2 \geqslant \mathcal{C}_{C(AB)}^2$ also violate the corresponding monogamy inequalities. Perhaps this violation is not a paradox by considering that the EOF cannot yet satisfy the monogamy inequality due to the concave log function [6].

Summarizing, we have shown that the monogamy inequality in qubit systems cannot be generalized to higher-dimensional objects such that a caveat is provided when the three-tangle is defined since it may exhibit a negative value for some state. However, in general the monogamy inequality in Eq. (1) also works, for example, for a state $|\Psi\rangle_{ABC} = \frac{1}{\sqrt{3}}(|111\rangle + |222\rangle + |333\rangle)$, we obtain $C_{AB}^2 + C_{AC}^2 = 0 \leq 4/3 = C_{A(BC)}^2$, satisfying the monogamy inequality. Consequently, the conditions of whether the monogamy inequality for higher dimensional objects is violated or not is still open. As stated in [7], the constrains by the monogamy inequality in Eq. (1) on the entanglement shared by parties lie at the heart of the success of many information-theoretic protocols, correspondingly the impacts on such protocols imposed by this violation of monogamy inequality deserve further investigations.

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