Static structure factor and quantum melting of vortex lattices in two-dimensional Bose-Einstein condensates

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(Received 20 January 2007; revised manuscript received 7 March 2007; published 28 March 2007)

The first-order quantum corrections to the static structure factor of vortex lattices in a rapidly rotating quasi-two-dimensional Bose-Einstein condensate are calculated. Furthermore, we estimate the melting filling fraction by using a criterion similar to that used in thermal melting related to the Debye-Waller factor of the smallest reciprocal lattice vector.

DOI: 10.1103/PhysRevA.75.033620 PACS number(s): 03.75.Lm, 03.75.Kk

I. INTRODUCTION

Rapidly rotating Bose-Einstein condensates have attracted increasing interest recently. Due to the analogy between the Hamiltonian for a rotating condensate in a harmonic trap and that for charged particles in a magnetic field, Landau levels can be conveniently used as the complete set of single-particle wave functions. The rapidly rotating limit makes the lowest Landau level (LLL) primarily occupied [1], which brings huge simplicity to the theoretical handling of rapidly rotating condensates. For condensates in the LLL, various states, ranging from well-organized vortex lattices (mean-field quantum Hall regime) to a series of highly correlated quantum Hall liquids [2–4], have been identified using the filling fraction ν, the ratio of particle numbers N to vortex numbers N_v. It is worth mentioning that, while the generally accepted criterion for the LLL approximation requires the average interaction energy per particle to be smaller than the Landau level spacing, calculations on type-II superconductors [5] and a recent numerical work on rotating gases [6] both indicated that this condition may be too restrictive.

The Gross-Pitaevskii energy functional provides accurate descriptions for the condensates in the mean-field quantum Hall regime [1,7–9], beyond which, however, quantum fluctuations become more and more important, eventually leading to the melting of the vortex lattice. In the rapidly rotating limit, Sinova *et al.* [10] first obtained the quadratic collective-excitation spectrum of a quasi-two-dimensional (2D) gas within the LLL approximation. This softer spectrum, or Tkachenko mode in the quantum Hall limit [13], is responsible for the lack of Bose-Einstein condensation of the system in the thermodynamic limit. Nevertheless, as was pointed out in Refs. [10,13], at zero temperature the single-vortex displacement does not exhibit divergent fluctuations even though the condensate fraction does, which indicates that the vortex lattice can still have positional long-range order before quantum melting in realistic systems. Several authors [10–12] have studied quantum melting of the vortex lattice by using elastic theory and the Lindemann criterion [14,15]. The filling fraction at which the lattice melts was estimated to be of the order of 10 [10,11]. A numerical calculation based on the exact-diagonalization method also indicated a similar result, $\nu \sim 6$ [16]. Another interesting work [17] using the vortex-coordinate approach suggests cooperative ring exchange as the mechanism of quantum melting of the same system and predicts a melting $\nu \sim 2$.

We attempt, in this paper, to tackle the problem of vortex lattice melting from a somewhat different angle, which focuses on the fluctuation effect on the static structure factor rather than single-vortex displacement. The structure factor, or equivalently the density-density correlation itself, is an important theoretical quantity for describing Bose-Einstein condensates [18] and has received considerably theoretical [19–23] and experimental [24,25] investigation for nonrotating condensates.

As is well known from solid state physics, for a crystal the thermal fluctuations diminish the intensity of the Bragg peaks but do not eliminate the peaks altogether [26]. The effect of these fluctuations is entirely contained in the Debye-Waller factor, the ratio of the structure factor at temperature T to its value at zero temperature. A phenomenological melting criterion related to the Debye-Waller factor of the smallest reciprocal lattice vector has been discussed in both a Yukawa system [27] and type-II superconductor vortex matter [28]. This criterion states, in the present case, that the vortex lattice is expected to melt when the reduction of the intensity of the Bragg peak due to quantum fluctuations reaches a certain fraction of its mean-field value.

In this paper, we calculate the effect of quantum fluctuations on the static structure factor of vortex lattices in a quasi-2D Bose-Einstein condensate, based on which we also estimate the melting filling fraction by using a criterion akin to that used in thermal melting related to the Debye-Waller factor. Our result is consistent with some other estimates [10,11,16]. The rest of the paper is organized as follows. In Sec. II we introduce the model and briefly review the mean-field and perturbation theories. In Sec. III a calculation of the static structure factor up to the first-order quantum correction is carried out thoroughly. The melting of the vortex lattice is discussed in Sec. IV.

II. MEAN-FIELD THEORY AND PERTURBATION FRAMEWORK

For completeness we will rederive, using the notation given in Ref. [5], some of the results obtained in Ref. [10]. Our starting point is the rotating-frame Hamiltonian [16] of the boson gas in an axisymmetric trap (with trap frequencies $\omega_\perp$ and ω_z) rotating with angular velocity $-\Omega\hat{\mathbf{z}}$,

$$H = \sum_{i=1}^{N}\left(\frac{(\mathbf{p}_i + m\Omega\hat{\mathbf{z}}\times\mathbf{r}_i)^2}{2m} + \frac{m}{2}[(\omega_\perp^2 - \Omega^2)(x_i^2 + y_i^2) + \omega_z^2 z_i^2]\right) + g\sum_{i<j=1}^{N}\delta(\mathbf{r}_i - \mathbf{r}_j), \quad (1)$$

where a hard-core potential of strength g is used to describe the two-body interaction in the low-energy limit. The kinetic part of this Hamiltonian is reminiscent of that for charged particles in a magnetic field, whose eigenfunctions are the well-known Landau wave functions. In the rapidly rotating limit $\Omega\to\omega_\perp$, the system approaches the quasi-two-dimensional limit and, assuming a strong confinement in the z direction, the Hamiltonian is reduced to a quasi-2D one. By first switching to the Landau gauge and then using the functional integral formalism, the partition function of this simplified model can be expressed as

$$\mathcal{Z} = \int \mathcal{D}\psi^*\mathcal{D}\psi e^{-\mathcal{S}[\psi^*,\psi]}, \quad (2)$$

where $\mathcal{S}$ is the action [29]

$$\mathcal{S}[\psi^*,\psi] = \int_0^{1/T} d\tau \int d\mathbf{x}\left[\psi^*\left(\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}D^2 - \mu\right)\psi + \frac{1}{2}g|\psi|^4\right], \quad (3)$$

T is the temperature, μ is the chemical potential, $D^2\equiv(\partial_x - iby)^2 + \partial_y^2$, and $b=2m\Omega/\hbar$.

The usual static mean-field equation can be obtained when the stationary approximation is applied,

$$\mathcal{H}\Phi - \delta\mu\Phi + g|\Phi|^2\Phi = 0, \quad (4)$$

where $\mathcal{H}\equiv-(\hbar^2/2m)D^2 - \hbar^2 b/2m$, $\delta\mu=\mu-\hbar^2 b/2m$. Within the LLL approximation the solution of this equation is the well-known Abrikosov solution and can be written as

$$\Phi = \zeta\varphi = \zeta\sqrt{\frac{2\sqrt{\pi}}{a}}\sum_{l=-\infty}^{\infty}\exp\left[i\left(\frac{\pi l(l-1)}{2} + \frac{2\pi\sqrt{b}x}{a}l\right) - \frac{1}{2}\left(y\sqrt{b} - \frac{2\pi}{b}l\right)^2\right], \quad (5)$$

where $a/\sqrt{b}=\sqrt{4\pi/\sqrt{3}b}$ is the lattice spacing. The condensate wave function Φ should minimize the free energy and one can then find the mean-field value of ζ:

$$\zeta_0 = \sqrt{\frac{\delta\mu}{\beta g}}, \quad (6)$$

where $\beta=(b/2\pi)\int_{cell}d\mathbf{x}|\varphi|^4$.

Perturbation theory of the weakly interacting Bose gas [30,31] can be used here to calculate the quantum corrections to ζ. First one parametrizes the quantum field ψ in terms of the time-independent condensate Φ and a quantum fluctuation field ϕ, $\psi=\Phi+\phi$, and decomposes the action into a classical term, an unperturbed free term, and a perturbation term:

$$\mathcal{S} = \mathcal{S}_c[\Phi^*,\Phi] + \mathcal{S}_{free}[\phi^*,\phi] + \mathcal{S}_{pert}[\phi^*,\phi], \quad (7)$$

where the perturbation part of the action $\mathcal{S}_{pert}$ includes all the cubic and quartic terms of ϕ. Next one diagonalizes the unperturbed part of the action to obtain the propagator, which will be detailed below. Then by using the loop expansion the free energy density can be given by

$$\mathcal{F}(\mu,\zeta) = \sum_i \mathcal{F}_i(\mu,\zeta), \quad (8)$$

where the subscript i denotes the contribution of the ith-order one-particle irreducible loop diagrams. The first quantum correction for ζ to its mean-field value ζ_0 is [30]

$$\zeta_1 = -\left.\frac{\partial\mathcal{F}_1(\mu,\zeta)}{\partial\zeta}\right|_{\zeta=\zeta_0}\Bigg/\left.\frac{\partial^2\mathcal{F}_0(\mu,\zeta)}{\partial\zeta^2}\right|_{\zeta=\zeta_0}. \quad (9)$$

The unperturbed part of the action includes all the quadratic terms of ϕ and can be written as

$$\mathcal{S}_{free}[\phi^*,\phi] = \int d\tau\int d\mathbf{x}\left[\phi^*\left(\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}D^2 - \mu\right)\phi + 2g|\Phi|^2|\phi|^2 + \frac{1}{2}g\Phi^2\phi^{*2} + \frac{1}{2}g\Phi^{*2}\phi^2\right]. \quad (10)$$

Within the LLL we expand ϕ in a basis of quasimomentum $\mathbf{k}$ eigenfunctions,

$$\varphi_{\mathbf{k}}(\mathbf{x}) = \sqrt{\frac{2\sqrt{\pi}}{a}}\sum_{l=-\infty}^{\infty}\times\exp\left[i\left(\frac{\pi l(l-1)}{2} + \frac{2\pi(\sqrt{b}x - k_y/\sqrt{b})}{a}l - xk_x\right) - \frac{1}{2}\left(y\sqrt{b} + \frac{k_x}{\sqrt{b}} - \frac{2\pi}{a}l\right)^2\right] \quad (11)$$

and

$$\phi(\mathbf{x},\tau) = \sqrt{\frac{T}{A}}\sum_{\omega_m,\mathbf{k}}\varphi_{\mathbf{k}}(\mathbf{x})e^{-i\omega_m\tau}a(\omega_m,\mathbf{k}), \quad (12)$$

where A is the area of the gas, the frequency $\omega_m=2m\pi T$, and the quasimomentum $\mathbf{k}$ is summed over the triangular-lattice Brillouin zone. For brevity, hereafter $(\omega_m,\mathbf{k})$ is denoted by p. Then $\mathcal{S}_{free}$ can be transformed into the quasimomentum and frequency representation

$$\mathcal{S}_{free}[a^*,a] = \sum_p\left((-i\omega_m - \delta\mu)a^*(p)a(p) + 2\beta_k g\zeta^2 a^*(p)a(p) + \frac{1}{2}\gamma_k^* g\zeta^2 a^*(-p)a^*(p) + \frac{1}{2}\gamma_k g\zeta^2 a(-p)a(p)\right) \quad (13)$$

where

$$\beta_k = \frac{b}{2\pi}\int_{cell} d\mathbf{x}|\varphi|^2\varphi_{\mathbf{k}}\varphi_{\mathbf{k}}^* = |\beta_k|,$$

$$\gamma_k = \frac{b}{2\pi}\int_{\mathrm{cell}} d\mathbf{x}(\varphi^*)^2\varphi_{-\mathbf{k}}\varphi_{\mathbf{k}} = |\gamma_k|e^{i\theta_k}. \tag{14}$$

The above action can be diagonalized by introducing the Bogoliubov transformation $a(p)=u_k b(p)+v_k b^*(-p)$, where u_k and v_k satisfy

$$|u_k|^2+|v_k|^2 = \frac{2g\zeta^2\beta_k-\delta\mu}{\sqrt{(2g\zeta^2\beta_k-\delta\mu)^2-(g\zeta^2|\gamma_k|)^2}},$$

$$|u_k||v_k| = \frac{g\zeta^2|\gamma_k|}{2\sqrt{(2g\zeta^2\beta_k-\delta\mu)^2-(g\zeta^2|\gamma_k|)^2}},$$

$$u_k=-|u_k|,\quad v_k=|v_k|e^{-i\theta_k}. \tag{15}$$

The diagonalized free action is

$$\mathcal{S}_{free} = \sum_p \{[-i\omega_m+\varepsilon(\mathbf{k})]b^*(p)b(p)\}. \tag{16}$$

Therefore the propagator corresponding to the free action is

$$\langle b^*(p)b(p)\rangle = \frac{1}{-i\omega_m+\varepsilon(\mathbf{k})}, \tag{17}$$

where $\varepsilon(\mathbf{k})$ is the dispersion relation

$$\varepsilon(\mathbf{k}) = g\zeta^2\left[\left(2\beta_k-\frac{\delta\mu}{g\zeta^2}\right)^2-|\gamma_k|^2\right]^{1/2}, \tag{18}$$

which gives the zero-order excitation spectrum at $\zeta=\zeta_0$,

$$\varepsilon_0(\mathbf{k}) = \frac{\delta\mu}{\beta}\tilde{\varepsilon}_0(\mathbf{k}) = \frac{\delta\mu}{\beta}[(2\beta_k-\beta)^2-|\gamma_k|^2]^{1/2}, \tag{19}$$

as was obtained first by Sinova *et al.* [10]. In particular, when $\mathbf{k}\to\mathbf{0}$,

$$\varepsilon_0(\mathbf{k}) \sim 0.0836\frac{2\pi}{b}\frac{\delta\mu}{\beta}\mathbf{k}^2. \tag{20}$$

This gapless spectrum is softer than the usual Goldstone modes in the homogeneous Bose system and will lead to a logarithmically infrared divergency of the order parameter even at zero temperature in the thermodynamic limit.

The mean-field and one-loop contributions to the free energy density are given, respectively, by

$$\mathcal{F}_0(\mu,\zeta) = -\delta\mu\zeta^2+\frac{1}{2}\beta g\zeta^4 \tag{21}$$

and

$$\mathcal{F}_1(\mu,\zeta) = \frac{g\zeta^2}{2}\int\frac{d\mathbf{k}}{(2\pi)^2}\left[\left(2\beta_k-\frac{\delta\mu}{g\zeta^2}\right)^2-|\gamma_k|^2\right]^{1/2}. \tag{22}$$

So we get the first-order quantum correction to ζ,

$$\zeta_1 = -\frac{1}{4}\left(\frac{\beta g}{\delta\mu}\right)^{1/2}\int\frac{d\mathbf{k}}{(2\pi)^2}\left(\frac{(2\beta_k-\beta)}{\tilde{\varepsilon}_0(\mathbf{k})}+\frac{\tilde{\varepsilon}_0(\mathbf{k})}{\beta}\right), \tag{23}$$

and the average particle density up to the first-order quantum correction,

$$\bar{n}(\mu) = -\frac{\partial}{\partial\mu}[\mathcal{F}_0(\mu,\zeta_0)+\mathcal{F}_1(\mu,\zeta_0)] = \frac{\delta\mu}{\beta g}-\frac{1}{2}\int\frac{d\mathbf{k}}{(2\pi)^2}\frac{\tilde{\varepsilon}_0(\mathbf{k})}{\beta}, \tag{24}$$

which lacks divergent fluctuations as does the single-vortex displacement.

III. STATIC STRUCTURE FACTOR

The static structure factor is defined by

$$S(\mathbf{q}) = \frac{1}{N}\int d\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}}\int d\mathbf{y}\, e^{i\mathbf{q}\cdot\mathbf{y}}\langle\rho(\mathbf{x},\tau)\rho(\mathbf{y},\tau)\rangle, \tag{25}$$

where $\rho\equiv|\psi|^2$ and $\langle\,\rangle$ indicates the quantum average. Within the LLL approximation the mean-field part of $S(\mathbf{q})$ is

$$S_{mf}(\mathbf{q}) = \frac{1}{N}\zeta_0^4\int d\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}}|\varphi(\mathbf{x})|^2\int d\mathbf{y}\, e^{i\mathbf{q}\cdot\mathbf{y}}|\varphi(\mathbf{y})|^2$$
$$= \frac{1}{\bar{n}^2}N\left(\frac{\delta\mu}{\beta g}\right)^2\exp\left(-\frac{\mathbf{q}^2}{2b}\right)\sum_{\mathbf{K}_m}\delta_{\mathbf{q},\mathbf{K}_m}, \tag{26}$$

where we have used the following formula [32]:

$$\int d\mathbf{x}\,\varphi(\mathbf{x})\varphi_{\mathbf{k}}^*(\mathbf{x})\exp(-i\mathbf{x}\cdot\mathbf{q})$$
$$= A\sum_{\mathbf{K}_m}\delta_{\mathbf{q}-\mathbf{k},\mathbf{K}_m}\exp\left(\frac{\pi i}{2}(m_1^2-m_1)\right)$$
$$\times\exp\left(-\frac{\mathbf{q}^2}{4b}-\frac{iq_xq_y}{2b}+\frac{ik_xq_y}{b}\right), \tag{27}$$

where the reciprocal lattice vector $\mathbf{K}_m=m_1\tilde{\mathbf{d}}_1+m_2\tilde{\mathbf{d}}_2$, and $\tilde{\mathbf{d}}_1=(2\pi\sqrt{b}/a)(1,-1/\sqrt{3})$, $\tilde{\mathbf{d}}_2=(0,4\pi\sqrt{b}/a\sqrt{3})$ are the reciprocal lattice basis vectors. The first-order quantum corrections to $S(\mathbf{q})$ consist of four terms $S_1,\ldots,S_4$, each of which will be calculated explicitly here. The first term is

$$
\begin{aligned}
S_1(\mathbf{q}) &= \frac{1}{N}\int d\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}}\int d\mathbf{y}\, e^{i\mathbf{q}\cdot\mathbf{y}}\Phi(\mathbf{x})\Phi(\mathbf{y})\langle\phi^*(\mathbf{x},\tau)\phi^*(\mathbf{y},\tau)\rangle + \text{c.c.}\\
&= \frac{1}{N}\frac{T}{A}\int d\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}}\int d\mathbf{y}\, e^{i\mathbf{q}\cdot\mathbf{y}}\Phi(\mathbf{x})\Phi(\mathbf{y})\sum_{p,p'}\varphi^*_{\mathbf{k}}(\mathbf{x})\varphi^*_{\mathbf{k}'}(\mathbf{y})\langle a^*(p)a^*(p')\rangle e^{i(\omega+\omega')\tau} + \text{c.c.}\\
&= -\frac{1}{NA}\frac{\delta\mu}{\beta g}\sum_{\mathbf{k}}\int d\mathbf{x}\,\varphi(\mathbf{x})\varphi^*_{\mathbf{k}}(\mathbf{x})e^{-i\mathbf{q}\cdot\mathbf{x}}\int d\mathbf{y}\,\varphi(\mathbf{y})\varphi^*_{-\mathbf{k}}(\mathbf{y})e^{i\mathbf{q}\cdot\mathbf{y}}\frac{\gamma_k^*}{2\tilde{\varepsilon}_0(\mathbf{k})} + \text{c.c.}
\end{aligned}
\tag{28}
$$

where Eqs. (15)–(20) are used and a frequency summation is performed. Using the formula (27) we get

$$
S_1(\mathbf{q}) = -\frac{1}{\bar{n}}\frac{\delta\mu|\gamma_k|}{\beta g\tilde{\varepsilon}_0(\mathbf{k})}\cos\left(\frac{k_x k_y + \mathbf{k}\times\mathbf{K}_m}{b} + \theta_k\right)\exp\left(-\frac{\mathbf{q}^2}{2b}\right),
\tag{29}
$$

where $\mathbf{k}$ is restricted to the first Brillouin zone and $\mathbf{q}=\mathbf{k}+\mathbf{K}_m$. The second and third fluctuation terms can be calculated in a similar way:

$$
\begin{aligned}
S_2(\mathbf{q}) &= \frac{1}{N}\int d\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}}\int d\mathbf{y}\, e^{i\mathbf{q}\cdot\mathbf{y}}\Phi(\mathbf{x})\Phi^*(\mathbf{y})\langle\phi^*(\mathbf{x},\tau)\phi(\mathbf{y},\tau)\rangle + \text{c.c.}\\
&= \frac{1}{N}\frac{T}{A}\int d\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}}\int d\mathbf{y}\, e^{i\mathbf{q}\cdot\mathbf{y}}\Phi(\mathbf{x})\Phi^*(\mathbf{y})\sum_{p,p'}\varphi^*_{\mathbf{k}}(\mathbf{x})\varphi_{\mathbf{k}'}(\mathbf{y})\langle a^*(p)a(p')\rangle e^{-i(\omega-\omega')\tau} + \text{c.c.}\\
&= \frac{1}{NA}\sum_{\mathbf{k}}\int d\mathbf{x}\,\varphi(\mathbf{x})\varphi^*_{\mathbf{k}}(\mathbf{x})e^{-i\mathbf{q}\cdot\mathbf{x}}\int d\mathbf{y}\,\varphi^*(\mathbf{y})\varphi_{\mathbf{k}}(\mathbf{y})e^{i\mathbf{q}\cdot\mathbf{y}}\frac{\delta\mu}{\beta g}\left(\frac{2\beta_k-\beta}{2\tilde{\varepsilon}_0(\mathbf{k})} - \frac{1}{2}\right) + \text{c.c.}\\
&= \frac{1}{\bar{n}}\frac{\delta\mu}{\beta g}\left(\frac{2\beta_k-\beta}{\tilde{\varepsilon}_0(\mathbf{k})} - 1\right)\exp\left(-\frac{\mathbf{q}^2}{2b}\right)
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
S_3(\mathbf{q}) &= \frac{1}{N}\int d\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}}\int d\mathbf{y}\, e^{i\mathbf{q}\cdot\mathbf{y}}|\Phi(\mathbf{x})|^2\langle|\phi(\mathbf{y},\tau)|^2\rangle + \mathbf{q}\to-\mathbf{q}\\
&= \frac{1}{N}\frac{T}{A}\int d\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}}\int d\mathbf{y}\, e^{i\mathbf{q}\cdot\mathbf{y}}|\Phi(\mathbf{x})|^2\sum_{p,p'}\varphi_{\mathbf{k}}(\mathbf{y})\varphi^*_{\mathbf{k}'}(\mathbf{y})\langle a^*(p)a(p')\rangle e^{-i(\omega-\omega')\tau} + \mathbf{q}\to-\mathbf{q}\\
&= \frac{1}{NA}\int d\mathbf{x}|\varphi(\mathbf{x})|^2 e^{-i\mathbf{q}\cdot\mathbf{x}}\sum_{\mathbf{k}}\int d\mathbf{y}|\varphi_{\mathbf{k}}(\mathbf{y})|^2 e^{i\mathbf{q}\cdot\mathbf{y}}\frac{\delta\mu}{\beta g}\left(\frac{2\beta_k-\beta}{2\tilde{\varepsilon}_0(\mathbf{k})} - \frac{1}{2}\right) + \mathbf{q}\to-\mathbf{q}\\
&= \frac{N}{\bar{n}^2}\exp\left(-\frac{\mathbf{q}^2}{2b}\right)\sum_{\mathbf{K}_m}\delta_{\mathbf{q},\mathbf{K}_m}\int\frac{d\mathbf{k}}{(2\pi)^2}\cos\left(\frac{\mathbf{k}\times\mathbf{K}_m}{b}\right)\frac{\delta\mu}{\beta g}\left(\frac{2\beta_k-\beta}{\tilde{\varepsilon}_0(\mathbf{k})} - 1\right),
\end{aligned}
\tag{31}
$$

where $\Sigma_{\mathbf{k}}$ is replaced by $A\int d\mathbf{k}/(2\pi)^2$ in the last line. The final term comes from the quantum correction of the condensate,

$$
S_4(\mathbf{q}) = 4\left(\frac{\delta\mu}{\beta g}\right)^{3/2}\frac{1}{N}\int d\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}}\int d\mathbf{y}\, e^{i\mathbf{q}\cdot\mathbf{y}}|\varphi(\mathbf{x})|^2|\varphi(\mathbf{y})|^2\zeta_1.
\tag{32}
$$

Substituting Eq. (23) into Eq. (32) one obtains

$$
\begin{aligned}
S_4(\mathbf{q}) &= -\frac{1}{N}\int d\mathbf{x}|\varphi(\mathbf{x})|^2 e^{-i\mathbf{q}\cdot\mathbf{x}}\int d\mathbf{y}|\varphi(\mathbf{y})|^2 e^{i\mathbf{q}\cdot\mathbf{y}}\int\frac{d\mathbf{k}}{(2\pi)^2}\frac{\delta\mu}{\beta g}\left(\frac{2\beta_k-\beta}{\tilde{\varepsilon}_0(\mathbf{k})} + \frac{\tilde{\varepsilon}_0(\mathbf{k})}{\beta}\right)\\
&= -\frac{N}{\bar{n}^2}\exp\left(-\frac{\mathbf{q}^2}{2b}\right)\sum_{\mathbf{K}_m}\delta_{\mathbf{q},\mathbf{K}_m}\int\frac{d\mathbf{k}}{(2\pi)^2}\frac{\delta\mu}{\beta g}\left(\frac{2\beta_k-\beta}{\tilde{\varepsilon}_0(\mathbf{k})} + \frac{\tilde{\varepsilon}_0(\mathbf{k})}{\beta}\right).
\end{aligned}
\tag{33}
$$

Combining the four terms, the static structure factor up to first-order quantum correction within the LLL can be put as

$$S(\mathbf{q}) = S_{mf}(\mathbf{q}) + \sum_{i=1}^{4} S_i(\mathbf{q}) = \frac{N}{\bar{n}^2}\left(\frac{\delta\mu}{\beta g}\right)^2 \exp\left(-\frac{\mathbf{q}^2}{2b}\right)\sum_{\mathbf{K}_m}\delta_{\mathbf{q},\mathbf{K}_m} + \frac{\delta\mu}{\beta g}\exp\left(-\frac{\mathbf{q}^2}{2b}\right)\left(f_1(\mathbf{q}) + \frac{N}{\bar{n}^2}\sum_{\mathbf{K}_m}\delta_{\mathbf{q},\mathbf{K}_m}[f_2(\mathbf{K}_m) + f_3]\right), \tag{34}$$

where

$$f_1(\mathbf{q}) = \frac{1}{\bar{n}}\left\{\left[(2\beta_k - \beta) - |\gamma_k|\cos\left(\frac{k_x k_y + \mathbf{k}\times\mathbf{K}_m}{b} + \theta_k\right)\right] \times \frac{1}{\tilde{\varepsilon}_0(\mathbf{k})} - 1\right\},$$

$$f_2(\mathbf{K}_m) = -\int \frac{d\mathbf{k}}{(2\pi)^2}\left\{\left[1 - \cos\left(\frac{\mathbf{k}\times\mathbf{K}_m}{b}\right)\right]\frac{2\beta_k - \beta}{\tilde{\varepsilon}_0(\mathbf{k})} + \cos\left(\frac{\mathbf{k}\times\mathbf{K}_m}{b}\right)\right\},$$

$$f_3 = -\int \frac{d\mathbf{k}}{(2\pi)^2}\frac{\tilde{\varepsilon}_0(\mathbf{k})}{\beta}. \tag{35}$$

Now we discuss the small-quasimomentum behavior of the static structure factor. Although each of the four terms $S_1, \ldots, S_4$ exhibits infrared divergency at any of the peaks, the sums of S_1, S_2 and S_3, S_4 do not. For example, in the function $f_1(\mathbf{q})$, it can be shown [32] that $\cos[(k_x k_y + \mathbf{k}\times\mathbf{K}_m)/b + \theta_k] \to 1 - (\mathbf{k}\times\mathbf{K}_m)^2$ when $\mathbf{k}\to\mathbf{0}$. Thus the $1/k^2$ singularity from $1/\tilde{\varepsilon}_0(\mathbf{k})$ will be canceled by $(\mathbf{k}\times\mathbf{K}_m)^2$, noting that $\beta_k, \gamma_k \to \beta$ at the same time. Similarly, the function $f_2(\mathbf{K}_m)$ is also not divergent.

IV. QUANTUM MELTING OF THE VORTEX LATTICE

The above calculation Eq. (35) indicates that the quantum fluctuations will reduce the intensity of the Bragg peaks of the static structure factor, and it will be shown below that the reduction is dependent only on the filling fraction ν:

$$\nu = \frac{N}{N_v} = \frac{2\pi\bar{n}}{b}. \tag{36}$$

We denote the ratio of the intensity of the Bragg peak up to first-order quantum correction at, say, the smallest reciprocal lattice vector $\mathbf{K}_1$ to its mean-field value by η,

$$\eta = 1 + \frac{f_2(\mathbf{K}_1) + f_3}{\delta\mu/\beta g} = 1 + 2\pi\frac{f_2(\mathbf{K}_1)/b + f_3/b}{\nu - \pi f_3/b}, \tag{37}$$

where Eq. (24) is used. Using rescaled quasimomenta $\mathbf{q}\to\sqrt{b}\mathbf{q}$, $\mathbf{k}\to\sqrt{b}\mathbf{k}$, one can find that $f_2(\mathbf{K}_1)/b$ and f_3/b are both constants independent of b, implying that the filling fraction ν can be used as the only parameter to control the fluctuation effects on the structure factor. Although the critical ratio η at which the vortex lattice is assumed to melt is not known *a priori* in our case, the numerical simulation of the Yukawa gas indicated a melting η of ~60% [27]. Moreover, for a triangular vortex lattice of a high-T_c superconductor the melting η was found to be ~50% [28] by using an equation similar to Eq. (37) and the exact melting temperature, which was calculated according to a direct comparison of the free energy of the solid phase to that of the liquid phase. Therefore we use this value and invert Eq. (37) to estimate the melting filling fraction. Our numerical calculation indicates that the vortex lattice melts due to quantum fluctuation for $\nu \sim 6$, consistent with some other estimates [10,11,16].

V. CONCLUSION

We have studied the effect of quantum fluctuations on the static structure factor of vortex lattices in a Bose-Einstein condensate in the rapidly rotating limit. We find that the softness of the collective excitation does not cause fluctuation divergency of the static structure factor. Our results also confirm the filling fraction ν to be the characteristic parameter controlling the melting of the lattice. We have shown that, in addition to the vortex-coordinate approach and the generally used elastic theory, the study of the structure factor can also be used to investigate the melting of the vortex lattice. Our estimate for the melting filling fraction, $\nu \sim 6$, is consistent with the exact-diagonaliztion calculation as well as the elastic theory estimates.

ACKNOWLEDGMENTS

We acknowledge fruitful discussions with Professor B. Rosenstein. The work is supported by the National Natural Science Foundation of China under Grant No. 90403002.

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